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TOPOLOGICAL EFFECTS IN MEDIUM

A.N.Sissakian, O.Yu.Shevchenko, S.B.Solganik***

Joint Institute for Nuclear Research, Dubna

*E-mail: shevch@nusun.jinr.ru

**E-mail: solganik@thsun1.jinr.ru

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Developing Fock's ideas, we consider here the topological effects in the gauge field theory. Two closely related topological phenomena are studied at finite density and temperature. These are chiral anomaly and the Chern–Simons term. It occurs that the chiral anomaly doesn't depend on density and temperature. The Chern–Simons term appearance in even dimensions is studied under two types of constraints: chiral and usual charges conservation. In odd dimensions, by using different methods, it is shown that $\mu^2 = m^2$ is the crucial point for Chern–Simons at zero temperature. So when $\mu^2 < m^2$, μ influence disappears and we get the usual Chern–Simons term. On the other hand, when $\mu^2 > m^2$, the Chern–Simons term vanishes because of nonzero density of background fermions. The connection between parity anomalous Chern–Simons in odd dimension and chiral anomaly in even dimension is established at arbitrary density and temperature. These results hold in any dimension both in Abelian and in non-Abelian cases.

Развивая идеи Фока, мы рассматриваем здесь топологические эффекты в теории калибровочных полей. Изучаются два тесно связанных топологических эффекта при конечных плотности и температуре. Это киральная аномалия и черн-саймоновский член. Оказывается, что киральная аномалия не зависит от температуры и плотности. Генерация черн-саймоновского члена изучается в четномерных пространствах при двух типах связей, отражающих сохранение кирального и обычного зарядов. В нечетномерных пространствах при использовании различных методов показывается, что $\mu^2 = m^2$ является критической точкой для черн-саймоновского члена при нулевой температуре. Так, в случае, когда $\mu^2 < m^2$, μ-зависимость исчезает и мы получаем обычный черн-саймоновский член. С другой стороны, когда $\mu^2 > m^2$, черн-саймоновский член исчезает благодаря ненулевой плотности фоновых фермионов. Устанавливается связь между P-аномальным черн-саймоновским членом в нечетномерных и киральной аномалией в четномерных пространствах при произвольных плотностях и температуре. Эти результаты имеют силу при любой размерности как в абелевом, так и в неабелевом случаях.

1. INTRODUCTION

Famous Russian theorist V.A. Fock was one of the first physicists who realized [1] the whole importance of topological phenomenons both in gauge field theory and in general relativity (which, as is well known, may also be viewed as a gauge theory). Namely this interesting subject is the topic of our review article.

[*]E-mail: shevch@nusun.jinr.ru
[**]E-mail: solganik@thsun1.jinr.ru

There is a lot of physical processes where density and temperature play essential role. These are processes occurred under large density background, for example, in quark-gluon plasma or in neutron stars. On the other hand, there exist processes where even negligible density or temperature may give rise to principal effects. One of the most interesting areas, where density and temperature influence could be considerable, is the area of topological effects. Here, even negligible density or temperature could change the topology of the problem as a whole, what could lead to considerable influence. In particular, here we are interested in the Chern–Pontriagin and the Chern–Simons secondary characteristic classes. That corresponds to chiral anomaly in even dimensions and to Chern–Simons (parity anomaly) in odd dimensions. Both phenomena are very important in quantum physics. So, chiral anomalies in quantum field theory have certain direct applications to the decay of π_0 into two photons ($\pi_0 \to \gamma\gamma$), in the understanding and solution of the U(1) problem and so on. On the other hand, there are many effects caused by the Chern–Simons secondary characteristic class. These are, for example, gauge particles mass appearance in quantum field theory, applications to condense matter physics such as the fractional quantum Hall effect and high T_c superconductivity, possibility of free of metric tensor theory construction, etc.

It must be emphasized that these two phenomena are closely related. As was shown (at zero density) in [2–4] the trace identities connect even dimensional anomaly with the odd dimensional Chern–Simons. The main goal of this article is to consider these anomalous objects at finite density and temperature.

It was shown [5, 6] in a conventional zero density and temperature gauge theory that the Chern–Simons term is generated in the Eulier–Heisenberg effective action by quantum corrections. Since the chemical potential term $\mu\bar{\psi}\gamma^0\psi$ is odd under charge conjugation we can expect that it would contribute to P- and CP-nonconserving quantity — the Chern–Simons term. As we will see, this expectation is completely justified. The zero density approach usually is a good quantum field approximation when the chemical potential is small as compared with characteristic energy scale of physical processes. Nevertheless, for investigation of topological effects it is not the case. As we will see below, even a small density could lead to principal effects.

In the excellent paper by Niemi [2] it was emphasized that the charge density at $\mu \neq 0$ becomes nontopological object, i.e., contains both topological part and nontopological one. The charge density at $\mu \neq 0$ (nontopological, neither parity odd nor parity even object)* in QED_3 at finite density was calculated and exploited in [8]. It must be emphasized that in [8] charge density (calculated in the constant pure magnetic field) contains as well parity odd part corresponding to

*For abbreviation, speaking about parity invariance properties of local objects, we will keep in mind symmetries of the corresponding action parts.

the Chern–Simons term, so as parity even part, which can't be covariantized and don't contribute to the mass of the gauge field. Here we are interested in finite density and temperature influence on covariant parity odd form in action leading to the gauge field mass generation — the Chern–Simons topological term. Deep insight on these phenomena at small densities was done in [2,4]. The result for the Chern–Simons term coefficient in QED_3 is

$$\left[\ \mathrm{th}\frac{1}{2}\beta(m-\mu)+\ \mathrm{th}\frac{1}{2}\beta(m+\mu)\right],$$

see [4], formulas (10.18). However, to get this result it was heuristically supposed that at small densities index theorem could still be used and only odd in energy part of spectral density is responsible for parity nonconserving effect. Because of this in [4] it had been stressed that the result holds only for small μ. However, as we'll see below this result holds for any values of chemical potential. Thus, to obtain trustful result at any values of μ one has to use transparent and free of any restrictions on μ procedure, which would allow one to perform calculations with arbitrary non-Abelian background gauge fields.

It was shown at zero chemical potential in [2,4,5] that the Chern–Simons term in odd dimensions is connected with chiral anomaly in even dimensions by trace identities. As we'll see below generalization of the trace identity on nonzero density is not trivial. It connects chiral anomaly with the Chern–Simons term, which has μ- and T-dependent coefficient. We will see below that despite chemical potential and temperature give rise to a coefficient in front of the Chern–Simons term [9] they don't influence chiral anomaly [10,11]. Indeed, anomaly is a short distance phenomenon, which should not be affected by medium (density and temperature) effects, or more quantitatively, so as the anomaly has ultraviolet nature, temperature and chemical potential should not give any ultraviolet effect since distribution functions decrease exponentially with energy in the ultraviolet limit.

The paper is organized as follows. In section 2 we briefly discuss the introducing of the chemical potential, chiral chemical potential and temperature to a theory. Section 3 is devoted to qualitative consideration of chiral anomaly in 2 and 4 dimensions. The rigorous proof of density and temperature independence of axial anomaly is presented in section 4. Also, it is shown in 2-dimensional Schwinger model that chiral anomaly is not influenced not only by chemical potential μ, but also by Lagrange multiplier κ at the constraint of chiral charge conservation. Section 5 is concerned to the Chern–Simons term in even dimensions and its reduction to odd dimension in high temperature limit. In section 6 we obtain the Chern–Simons term in 3-dimensional theory at finite density and temperature by use of a few different methods. In section 7 we evaluate Chern–Simons term in the presence of nonzero temperature and density in 5-dimensional theory and generalize this result on arbitrary non-Abelian odd-dimensional theory.

Nonrelativistic consideration is presented in section 8. In section 9 we generalize trace identity on arbitrary density of background fermions on the basis of the previous calculations. Section 10 is devoted to concluding remarks.

2. CHEMICAL POTENTIAL

As is well known, chemical potential can be introduced in a theory as Lagrange multiplier at corresponding conservation laws. In nonrelativistic physics this is conservation of full number of particles. In relativistic quantum field theory these are conserving charges. The ground state energy can be obtained by use of variational principle

$$\langle \psi^* \hat{H} \psi \rangle = \min \tag{1}$$

under charge conservation constraint for relativistic equilibrium system

$$\langle \psi^* \hat{Q} \psi \rangle = \mathrm{const}, \tag{2}$$

where $\hat{H}$ and $\hat{Q}$ are Hamiltonian and charge operators. Instead, we can use method of undetermined Lagrange multipliers and seek absolute minimum of expression

$$\langle \psi^* (\hat{H} - \mu \hat{Q}) \psi \rangle, \tag{3}$$

where μ is Lagrange multiplier. Since $\hat{Q}$ commute with the Hamiltonian, $\langle \hat{Q} \rangle$ is conserved.

On the other hand, we can impose another constraint, which implies chiral charge conservation

$$\langle \psi^* \hat{Q}_5 \psi \rangle = \mathrm{const}, \tag{4}$$

or in Lagrange approach we have

$$\langle \psi^* (\hat{H} - \kappa \hat{Q}_5) \psi \rangle = \min, \tag{5}$$

where κ arises as Lagrange multiplier at $\langle \hat{Q}_5 \rangle = \mathrm{const}$ constraint. Thus, μ corresponds to nonvanishing fermion density (number of particles minus number of antiparticles) in background. Meanwhile, κ is responsible for conserving asymmetry in numbers of left- and right-handed background fermions.

It must be emphasized that the formal addition of a chemical potential in the theory looks like a simple gauge transformation with the gauge function μt. However, it doesn't only shift the time component of a vector potential but also gives corresponding prescription for handling Green's function poles. The correct introduction of a chemical potential redefines the ground state (Fermi energy), which leads to a new spinor propagator with the correct ϵ prescription for poles. So, for the free spinor propagator we have (see, for example, [12, 13])

$$G(p;\mu) = \frac{\tilde{\not p} + m}{(\tilde{p_0} + i\epsilon\ \mathrm{sgn} p_0)^2 - \vec{p}^{\,2} - m^2}, \tag{6}$$

where $\tilde{p} = (p_0+\mu, \vec{p})$. Thus, when $\mu = 0$ one at once gets the usual ϵ prescription because of the positivity of $p_0\ \mathrm{sgn}\, p_0$. In the presence of a background Yang–Mills field we consequently have for the Green function operator (in Minkovski space)

$$\hat{G} = (\gamma\tilde{\pi} - m)\frac{1}{(\gamma\tilde{\pi})^2 - m^2 + i\epsilon(p_0+\mu)\ \mathrm{sgn}(p_0)}, \tag{7}$$

where $\tilde{\pi_\nu} = \pi_\nu + \mu\delta_{\nu 0}$, $\pi_\nu = p_\nu - gA_\nu(x)$.

In Euclidian metric one has

$$G(p;\mu) = \frac{\tilde{\not p} + m}{\tilde{p_0}^2 + \vec{p}^{\,2} + m^2}, \tag{8}$$

where $\tilde{p} = (p_0 + i\mu, \vec{p})$.

For temperature introduction we will use a standard Matzubara approach valid for systems in equilibrium. That is Euclidian generating functional with temperature instead of time, and antiperiodic conditions on fermion fields $\psi(0,\vec{x}) = -\psi(\beta,\vec{x})$ and periodic for boson ones $A(0,\vec{x}) = A(\beta,\vec{x})$. Thus, for transfer to finite temperature case we will use

$$\int d^D x \to i\int_0^\beta dx_0 \int d^{D-1}x,$$

$$\int \frac{d^D k}{(2\pi)^D} \to \frac{i}{\beta}\sum_{n=-\infty}^{\infty}\int \frac{d^{D-1}k}{(2\pi)^{D-1}},$$

together with $p_0 \to \omega_n = (2n+1)\pi/\beta$. Here, the chemical potential also can be introduced by adding it to a Matzubara frequency $p_0 \to \omega_n = (2n+1)\pi/\beta + i\mu$.

3. CHIRAL ANOMALY. QUALITATIVE CONSIDERATION

First of all let us consider simple but rather intuitive than rigorous derivation of axial anomaly [14]. Let us start with 2–dimensional right-handed Weyl fermion theory coupled to a uniform electric field $\dot{A}_1 = E$ in the temporal gauge. The one component right-handed Weyl equation for $\psi_R = 1/2(1+\gamma_5)\psi$ reads

$$i\dot{\psi}_R(x) = (-i\partial_x - A^1)\psi_R(x). \tag{9}$$

The dispersion law is $\omega(P) = P$. Corresponding to the classical equation of a charged particle in the presence of an electric field where $\dot{P} = eE$, the acceleration of the right-handed particles in quantum theory is given by $\dot{\omega} = \dot{P} = eE$. The creation rate of the right-handed particles per unit time and unit length is determined by a charge of the Fermi surface, which distinguishes the filled and unfilled states. Let the quantization length be L; the density of states per length L is $L/2\pi$ and the rate of change of right-handed particle number N_R is

$$\dot{N}_R = L^{-1}(L/2\pi)\dot{\omega} = (e/2\pi)E. \tag{10}$$

This particle creation is the axial anomaly. Consequently the chiral charge Q_R is not conserved and $\dot{Q}_R = \dot{N}_R = (e/2\pi)E$. It follows from an analogous reasoning that the annihilation rate of left-handed particles with the dispersion law $\omega = -P$ is

$$\dot{N}_L = -(e/2\pi)E. \tag{11}$$

Therefore the anomaly for the Dirac particles is

$$\dot{N}_R - \dot{N}_L = (e/\pi)E, \tag{12}$$

which gives $\dot{Q}_5 = (e/\pi)E$.

In 4 dimensions we first calculate the energy levels of the right-handed Weyl fermion in the presence of the applied uniform magnetic field along the third direction given by

$$A^2 = Hx^1 \quad \text{and} \quad A^\mu = 0 \quad \text{otherwise.}$$

The solution to the equation for two-component right-handed field ψ_R of the form

$$[i\partial/\partial t - (\mathbf{P} - e\mathbf{A})\sigma]\,\psi_R(x) = 0 \tag{13}$$

is expressed in terms of a solution of the auxiliary equation

$$[i\partial/\partial t - (\mathbf{P} - e\mathbf{A})\sigma]\,[i\partial/\partial t + (\mathbf{P} - e\mathbf{A})\sigma]\,\Phi = 0 \tag{14}$$

as

$$\psi_R = [i\partial/\partial t + (\mathbf{P} - e\mathbf{A})\sigma]\,\Phi. \tag{15}$$

From Eq. (14) the energy and the P_2, P_3 eigenfunction satisfies an equation of the harmonic oscillator type

$$\left[-(\partial/\partial^1)^2 + (eH)^2(x^1 + P_2/eH) + (P_3)^2 + eH\sigma\right]\Phi = \omega^2\Phi,$$

where $\sigma = \pm 1$. The energy levels are given by the Landau levels,

$$\omega(n, \sigma, P_3) = \pm \left[eH(2n+1) + (P_3)^2 + eH\sigma\right]^{1/2}, \quad (n = 0, 1, 2, ...) \tag{16}$$

except for the $n = 0$ and $\sigma = -1$ mode, where

$$\omega(n = 0, \sigma = -1, P_3) = \pm P_3. \tag{17}$$

The eigenfunction takes the form

$$\Phi_{n\sigma}(x) = Nn\sigma \exp(-iP_2 x^2 - iP_3 x^3) \exp\left[-1/2eH(x_1 + P_2/eH)^2\right] H_n(x_1 + P_2/eH)\xi(\sigma), \tag{18}$$

with $Nn\sigma$ as the normalization constant. Here $\xi(\sigma)$ denotes the eigenfunctions of the Pauli spin σ_3 which can be taken as $\xi(1) = \binom{1}{0}$ and $\xi(-1) = \binom{0}{1}$. The solution of (14) is obtained by inserting (18) into (15). This leads to the relations

$$\psi_R^{(n+1,\sigma=-1)} = (N_{n+1,\sigma=-1}/N_{n,\sigma=1})\psi_R^{(n,\sigma=1)} \quad , \quad n = 0, 1, ...$$

and

$$\psi_R^{(n=0,\sigma=-1)} = 0, \;\; \text{with} \;\; \omega = -P_3.$$

Thus the energy levels of ψ_R are (16) and

$$\omega(n = 0, \sigma = -1, P_3) = P_3. \tag{19}$$

Next a uniform electric field is turned on along the third direction parallel to H. As for the zero mode $(n = 0, \sigma = -1)$ the dispersion law is the same as that for 2 dimensions and the creation rate of the particles is calculated in a similar manner. It should be noted that when E varies adiabatically there is no particle creation in the $n = 0$ modes. The density of the state per length L is $LeH/4\pi^2$ and the creation rate is given by

$$\dot{N}_R = L^{-1}(LeH/4\pi^2)\dot{\omega}(n = 0, \sigma = -1, P_3) = (e^2/4\pi^2)EH, \tag{20}$$

which equals to $\dot{Q}_R$.

For the left-handed fermions the annihilation rate of the left-handed particles is

$$\dot{N}_L = -(e^2/4\pi^2)EH, \tag{21}$$

which is $\dot{Q}_L$.

We then have for the Dirac field

$$\dot{Q}_5 = \dot{Q}_R - \dot{Q}_L = (e^2/2\pi^2)EH, \tag{22}$$

that is the chiral anomaly.

Now we can easily estimate influence of background density on the anomaly in this approach. From the above consideration we can see that the anomaly is proportional to the time derivative of the zero mode energy. Taking into account that nonzero fermion density influence just reduces to the shift of the Landau levels on μ which doesn't depend on time, we can conclude that finite density doesn't influence the chiral anomaly. The same arguments are just for the Lagrange multiplier at the axial charge κ, the only difference is that κ makes shift for left- and right-handed fermions with opposite sign. So, until μ (κ) is time independent it won't affect the chiral anomaly. We would like to stress that here there was made adiabatic approximation, when we turn on electric field. So, this consideration is just a plot and it needs a strict proof.

4. CHIRAL ANOMALY AT FINITE TEMPERATURE AND DENSITY

4.1. Two Dimensions. Since anomaly term originates from the ultraviolet divergent part, it is not expected to be changed by the temperature. Indeed, it was shown in several papers (see [10] and references therein). Moreover, the same can be said about the influence of background fermion density that has been checked in the works [11].

To clear understand the nature of anomaly μ independence we'll first consider the simplest case — 2-dimensional QED by the use of the Schwinger nonperturbative method [16]. Thus, following Schwinger one writes

$$J^{\mu} = -ig \ \mathrm{tr} \left[\gamma^{\mu} G(x, x^{'}) \exp \left(-ig \int_{x^{'}}^{x} d\xi^{\mu} A_{\mu}(\xi) \right) \right]_{x^{'} \to x}, \tag{23}$$

where $G(x, x^{'})$ is a propagator satisfying the following equation

$$\gamma^{\mu} \left(\partial_{\mu}^{x} - igA_{\mu}(x) \right) G(x, x^{'}) = \delta(x - x^{'}). \tag{24}$$

Further we use Schwinger's anzats

$$G(x, x^{'}) = G^{0}(x, x^{'}) \exp \left[ig(\phi(x) - \phi(x^{'})) \right], \tag{25}$$

where $G^{0}(x, x^{'})$ is a free propagator

$$\gamma^{\mu} \partial_{\mu}^{x} G^{0}(x, x^{'}) = \delta(x - x^{'}).$$

Thus, for ϕ we can write $\gamma^\mu \partial_\mu \phi = \gamma^\mu A_\mu$. At finite density $G^0(x,x^{'})$ has the form

$$
\begin{aligned}
G^0(x,x^{'}) = \int \frac{d^2p}{(2\pi)^2}\, \mathrm{e}^{ip(x-x^{'})} \frac{\not{p}}{p^2 + i\varepsilon(p_0+\mu)\,\mathrm{sgn}p_0} = \\
= -i\, \not{\partial} \Bigg[\int \frac{d^2p}{(2\pi)^2}\, \mathrm{e}^{ip(x-x^{'})} \frac{1}{p^2+i\varepsilon} - \\
-2\int_{-\infty}^{+\infty} \frac{dp_1}{2\pi} \int_{-\infty}^{+\infty} \frac{dp_0}{2\pi} \theta(-\tilde{p}_0\,\mathrm{sgn}p_0)\, \mathrm{e}^{ip(x-x^{'})} \Im m \frac{1}{p^2+i\varepsilon} \Bigg].
\end{aligned}
\tag{26}
$$

So, beside the usual zero density part μ–dependent one appears. Further, we have to regularize current by use of symmetrical limit $x \to x^{'}$. After some simple algebra it is clearly seen that all μ–dependent terms after taking off the limit will disappear. Thus, contribution to the current arises from the Schwinger part only, so

$$
\begin{aligned}
J^\mu &= i\frac{g^2}{2\pi}\left(\delta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\partial^2}\right) A_\nu \\
J^\mu_5 &= i\frac{g^2}{2\pi}\left(\varepsilon^{\mu\nu} - \varepsilon^{\mu\alpha}\frac{\partial^\alpha \partial^\nu}{\partial^2}\right) A_\nu ,
\end{aligned}
\tag{27}
$$

and we get usual anomaly in chiral current

$$
\partial_\mu J^\mu = 0 \ , \qquad \partial_\mu J^\mu_5 = i\frac{g^2}{2\pi}\varepsilon^{\mu\nu}\partial_\mu A_\nu = i\frac{g^2}{4\pi}{}^*F. \tag{28}
$$

It is natural to introduce Lagrange multiplier κ at corresponding constraint to support the conservation of the Q^5 charge, i.e., the difference of left and right fermion densities $Q_L - Q_R$. Since κ and μ are Lagrange multipliers at corresponding conservation laws they, in principle, have to influence some way a symmetry violation by a quantum corrections, i.e., anomalies. However, the rather amazing situation occurs. The demand of chiral charge conservation (instead of the usual charge conservation) on the quantum level doesn't influence the chiral anomaly. Really, in 2-dimensions introduction of Lagrange multiplier κ at the chiral charge conservation gives the term $\kappa\bar{\psi}\gamma^5\gamma^0\psi = \kappa\bar{\psi}\gamma^1\psi$ in Lagrangian. So, κ affects in the same way as μ, i.e., κ doesn't influence the chiral anomaly (it is also seen in direct calculations, which are similar to presented above for the case with μ). That could be explained due to ultraviolet nature of the chiral anomaly, while κ (μ) doesn't introduce new divergences in the theory.

From the calculations it is clearly seen the principal difference of the chiral anomaly and Chern–Simons. The ultraviolet regulator — P exponent — gives rise to the anomaly, but (as we'll see below) doesn't influence Chern–Simons.

Thus, it is natural that the anomaly doesn't depend on μ, κ, and T because it has ultraviolet regularization nature, while neither density nor temperature does influence ultraviolet behavior of the theory. The general and clear proof of axial anomaly temperature independence in any even dimension will be presented in section 9 on the basis of the trace identities.

4.2. Four Dimensions. In [11] direct calculations of axial anomaly at finite temperature and density in 4-dimensional gauge theory were performed by using imaginary and real time formalism by Fujikawa method [15]. Here we present the derivation of the axial anomaly using the elegant Fujikawa procedure. Considering a system of fermions and gauge bosons in thermodynamical equilibrium at temperature $T=\beta^{-1}$ and nonzero chemical potential μ in the imaginary time formalism one reads the generating functional of correlation functions

$$Z[J_\nu,\eta,\bar\eta] = \int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar c\mathcal{D}\psi\mathcal{D}\bar\psi$$
$$\exp\left[\int_0^\beta d\tau\int d^3x\left(\mathcal{L}(\vec x,\tau)+J_\nu A^\nu+\bar\psi\eta+\psi\bar\eta\right)\right], \qquad (29)$$

where

$$\mathcal{L}(\vec x,\tau)=\mathcal{L}_\psi+\mathcal{L}_{YM}+\mathcal{L}_c+\mathcal{L}_{GF}$$

represents the effective Lagrangian density of the $SU(N)$ Yang–Mills field $A = (A^j_\rho)$ coupled to fermion fields $\psi=(\psi^a_\alpha)$, $\bar\psi=(\bar\psi^a_\alpha)$ and to Faddeev–Popov ghost fields $c=(c_\alpha)$, $\bar c=(\bar c_\alpha)$. $\eta=(\eta^a_\alpha)$, $\bar\eta=(\bar\eta^a_\alpha)$ and $J=(J^j_\rho)$ are external sources. A^j_ρ, J^j_ρ, c_α, $\bar c_\alpha$ are periodic in τ with period β, while ψ^a_α, $\bar\psi^a_\alpha$, η^a_α, $\bar\eta^a_\alpha$ are antiperiodic. Upper latin indices and lower Greek ones indicate flavor and $SU(N)$ internal (color) indices respectively, and $j=1,...,N^2-1$, the number of standard $SU(N)$ generators (T^j). $\mathcal{L}_{YM}$ and $\mathcal{L}_c$ are standard Lagrangian densities for Yang–Mills bosons and ghosts, while $\mathcal{L}_{GF}$ describes gauge fixing. On the other hand, one has

$$\mathcal{L}_\psi=\sum_{a=1}^{N_f}\bar\psi^a(i\,\not{D}^a_{T,\mu}-m^a)\psi^a$$

(lower color indices being also implicitly contracted), with N_f the number of flavors,

$$i\not{D}^a_{T,\mu}=i\not{D}_T+\mu^a\gamma_0$$

and

$$\not{D}_T=i\gamma^0(\partial/\partial\tau+A_4)-\gamma^k(\partial/\partial x^k+A_k),$$

where $iA_\rho=gT^jA^j_\rho$ and the Wick rotation has been performed in the imaginary time formalism ($x_0\to\tau=ix_0,\ A_0\to A_4=-iA_0$) so that $\not{D}_T$ becomes

Hermitian. It is considered a chemical potential μ^a for each flavor (there is no flavor mixing).

Following Fujikawa [15] we are interested in the chiral transformation of the fermion fields:

$$\psi_\alpha^a \rightarrow \exp[i\delta(\vec{x},\tau)\gamma_5]\psi_\alpha^a\ , \quad \bar{\psi}_\alpha^a \rightarrow \bar{\psi}_\alpha^a \exp[i\delta(\vec{x},\tau)\gamma_5],$$

which produces a change in the fermion measure $\mathcal{D}\psi\mathcal{D}\bar{\psi} \rightarrow C\mathcal{D}\psi\mathcal{D}\bar{\psi}$, giving rise to the anomaly factor in the chiral current conservation law, C, which is the direct finite temperature and density extension of the zero temperature and density factor appearing in [15]. The chemical potential term is invariant under the above chiral transformation. Then the only possible finite temperature and density effects must be contained in C. In order to display them, let us expand

$$\psi^a(\vec{x},\tau) = \sum_n a_n \phi_n^a(\vec{x},\tau)\ \ , \quad \bar{\psi}^a(\vec{x},\tau) = \sum_n \bar{b}_n \phi_n^{a\,+}(\vec{x},\tau),$$

$a_n, \bar{b}_n$ being elements of the Grassmann algebra. On the other hand, $\phi_n^a(\vec{x},\tau)$, which is antiperiodic in τ, is an eigenfunction of the Hermitian operator $\not{D}_{T,\mu}^a = = \not{D}_T + \mu^a\gamma_0$, i.e., $\not{D}_{T,\mu}^a \phi_n^a = \lambda_n \phi_n^a$, ($\lambda_n$ being real and color indices being omitted) and it fulfills

$$\int_0^\beta d\tau \int d^3x \phi_n^{a\,+}\phi_r^a = \delta_{nr}.$$

Moreover, it can be Fourier–expanded as

$$\phi_n^a(\vec{x},\tau) = \frac{1}{\beta}\sum_{j=-\infty}^{\infty}\int d^3k\ \mathrm{e}^{i\omega_n\tau}\ \mathrm{e}^{-i\vec{k}\vec{x}}\phi_j^a(\vec{k}), \tag{30}$$

$$\omega_n = \frac{(2n+1)\pi}{\beta}.$$

Then, the measure $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ becomes $\prod_n da_n \prod_m d\bar{b}_m$, and by extending directly the zero temperature and density calculations [15], one finds

$$C = \exp\left[-2i\int_0^\beta d\tau \int d^3x \delta(\vec{x},\tau)a(\vec{x},\tau)\right] \tag{31}$$

with

$$a(\vec{x},\tau) = \sum_{a=1}^{N_f}\sum_n \phi_n^{a\,+}(\vec{x},\tau)\gamma_5\phi_n^a(\vec{x},\tau). \tag{32}$$

The finite temperature and density anomaly $a(\vec{x},\tau)$ can be regularized by extending again Fujikawa's trick as

$$a(\vec{x},\tau) = \lim_{M\to\infty} \sum_{a=1}^{N_f} \sum_n \phi_n^{a+}(\vec{x},\tau)\gamma_5 \exp\left[-M^{-2}(\not{D}^a_{T,\mu})^2\right] \phi_n^a(\vec{x},\tau), \qquad (33)$$

and by changing the basis vectors to "plane waves" with (30). We remark that $\not{D}^a_{T,\mu}$ is equivalent to $\not{D}_T$ provided that, in the latter, one replaces A_4 by $A_4 - i\mu^a$. This replacement leaves $F_{\rho\nu}$ invariant ($F_{\rho\nu} = \partial_\rho A_\nu - \partial_\nu A_\rho + [A_\rho, A_\nu]$): notice that μ^a is constant and $[\mu^a, T^j] = 0$ for $a = 1...N_f$, $j = 1...N^2-1$. One obtains

$$a(\vec{x},\tau) = N_f \lim_{M\to\infty} \mathrm{tr}\left(\gamma_5 \left\{[\gamma^\rho,\gamma^\nu]F_{\rho\nu}\right\}^2\right) \frac{1}{8M^2}\frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \exp\left[-\left(\frac{\omega_n^2+\vec{k}^2}{M^2}\right)\right], \qquad (34)$$

where the trace tr runs over both internal and γ matrices indices.

The infinite series on the right-hand side of (34) displays what is, quite likely, the most important difference between the actual finite temperature and density case and the zero temperature and density one treated in [15]. We recall the following formula valid for any M:

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty} \exp\left[-\frac{\pi^2}{M^2\beta^2}(2n+1)^2\right] = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \exp\left[-\left(\frac{k^0}{M}\right)^2\right]. \qquad (35)$$

A simple derivation of (35) can be found in [25].

By using (35) in (34) and taking the trace over the γ matrices, we arrive at the final formula:

$$a(\vec{x},\tau) = -\frac{N_f}{16\pi^2}\, \mathrm{tr}\left(\frac{1}{2}\epsilon^{\sigma\nu\rho\lambda}F_{\rho\lambda}F_{\sigma\nu}\right), \qquad (36)$$

where the trace now runs only over internal indices. We can see that there are no finite temperature and density corrections to the chiral anomaly, as we have expected from the previous considerations.

5. CHERN–SIMONS IN EVEN DIMENSIONAL THEORY

It will be natural to introduce in Lagrangian the classical conservation law — the conservation of the Q^5 charge, i.e., the difference of left and right fermion densities $Q_L - Q_R$. Thus the Lagrangian with constraint on Q_5 has the form

$$\mathcal{L} = \frac{1}{8}\,\mathrm{tr}FF + \bar{\psi}\left(i\hat{\partial} - g\hat{A} + i\kappa\gamma^0\gamma^5\right)\psi. \qquad (37)$$

Thus, if we will deal with such a Lagrangian we must get theory in which Q^5-charge is conserved. To get effective action only background field dependent we have to take over $d\bar{\psi}\, d\psi$ integration. There are two ways to do it: one can calculate straight forward by using the perturbation theory and get the effective action, another one is proper time method. Certainly, we have to take into account that at high temperatures dimensional reduction takes place. Thus, for example, vacuum polarization tensor in reduced — 3-dimensional — theory can be written as

$$\Pi_{ij}(p^2) = \left(g_{ij}p^2 - p_i p_j\right)\Pi^{(1)}(p^2) + ie_{ijk}p^k\Pi^{(2)}(p^2) + p_i p_j \Pi^{(3)}(p^2). \tag{38}$$

The part of the vacuum polarization tensor containing Levi–Chivita tensor $ie_{ijk}p^k\Pi^{(2)}(p^2)$ gives rise to the Chern–Simons term.

It is convenient to rewrite Lagrangian in more appropriate form using projection operators

$$\mathcal{L} = \frac{1}{2}\,\mathrm{tr}FF + \bar{\psi}_L\left(i\hat{\partial} - g\hat{A} + i\kappa\gamma^0\right)\psi_L + \bar{\psi}_R\left(i\hat{\partial} - g\hat{A} - i\kappa\gamma^0\right)\psi_R, \tag{39}$$

where we have used $I = P_+ + P_-\,,\ \gamma^5 = P_+ - P_-\,,\ P_+ = \frac{1+\gamma^5}{2}\,,\ P_- = \frac{1-\gamma^5}{2}$.

So, now we can evaluate J_L and J_R separately. One can easily see that the Lagrangian we have got is absolutely analogous to finite temperature and density Lagrangian with left(right)-handed fermions which was considered in [17] using perturbative expansion.

Thus we can immediately write the answer for J_L and J_R currents

$$J^\mu_{(L/R)} = (\pm)\frac{\kappa}{4\pi}\beta W[A], \tag{40}$$

where $W[A]$ is the Chern–Simons term. And consequently for full current and chiral current we'll get correspondingly

$$J^\mu = 0, \tag{41}$$

$$J^\mu_5 = 2\frac{\kappa}{4\pi}\beta W[A]. \tag{42}$$

It is also possible to obtain Chern–Simons at zero temperature for $\kappa \neq 0$ with clear physical sense (see, for example, [18] where chiral fermions are considered at finite density and [14] where Weyl particles are considered). In the 2-dimensional Schwinger model there is chiral anomaly

$$\partial_\mu J^5_\mu = -\frac{1}{\pi}\varepsilon_{\mu\nu}F^{\mu\nu}. \tag{43}$$

It could be derived by using the picture of energy levels crossing, see for example [14,19]. Here, we will exploit this method for consideration of the Chern–Simons term. Thus we will consider the Schwinger model (37) on a ring with periodic for A_μ and antiperiodic for ψ boundary conditions

$$A(x=-L/2,t)=A(x=L/2,t),$$
$$\psi(x=-L/2,t)=-\psi(x=L/2,t). \tag{44}$$

Thus, fields A and ψ could be expanded in Fourier modes $\exp(ikx2\pi/L)$ for bosons and $\exp(i[k+1/2]x2\pi/L)$ for fermions. The Lagrangian (37) is invariant under local gauge transformations:

$$\psi \to \psi\ \mathrm{e}^{i\alpha(x,t)}\ , \quad A_\mu \to A_\mu + \partial_\mu \alpha(x,t).$$

It is easily seen that due to local gauge transformations, we can put all modes of A_1 to be zero except for the zero-mode. Thus, we can consider A_1 to be x-independent. There exists another type of gauge transformations (large gauge transformations)

$$\alpha = \frac{2\pi}{L} n x,$$

where n is an integer number. Nevertheless, this gauge is not periodic, it satisfies condition (44). Really, $\partial\alpha/\partial x = \mathrm{const}$ and $\partial\alpha/\partial t = 0$, thus periodicity of A_μ is conserved, the same is also true for ψ. So, we can consider the model on the circle $[0, 2\pi/L]$. Further, we use adiabatic approximation, putting that A_1 is independent of time (to a slight time dependence we will turn on later), and that $A_0 = 0$. This adiabatic approximation is quite natural from the physical point of view, see for example elegant consideration by Shifman [19]. We now calculate number density of real left(right) fermions $n_{L/R}[A_1]$ and fermionic energy density $\varepsilon_{L/R}[A_1]$, assuming that number density $n_{L/R}$ at $\dot{A}_1 = 0$ is fixed. Note that system with fixed $n_{L/R}$ can be prepared by inserting fermions into the box, which is initially empty.

It is straightforward to calculate the fermionic spectrum at $A_1 \neq 0$,

$$E_{L/R} = \frac{2\pi}{L}(k \mp N_{CS}) \quad , \qquad k = 0, \pm 1, \pm 2, ... \ , \tag{45}$$

where

$$N_{CS} = \frac{1}{2\pi}\int A_1 dx^1 \tag{46}$$

is the Chern–Simons number in (1+1) dimensions. As the gauge field changes from zero to some fixed A_1, $[N_{CS}]$ levels of left-handed fermions cross zero from above and the same number of right-handed fermionic levels cross zero from below. This means that $[N_{CS}]$ left-handed fermions fill the negative energy levels in the Dirac sea, see Fig.1, and the same number of right-handed fermions leave it. We would like to stress, that in this physical clear picture it is essential to use the adiabatic approximation. The number densities for left(right)-handed fermions are

$$n_{L/R}[A_1] = n^0_{L/R} \mp n_{CS} + O(L^{-1}), \tag{47}$$

where $n_{CS} = N_{CS}/L$ is the average Chern–Simons density.

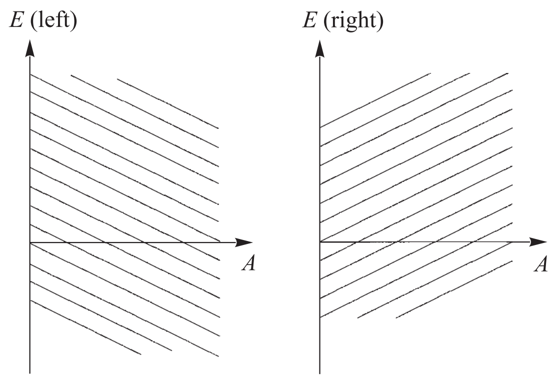

Fig. 1. Behaviour of the left(right)-handed fermionic levels

Note that equation (47) is essentially the integral form of the anomaly equation (43). The average energy density of real fermions is

$$\varepsilon_{L/R} = \frac{2}{L} \sum_{[N_{CS}]+1}^{N^0_{L/R}/2} \varepsilon_k = \frac{\pi}{2}(n^0_{CS} \mp n_{CS})^2 + O(L^{-2}). \tag{48}$$

We can introduce chemical potential for left(right)-handed fermions in a standard way

$$\mu_{L/R} = \frac{\partial \varepsilon_{L/R}}{\partial n_{L/R}} \tag{49}$$

and we obtain

$$\mu_{L/R} = \pi(n_{L/R} \mp n_{CS}). \tag{50}$$

Introducing the standard Legendre transform

$$\tilde{E}_{L/R}[\mu_{L/R}, A_1] = E_{L/R} \mp \mu_{L/R} N_{L/R} \tag{51}$$

we find

$$\Delta E = (\mu_R - \mu_L) N_{CS}, \tag{52}$$

so, for the case when $\mu_R = \mu_L = \mu$ we'll get energy unchanged. On the other hand, for chiral fermions [18] sign of term μ_R will change and

$$\Delta E = -2\mu N_{CS}. \tag{53}$$

If we impose conservation of the left- and right-handed fermions (with Lagrangian multiple κ) instead of separate conservation of left (right)-fermions, we'll get

$$\Delta E = -2\kappa N_{CS}. \tag{54}$$

Thus, the same result arises both for chiral fermions at finite density, and for usual fermions under conservation of chiral charge. One should notice that here there were used two approximations. The first one is time independence of A_0, the second is adiabatic approximation. Nevertheless, this consideration is valuable due to construction of clear physical picture of the phenomenon.

The Chern–Simons term appearance in even dimensional theory could be shown in simple and clear way. The only thing we need for it is temperature and density independence of chiral anomaly (see previous sections). From the definition one has

$$\frac{\partial I_{eff}}{\partial \kappa} = \int d^D x \langle J_5^0 \rangle. \tag{55}$$

Since axial anomaly doesn't depend on κ, effective action contains the term proportional to anomalous Q_5 charge with κ as a coefficient. The same is for a chiral theory, there effective action contains the term proportional to anomalous Q charge with μ as a coefficient, see for example [17, 18, 20]. So, we have

$$\Delta I_{eff} = -\kappa \int dx_0 W[A] \tag{56}$$

in conventional gauge theory and

$$\Delta I_{eff}^{\mathrm{chiral}} = -\mu \int dx_0 W[A] \tag{57}$$

in the chiral theory. Here $W[A]$ is the Chern–Simons term. Thus we get Chern–Simons with Lagrange multiplier as a coefficient.

It is well known that at nonzero temperature in $\beta \to 0$ limit the dimensional reduction effect occurs. So, extra t-dependence of Chern–Simons term in (56) disappears and Chern–Simons can be treated as a mass term in 3-dimensional theory with $i\kappa/T$ coefficient (the same for chiral theory with μ, see [17]). For anomalous parts of effective action we have

$$\Delta I_{eff} = -i\kappa\beta W[A] \ , \qquad \Delta I_{eff}^{\text{chiral}} = -i\mu\beta W[A] \tag{58}$$

in conventional and chiral gauge theories correspondingly. The only problem arises in treating Chern–Simons as a mass term is that the coefficient is imaginary, see discussions on the theme in [17,20]. The other problem is that the coefficient is not the integer function, see discussions in conclusion. One can notice, that results (56), (57) and (58) hold in arbitrary even dimension. Let us stress, that we don't need any complicated calculations to obtain (56)–(58). The only thing we need is the knowledge of chiral anomaly independence on μ, κ, and β.

This result also can be derived by use of the proper-time method. Chiral current reads as follows

$$J_5^\mu = -ig\ \mathrm{tr}\left[\gamma^5\gamma^\mu G(x,x^{'})\right] = -ig\ \mathrm{tr}\left[\gamma^5\gamma^\mu \frac{1}{i\ \partial\!\!\!/ - g\ A\!\!\!/ + \gamma^5\gamma^0\kappa}\right] =$$
$$= -ig\ \mathrm{tr}\left[\gamma^\mu \frac{1}{i\ \partial\!\!\!/ - g\gamma^5\ A\!\!\!/ + \gamma^0\kappa}\right]. \tag{59}$$

The propagator has the following form

$$G(x,x^{'}) = \left[i\ \partial\!\!\!/ - g\gamma^5\ A\!\!\!/\right](-i)\int_{-\infty}^{0} d\tau U(x,x^{'};\tau), \tag{60}$$

where $U(x,x^{'};\tau)$ is the evolution operator in a proper time. The propagator after substitution of the evolution operator can be rewritten as

$$G(x,x^{'}) = \exp\left(-ig\gamma^5\int_{x^{'}}^{x} d\zeta_\mu A^\mu\right)\int \frac{d^4p}{(2\pi)^4}\ \mathrm{e}^{ip(x-x^{'})}G(p), \tag{61}$$

where $G(p)$ has the form

$$G(p) = -ia_d\int_{-\infty}^{0}\frac{d\tau}{\tau^{d/2}}\exp\Big(-\frac{1}{2}\ \mathrm{tr}\ln\left[\frac{i}{4\tau}\ \mathrm{ch}(gF\tau)\right] -$$
$$-ip(gF)^{-1}\ \mathrm{th}(gF\tau)p\Big)\left[\gamma^5\gamma^\alpha\left(\ \mathrm{th}(gF\tau)\right)^{\alpha\nu}p^\nu - p\!\!\!/\right]$$
$$\exp\left(i\frac{g}{2}\gamma^5\sigma_{\mu\nu}F^{\mu\nu}\tau\right), \tag{62}$$

where $a_d = \ \mathrm{e}^{i\pi d/4}/(2\pi)^{d/2}$. Substituting expression for the propagator in (59), we will get for the chiral current

$$J_5^\mu = ga_d\int_{-\infty}^{0}\frac{d\tau}{\tau^{d/2}}\exp\left(-\frac{1}{2}\ \mathrm{tr}\ln\left[\frac{i}{4\tau}\ \mathrm{ch}(gF\tau)\right]\right)$$

$$\int \frac{d^4p}{(2\pi)^4} \exp\left(-ip(gF)^{-1}\,\mathrm{th}(gF\tau)p\right) p^\nu$$
$$\mathrm{tr}\Big[\left(\gamma^\mu\gamma^5\gamma^\alpha\,[\,\mathrm{th}(gF\tau)]^{\alpha\nu} - \gamma^\mu\gamma^\alpha g^{\alpha\nu}\right)\exp\left(i\frac{g}{2}\gamma^5\sigma_{\mu\nu}F^{\mu\nu}\tau\right)\Big]. \tag{63}$$

Taking into account that p integration is at finite density and temperature, i.e., integral in p_0 is changed on sum, and extracting the Levi–Chivita tensor containing part (it is really simple, if one takes traces in covariant form) we obtain

$$J_5^\mu = \frac{g^2}{8\pi^2}\beta \int \frac{d\tau}{\tau^2} \sum_{m=1}^{\infty} (-1)^m m \exp\left(i\frac{\beta^2 m^2}{4\tau}\right)\ \mathrm{sh}(\beta m\kappa)\ {}^*F^{\mu 0}. \tag{64}$$

To regulate this expression we use dimensional regularization, which can be expressed in terms of generalized Riemann zeta functions. Also, we take high temperature limit, i.e., $\beta \to 0$, and finally get

$$J_5^\mu = i\kappa\frac{g^2}{2\pi^2}\ {}^*F^{\mu 0}. \tag{65}$$

6. CHERN–SIMONS IN THREE-DIMENSIONAL THEORY

6.1. Constant Magnetic Field. Let us first consider a (2+1)-dimensional Abelian theory in the external constant magnetic field. We will evaluate fermion density by performing the direct summation over Landau levels. As a starting point, we will use the formulae for fermion number at finite density and temperature [2]

$$N = -\frac{1}{2}\sum_n\ \mathrm{sgn}(\lambda_n) + \sum_n \Big[\frac{\theta(\lambda_n)}{\exp(-\beta(\mu-\lambda_n))+1} - $$
$$-\frac{\theta(-\lambda_n)}{\exp(-\beta(\lambda_n-\mu))+1}\Big] =$$
$$= \frac{1}{2}\sum_n\ \mathrm{th}\frac{1}{2}\beta(\mu-\lambda_n) \stackrel{\beta\to\infty}{\longrightarrow} \frac{1}{2}\sum_n\ \mathrm{sgn}(\mu-\lambda_n). \tag{66}$$

Landau levels in the constant magnetic field have the form [21]

$$\lambda_0 = -m\,\mathrm{sgn}(eB) \quad , \qquad \lambda_n = \pm\sqrt{2n|eB|+m^2}, \tag{67}$$

where n=1,2, ... It is also necessary to take into account in (66) the degeneracy of Landau levels. Namely, the number of degenerate states for each Landau level is $|eB|/2\pi$ per unit area. Even now we can see that only zero modes (because of

$\operatorname{sgn}(eB))$ could contribute to the parity odd quantity. So, for zero temperature, by using the identity

$$\operatorname{sgn}(a-b)+\operatorname{sgn}(a+b)=2\operatorname{sgn}(a)\theta(|a|-|b|),$$

one gets for zero modes

$$\begin{aligned}\frac{|eB|}{4\pi}\operatorname{sgn}\left(\mu+m\operatorname{sgn}(eB)\right)&=\frac{|eB|}{4\pi}\operatorname{sgn}(\mu)\theta(|\mu|-|m|)+\\&+\frac{|eB|}{4\pi}\operatorname{sgn}(eB)\operatorname{sgn}(m)\theta(|m|-|\mu|),\end{aligned}\tag{68}$$

and for nonzero modes

$$\begin{aligned}\frac{1}{2}\frac{|eB|}{2\pi}\sum_{n=1}^{\infty}\operatorname{sgn}(\mu-\sqrt{2n|eB|+m^2})+\operatorname{sgn}(\mu+\sqrt{2n|eB|+m^2})=\\=\frac{|eB|}{2\pi}\operatorname{sgn}(\mu)\sum_{n=1}^{\infty}\theta(|\mu|-\sqrt{2n|eB|+m^2}).\end{aligned}\tag{69}$$

Combining contributions of all modes we get for fermion density

$$\begin{aligned}\rho=\frac{|eB|}{2\pi}\operatorname{sgn}(\mu)\sum_{n=1}^{\infty}\theta\left(|\mu|-\sqrt{2n|eB|+m^2}\right)+\\+\frac{1}{2}\frac{|eB|}{2\pi}\operatorname{sgn}(\mu)\theta(|\mu|-|m|)+\frac{1}{2}\frac{eB}{2\pi}\operatorname{sgn}(m)\theta(|m|-|\mu|)=\\=\frac{|eB|}{2\pi}\operatorname{sgn}(\mu)\left(\operatorname{Int}\left[\frac{\mu^2-m^2}{2|eB|}\right]+\frac{1}{2}\right)\theta(|\mu|-|m|)+\\+\frac{eB}{4\pi}\operatorname{sgn}(m)\theta(|m|-|\mu|).\end{aligned}\tag{70}$$

Here we see that zero modes contribute both to parity odd and to parity even part, while nonzero modes contribute to the parity even part only (note that under parity transformation $B\to-B$). Thus, fermion density contains both Chern–Simons part and parity even part. At finite temperature it is also possible to get Chern–Simons. Substituting zero modes into (66) one gets

$$\begin{aligned}N_0=\frac{|eB|}{2\pi}\frac{1}{2}\operatorname{th}\left[\frac{1}{2}\beta\left(\mu+m\operatorname{sgn}(eB)\right)\right]=\\=\frac{|eB|}{4\pi}\left[\frac{\operatorname{sh}(\beta\mu)}{\operatorname{ch}(\beta\mu)+\operatorname{ch}(\beta m)}+\operatorname{sgn}(eB)\frac{\operatorname{sh}(\beta m)}{\operatorname{ch}(\beta\mu)+\operatorname{ch}(\beta m)}\right],\end{aligned}\tag{71}$$

so, excluding parity odd part, one gets for Chern–Simons at finite temperature and density

$$N_{CS}=\frac{eB}{4\pi}\frac{\operatorname{sh}(\beta m)}{\operatorname{ch}(\beta\mu)+\operatorname{ch}(\beta m)}=\frac{eB}{4\pi}\operatorname{th}(\beta m)\frac{1}{1+\operatorname{ch}(\beta\mu)/\operatorname{ch}(\beta m)}.\tag{72}$$

So, the result coincides with the result for Chern–Simons term coefficient by Niemi and Semenoff [4] obtained for small μ

$$\left[\ \mathrm{th}\frac{1}{2}\beta(m-\mu)+\ \mathrm{th}\frac{1}{2}\beta(m+\mu)\right].$$

It is obviously the limit to zero temperature. The lack of this method is that it works only for Abelian and constant field case.

This result at zero temperature can be obtained using the Schwinger proper–time method. Consider $(2+1)$-dimensional theory in the Abelian case and choose background field in the form

$$A^{\mu}=\frac{1}{2}x_{\nu}F^{\nu\mu},\qquad F^{\nu\mu}=\mathrm{const}.$$

To obtain the Chern–Simons term in this case, it is necessary to consider the background current

$$\langle J^{\mu}\rangle=\frac{\delta S_{eff}}{\delta A_{\mu}}$$

rather than the effective action itself. This is because the Chern–Simons term formally vanishes for such the choice of A^{μ} but its variation with respect to A^{μ} produces a nonvanishing current. So, consider

$$\langle J^{\mu}\rangle=-ig\ \mathrm{tr}\left[\gamma^{\mu}G(x,x^{'})\right]_{x\to x^{'}}, \tag{73}$$

where

$$G(x,x^{'})=\exp\left(-ig\int_{x^{'}}^{x}d\zeta_{\mu}A^{\mu}(\zeta)\right)\langle x|\hat{G}|x^{'}\rangle. \tag{74}$$

Let us rewrite Green function (7) in a more appropriate form

$$\hat{G}=(\gamma\tilde{\pi}-m)\Big[\frac{\theta((p_0+\mu)\ \mathrm{sgn}(p_0))}{(\gamma\tilde{\pi})^2-m^2+i\epsilon}+\frac{\theta(-(p_0+\mu)\ \mathrm{sgn}(p_0))}{(\gamma\tilde{\pi})^2-m^2-i\epsilon}\Big]. \tag{75}$$

Now, we use the well-known integral representation of denominators

$$\frac{1}{\alpha\pm i0}=\mp i\int_{0}^{\infty}ds\ \mathrm{e}^{\pm i\alpha s},$$

which corresponds to introducing the "proper–time" s into the calculation of the Eulier–Heisenberg Lagrangian by the Schwinger method [22]. We obtain

$$\begin{aligned}\hat{G}=i(\gamma\tilde{\pi}-m)\int_{0}^{\infty}ds\Big[-\exp\left(is\left[(\gamma\tilde{\pi})^2-m^2+i\epsilon\right]\right)\\ \theta((p_0+\mu)\ \mathrm{sgn}(p_0))+\\ +\exp\Big(-is\left[(\gamma\tilde{\pi})^2-m^2-i\epsilon\right]\Big)\theta(-(p_0+\mu)\ \mathrm{sgn}(p_0))\Big].\end{aligned} \tag{76}$$

For simplicity, we restrict ourselves only to the magnetic field case, where $A_0 = 0, [\tilde{\pi}_0, \tilde{\pi}_\mu] = 0$. Then we easily can factorize the time dependent part of Green function

$$G(x,x^{'}) = \int \frac{d^3p}{(2\pi)^3}\hat{G}\ \mathrm{e}^{ip(x-x^{'})} =$$
$$= \int \frac{d^2p}{(2\pi)^2}\hat{G}_{\vec{x}}\ \mathrm{e}^{i\vec{p}(\vec{x}-\vec{x}^{'})} \int \frac{dp_0}{2\pi}\hat{G}_{x_0}\ \mathrm{e}^{ip_0(x_0-x_0^{'})}. \tag{77}$$

By using the obvious relation

$$(\gamma\tilde{\pi})^2 = (p_0+\mu)^2 - \vec{\pi}^2 + \frac{1}{2}g\sigma_{\mu\nu}F^{\mu\nu} \tag{78}$$

one gets

$$G(x,x^{'})|_{x\to x^{'}} = -i\int \frac{dp_0}{2\pi}\frac{d^2p}{(2\pi)^2}(\gamma\tilde{\pi}-m)\int_0^\infty ds$$
$$\Bigg[\ \mathrm{e}^{is(\tilde{p}_0^2-m^2)}\ \mathrm{e}^{-is\vec{\pi}^2}\ \mathrm{e}^{isg\sigma F/2} - \theta(-(p_0+\mu)\ \mathrm{sgn}(p_0))$$
$$\Big(\ \mathrm{e}^{is(\tilde{p}_0^2-m^2)}\ \mathrm{e}^{-is\vec{\pi}^2}\ \mathrm{e}^{isg\sigma F/2} +\ \mathrm{e}^{-is(\tilde{p}_0^2-m^2)}\ \mathrm{e}^{is\vec{\pi}^2}\ \mathrm{e}^{-isg\sigma F/2}\Big)\Bigg]. \tag{79}$$

Here the first term corresponds to the usual μ–independent case and there are two additional μ–dependent terms. In the calculation of the current the following trace arises:

$$\mathrm{tr}\left[\gamma^\mu(\gamma\tilde{\pi}-m)\ \mathrm{e}^{isg\sigma F/2}\right] = 2\pi^\nu g^{\nu\mu}\cos(g|^*F|s) +$$
$$+2\frac{\pi^\nu F^{\nu\mu}}{|^*F|}\sin(g|^*F|s) - 2im\frac{^*F^\mu}{|^*F|}\sin(g|^*F|s),$$

where $^*F^\mu = \varepsilon^{\mu\alpha\beta}F_{\alpha\beta}/2$ and $|^*F| = \sqrt{B^2-E^2}$. Since we are interested in calculation of the parity odd part (Chern–Simons term) it is enough to consider only terms proportional to the dual strength tensor $^*F^\mu$. On the other hand the term $2\pi^\nu g^{\nu\mu}\cos(g|^*F|s)$ at $\nu = 0$ (see expression for the trace, we take in mind that here there is only magnetic field) also gives nonzero contribution to the current J^0 [8]

$$J^0_{\mathrm{even}} = g\frac{|gB|}{2\pi}\left(\mathrm{Int}\left[\frac{\mu^2-m^2}{2|gB|}\right]+\frac{1}{2}\right)\theta(|\mu|-|m|). \tag{80}$$

This part of current is parity invariant because under parity $B\to -B$. It is clear that this parity even object does contribute neither to the parity anomaly nor to the mass of the gauge field. Moreover, this term has been obtained [8] in the pure magnetic background and scalar magnetic field occurs in the argument's denominator of the cumbersome function — integer part. So, the parity even term seems to be "noncovariantizable", i.e., it can't be converted in covariant form in effective action. For a pity, in papers [8] charge density consisting of both parity odd and parity even parts is dubbed Chern–Simons, what leads to misunderstanding. The main goal of this article is to explore the parity anomalous topological Chern–Simons term in the effective action at finite density. So, just the term proportional to the dual strength tensor ${}^*F^\mu$ will be considered. The relevant part of the current reads

$$\begin{aligned} J^\mu_{CS} = \frac{g}{2\pi}\int dp_0 \int \frac{d^2p}{(2\pi)^2}\int_0^\infty ds \frac{2im{}^*F^\mu}{|{}^*F|}\sin\left(g|{}^*F|s\right) \\ \Bigg[\mathrm{e}^{is(\tilde{p}_0^2-m^2)}\ \mathrm{e}^{-is\vec{\pi}^2} - \theta(-(p_0+\mu)\ \mathrm{sgn}(p_0)) \\ \left(\mathrm{e}^{is(\tilde{p}_0^2-m^2)}\ \mathrm{e}^{-is\vec{\pi}^2} -\ \mathrm{e}^{-is(\tilde{p}_0^2-m^2)}\ \mathrm{e}^{is\vec{\pi}^2}\right)\Bigg]. \end{aligned} \tag{81}$$

Evaluating integral over spatial momentum we derive

$$\begin{aligned} J^\mu_{CS} = \frac{g^2}{4\pi^2} m{}^*F^\mu \int_{-\infty}^{+\infty} dp_0 \int_0^\infty ds \Bigg[\mathrm{e}^{is(\tilde{p}_0^2-m^2)} - \\ -\theta(-\tilde{p}_0\ \mathrm{sgn}(p_0)) \left(\mathrm{e}^{is(\tilde{p}_0^2-m^2)} +\ \mathrm{e}^{-is(\tilde{p}_0^2-m^2)m}\right)\Bigg]. \end{aligned} \tag{82}$$

Thus, we have got besides the usual Chern–Simons part [6], also the μ–dependent one. It is easy to calculate it by use of the formula

$$\int_0^\infty ds\ \mathrm{e}^{is(x^2-m^2)} = \pi\left(\delta(x^2-m^2) + \frac{i}{\pi}\mathcal{P}\frac{1}{x^2-m^2}\right)$$

and we get eventually

$$\begin{aligned} J^\mu_{CS} &= \frac{m}{|m|}\frac{g^2}{4\pi}{}^*F^\mu\big[1-\theta(-(m+\mu)\ \mathrm{sgn}(m)) - \theta(-(m-\mu)\ \mathrm{sgn}(m))\big] \\ &= \frac{m}{|m|}\theta(m^2-\mu^2)\frac{g^2}{4\pi}{}^*F^\mu. \end{aligned} \tag{83}$$

Let us now discuss the non-Abelian case. Then $A^\mu = T_a A^\mu_a$ and current reads

$$\langle J^\mu_a\rangle = -ig\ \mathrm{tr}\left[\gamma^\mu T_a G(x,x^{'})\right]_{x\to x^{'}}.$$

It is well known [6,23] that there exist only two types of the constant background fields. The first is the "Abelian" type (it is easy to see that the self–interaction $f^{abc}A_b^\mu A_c^\nu$ disappears under that choice of the background field)

$$A_a^\mu = \eta_a \frac{1}{2} x_\nu F^{\nu\mu}, \tag{84}$$

where η_a is an arbitrary constant vector in the color space, $F^{\nu\mu} = \mathrm{const.}$ The second is the pure "non-Abelian" type

$$A^\mu = \mathrm{const.} \tag{85}$$

Here the derivative terms (Abelian part) vanish from the strength tensor and it contains only the self–interaction part $F_a^{\mu\nu} = gf^{abc}A_b^\mu A_c^\nu$. It is clear that to catch the Abelian part of the Chern–Simons term we should consider the background field (84), whereas for the non-Abelian (derivative noncontaining, cubic in A) part we have to use the case (85).

Calculations in the "Abelian" case reduces to the previous analysis, except the trivial adding of the color indices in the formula (83):

$$J_a^\mu = \frac{m}{|m|}\theta(m^2 - \mu^2)\frac{g^2}{4\pi}{}^*F_a^\mu. \tag{86}$$

In the case (85) all calculations are similar. The only difference is that the origin of term $\sigma_{\mu\nu}F^{\mu\nu}$ in (78) is not the linearity A in x (as in Abelian case) but the pure non-Abelian $A^\mu = \mathrm{const.}$ Here term $\sigma_{\mu\nu}F^{\mu\nu}$ in (78) becomes quadratic in A and we have

$$J_a^\mu = \frac{m}{|m|}\theta(m^2 - \mu^2)\frac{g^3}{4\pi}\varepsilon^{\mu\alpha\beta}\ \mathrm{tr}\left[T_a A^\alpha A^\beta\right]. \tag{87}$$

Combining formulas (86) and (87) and integrating over field A_a^μ we obtain eventually

$$S_{eff}^{\mathrm{CS}} = \frac{m}{|m|}\theta(m^2 - \mu^2)\pi W[A], \tag{88}$$

where $W[A]$ is the Chern–Simons term

$$W[A] = \frac{g^2}{8\pi^2}\int d^3x \varepsilon^{\mu\nu\alpha}\ \mathrm{tr}\left(F_{\mu\nu}A_\alpha - \frac{2}{3}gA_\mu A_\nu A_\alpha\right).$$

It may seem that covariant notation is rather artificial. However, it helps us to extract the Levi–Chivita tensor containing part of action, i.e., parity anomalous Chern–Simons term.

6.2. Arbitrary External Field. One can see that the methods we have used above for calculation of the Chern–Simons term are noncovariant. Indeed, both of them use the constant magnetic background. Therefore, here we will use completely covariant approach, which allows an arbitrary initial field configuration and non-Abelian fields. We will employ the perturbative expansion at once in the non-Abelian case.

Let us first consider non-Abelian 3-dimensional gauge theory. The only graphs whose P-odd parts contribute to the parity anomalous Chern–Simons term are shown in Fig. 2.

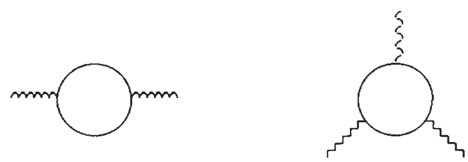

Fig. 2. Graphs whose P-odd parts contribute to the Chern–Simons term in non-Abelian 3D gauge theory

Thus, the part of effective action containing the Chern–Simons term looks as

$$I_{eff}^{CS} = \frac{1}{2}\int_x A_\mu(x)\int_p \mathrm{e}^{-ixp} A_\nu(p)\Pi^{\mu\nu}(p)$$
$$+\frac{1}{3}\int_x A_\mu(x)\int_{p,r} \mathrm{e}^{-ix(p+r)} A_\nu(p)A_\alpha(r)\Pi^{\mu\nu\alpha}(p,r), \tag{89}$$

where polarization operator and vertices have a standard form

$$\begin{aligned}\Pi^{\mu\nu}(p) &= g^2\int_k \mathrm{tr}\,[\gamma^\mu S(p+k;\mu)\gamma^\nu S(k;\mu)]\\ \Pi^{\mu\nu\alpha}(p,r) &= g^3\int_k \mathrm{tr}\,[\gamma^\mu S(p+r+k;\mu)\gamma^\nu S(r+k;\mu)\gamma^\alpha S(k;\mu)]\,,\end{aligned} \tag{90}$$

here, under integration we understand

$$\int_x = i\int_0^\beta dx_0\int d\vec{x} \quad \text{and} \quad \int_k = \frac{i}{\beta}\sum_{n=-\infty}^{\infty}\int \frac{d\vec{k}}{(2\pi)^2}.$$

First consider the second order term (Fig. 2, graph (a)). It is well known that the only object giving us the possibility of constructing P- and T-odd form in action is Levi–Chivita tensor*. Thus, we will drop all terms noncontaining Levi–Chivita

*In three dimensions it arises as a trace of three γ matrices (Pauli matrices).

tensor. Signal for the mass generation (Chern–Simons term) is $\Pi^{\mu\nu}(p^2=0)\neq 0$. So we get

$$\Pi^{\mu\nu} = g^2 \int_k (-i2me^{\mu\nu\alpha}p_\alpha)\frac{1}{(\tilde{k}^2+m^2)^2}. \tag{91}$$

After some simple algebra one obtains

$$\begin{aligned}\Pi^{\mu\nu} &= -i2mg^2e^{\mu\nu\alpha}p_\alpha \frac{i}{\beta}\sum_{n=-\infty}^{\infty}\int \frac{d^2k}{(2\pi)^2}\frac{1}{(\tilde{k}^2+m^2)^2} = \\ &= -i2mg^2e^{\mu\nu\alpha}p_\alpha \frac{i}{\beta}\sum_{n=-\infty}^{\infty}\frac{i}{4\pi}\frac{1}{\omega_n^2+m^2},\end{aligned} \tag{92}$$

where $\omega_n = (2n+1)\pi/\beta + i\mu$. Performing summation we get

$$\Pi^{\mu\nu} = i\frac{g^2}{4\pi}e^{\mu\nu\alpha}p_\alpha \ \mathrm{th}(\beta m)\frac{1}{1+\ \mathrm{ch}(\beta\mu)/\ \mathrm{ch}(\beta m)}. \tag{93}$$

It is easily seen that at $\beta\to\infty$ limit we'll get zero temperature result [9]

$$\Pi^{\mu\nu} = i\frac{m}{|m|}\frac{g^2}{4\pi}e^{\mu\nu\alpha}p_\alpha\theta(m^2-\mu^2). \tag{94}$$

In the same manner handling the third order contribution (Fig. 2b) one gets

$$\begin{aligned}\Pi^{\mu\nu\alpha} &= -2g^3ie^{\mu\nu\alpha}\frac{i}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^2k}{(2\pi)^2}\frac{m(\tilde{k}^2+m^2)}{(\tilde{k}^2+m^2)^3} = \\ &= -i2mg^3e^{\mu\nu\alpha}\frac{i}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^2k}{(2\pi)^2}\frac{1}{(\tilde{k}^2+m^2)^2}\end{aligned} \tag{95}$$

and further all calculations are identical to the second order

$$\Pi^{\mu\nu\alpha} = i\frac{g^3}{4\pi}e^{\mu\nu\alpha}\ \mathrm{th}(\beta m)\frac{1}{1+\ \mathrm{ch}(\beta\mu)/\ \mathrm{ch}(\beta m)}. \tag{96}$$

Substituting (93), (96) in the effective action (89) we get eventually

$$\begin{aligned}I_{eff}^{CS} = \ \mathrm{th}(\beta m)\frac{1}{1+\ \mathrm{ch}(\beta\mu)/\ \mathrm{ch}(\beta m)}\frac{g^2}{8\pi}\int d^3xe^{\mu\nu\alpha} \\ \mathrm{tr}\left(A_\mu\partial_\nu A_\alpha - \frac{2}{3}gA_\mu A_\nu A_\alpha\right).\end{aligned} \tag{97}$$

Thus, we get Chern–Simons term with temperature and density dependent coefficient.

7. CHERN–SIMONS TERM IN ARBITRARY ODD DIMENSION

Let's now consider 5-dimensional gauge theory. Here the Levi–Chivita tensor is 5-dimensional $e^{\mu\nu\alpha\beta\gamma}$ and the relevant graphs are shown in Fig. 3.

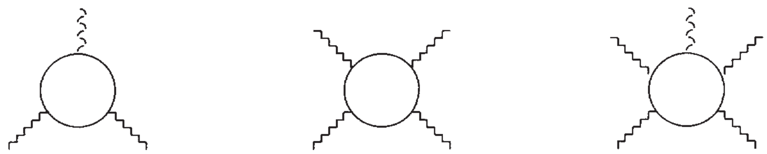

Fig. 3. Graphs whose P-odd parts contribute to the Chern–Simons term in non-Abelian 5D theory

The part of effective action containing the Chern–Simons term reads

$$I_{eff}^{CS} = \frac{1}{3}\int_x A_\mu(x)\int_{p,r} \mathrm{e}^{-ix(p+r)} A_\nu(p)A_\alpha(r)\Pi^{\mu\nu\alpha}(p,r)$$
$$+\frac{1}{4}\int_x A_\mu(x)\int_{p,r} \mathrm{e}^{-ix(p+r+s)} A_\nu(p)A_\alpha(r)A_\beta(s)\Pi^{\mu\nu\alpha\beta}(p,r,s)$$
$$+\frac{1}{5}\int_x A_\mu(x)\int_{p,r} \mathrm{e}^{-ix(p+r+s+q)} A_\nu(p)A_\alpha(r)A_\beta(s)A_\gamma(s)$$
$$\times\Pi^{\mu\nu\alpha\beta\gamma}(p,r,s,q). \qquad (98)$$

All calculations are similar to 3-dimensional case. First consider third order contribution (Fig. 3a)

$$\Pi^{\mu\nu\alpha}(p,r) = g^3\int_k \mathrm{tr}\left[\gamma^\mu S(p+r+k;\mu)\gamma^\nu S(r+k;\mu)\gamma^\alpha S(k;\mu)\right]. \qquad (99)$$

Taking into account that trace of five γ matrices in 5-dimensions is

$$\mathrm{tr}\left[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\rho\right] = 4ie^{\mu\nu\alpha\beta\rho},$$

we extract the parity odd part of the vertices

$$\Pi^{\mu\nu\alpha} = g^3\frac{i}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^4k}{(2\pi)^4}(i4me^{\mu\nu\alpha\beta\sigma}p_\beta r_\sigma)\frac{1}{(\tilde{k}^2+m^2)^3}, \qquad (100)$$

or in more transparent way

$$\Pi^{\mu\nu\alpha} \quad = \quad i4mg^3e^{\mu\nu\alpha\beta\sigma}p_\alpha r_\sigma\frac{i}{\beta}\sum_{n=-\infty}^{+\infty}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(\omega_n^2+\vec{k}^2+m^2)^3} =$$

$$= \quad i4mg^3 e^{\mu\nu\alpha\beta\sigma} p_\alpha r_\sigma \frac{i}{\beta} \sum_{n=-\infty}^{+\infty} \frac{-i}{64\pi^2} \frac{1}{\omega_n^2 + m^2}. \tag{101}$$

Evaluating summation one comes to

$$\Pi^{\mu\nu\alpha} = i \ \mathrm{th}(\beta m) \frac{1}{1 + \ \mathrm{ch}(\beta\mu)/\ \mathrm{ch}(\beta m)} \frac{g^3}{16\pi^2} e^{\mu\nu\alpha\beta\sigma} p_\alpha r_\sigma. \tag{102}$$

In the same way operating graphs (b) and (c) (Fig. 3) one will obtain

$$\Pi^{\mu\nu\alpha\beta} = i \ \mathrm{th}(\beta m) \frac{1}{1 + \ \mathrm{ch}(\beta\mu)/\ \mathrm{ch}(\beta m)} \frac{g^4}{8\pi^2} e^{\mu\nu\alpha\beta\sigma} s_\sigma \tag{103}$$

and

$$\Pi^{\mu\nu\alpha\beta\gamma} = i \ \mathrm{th}(\beta m) \frac{1}{1 + \ \mathrm{ch}(\beta\mu)/\ \mathrm{ch}(\beta m)} \frac{g^5}{16\pi^2} e^{\mu\nu\alpha\beta\sigma}. \tag{104}$$

Substituting (102)–(104) in the effective action (98) we get the final result for Chern–Simons in 5-dimensional theory

$$\begin{aligned} I_{eff}^{CS} = \ & \mathrm{th}(\beta m) \frac{1}{1 + \ \mathrm{ch}(\beta\mu)/\ \mathrm{ch}(\beta m)} \frac{g^3}{48\pi^2} \int_x e^{\mu\nu\alpha\beta\gamma} \\ \mathrm{tr} & \left(A_\mu \partial_\nu A_\alpha \partial_\beta A_\gamma + \frac{3}{2} g A_\mu A_\nu A_\alpha \partial_\beta A_\gamma + \frac{3}{5} g^2 A_\mu A_\nu A_\alpha A_\beta A_\gamma \right). \end{aligned} \tag{105}$$

It is remarkable that all parity odd contributions are finite both in 3-dimensional and in 5-dimensional cases. Thus, all values in the effective action are renormalized in a standard way, i.e., the renormalizations are determined by conventional (parity even) parts of vertices.

From the above direct calculations it is clearly seen that the chemical potential and temperature-dependent coefficient is the same for all parity odd parts of diagrams and doesn't depend on space dimension. So, the influence of finite density and temperature on the Chern–Simons term generation is the same in any odd dimension:

$$I_{eff}^{\mathrm{CS}} = \ \mathrm{th}(\beta m) \frac{1}{1 + \ \mathrm{ch}(\beta\mu)/\ \mathrm{ch}(\beta m)} \pi W[A] \overset{\beta\to\infty}{\longrightarrow} \frac{m}{|m|} \theta(m^2 - \mu^2) \pi W[A], \tag{106}$$

where $W[A]$ is the Chern–Simons secondary characteristic class in any odd dimension. Since only the lowest orders of perturbative series contribute to the Chern–Simons term at finite density and temperature (the same situation is well known at zero density), the result obtained by using formally perturbative technique appears to be nonperturbative. Thus, the μ- and T-dependent Chern–Simons term coefficient reveals the amazing property of universality. Namely, it does depend on neither dimension of the theory nor Abelian or non-Abelian gauge theory is studied.

The arbitrariness of μ gives us the possibility to see Chern–Simons coefficient behaviour at any masses. It is very interesting that $\mu^2 = m^2$ is the crucial point for Chern–Simons at zero temperature. Indeed, it is clearly seen from (106) that when $\mu^2 < m^2$, μ influence disappears and we get the usual Chern–Simons term

$$I_{eff}^{\mathrm{CS}} = \pi W[A].$$

On the other hand, when $\mu^2 > m^2$, the situation is absolutely different. One can see that here the Chern–Simons term disappears because of nonzero density of background fermions. We would like to emphasize the important massless case $m = 0$ considered in many papers, see for example [4,6,24]. Here even negligible density or temperature, which always take place in any physical processes, leads to vanishing of the parity anomaly. Let us stress again that we nowhere have used any restrictions on μ. Thus we not only confirm result of [4] for Chern–Simons in QED_3 at small density, but also expand it on arbitrary μ, non-Abelian case and arbitrary odd dimension.

8. NONRELATIVISTIC CONSIDERATION

Here, we will show that in nonrelativistic case there is no Chern–Simons term, there is only pseudo Chern–Simons, which is even under parity transformation. It is also presented the possibility of getting mixed Chern–Simons term in nontrivial external field.

First, we would like to notice that there are two approaches in fermion number definition. The first one is (see for example [32])

$$\langle Q \rangle_{\beta,\mu} = \sum_n \frac{1}{e^{\beta(\lambda_n - \mu)} + 1}, \tag{107}$$

and a normal ordering is performed at the given value of the chemical potential μ. (This normal ordering is suppressed here since it is inessential to the present discussion.) The other definition (see [2]) is related to the above by

$$\langle Q \rangle_{\beta,\mu} = \langle N \rangle_{\beta,\mu} + \frac{1}{2}\zeta_H(0), \tag{108}$$

where ζ_H is the Riemann ζ function related to the even part of the spectral density of the Hamiltonian H

$$\zeta_H(s) = \int_0^\infty d\lambda \left[\rho_H(\lambda) + \rho_H(-\lambda)\right] \lambda^{-s}. \tag{109}$$

So, the difference in the definitions is given by a β and μ independent constant, $\zeta_H(0)$. Indeed, one can easily check that at the operator level, these two definitions are related as

$$\mathbf{Q} = \mathbf{N} + \frac{1}{2}\int dx \left\{\psi^+(x), \psi(x)\right\} \quad , \quad \mathbf{N} = \frac{1}{2}\int dx \left[\psi^+(x), \psi(x)\right] . \tag{110}$$

As we have seen above, the fermion number density has the following form

$$N = \frac{1}{2}\sum_n \, \mathrm{th}\frac{1}{2}\beta(\mu - \lambda_n) \stackrel{\beta\to\infty}{\longrightarrow} \frac{1}{2}\sum_n \, \mathrm{sgn}(\mu - \lambda_n). \tag{111}$$

Landau levels in the relativistic case are

$$\lambda_0 = -m \, \mathrm{sgn}(eB) \quad , \qquad \lambda_n = \pm\sqrt{2n|eB| + m^2}, \tag{112}$$

where $n = 1, 2, \ldots$. On the other hand, in the nonrelativistic case energy levels have the form

$$\lambda_n = (n + \frac{1}{2})\Omega, \tag{113}$$

where $\Omega = |eB|/m$ cyclotron frequency, $n = 0, 1, 2,$...

As we have seen above in the relativistic case fermion density has the form

$$N = \frac{|eB|}{2\pi} \, \mathrm{sgn}(\mu) \left(\mathrm{Int}\left[\frac{\mu^2 - m^2}{2|eB|}\right] + \frac{1}{2}\right) \theta(|\mu| - |m|) + \\ + \frac{eB}{4\pi} \, \mathrm{sgn}(m)\theta(|m| - |\mu|). \tag{114}$$

Thus we can see that in the relativistic case there is especial zero mode, the only mode which contributes to parity-odd part of fermion number. On the contrary, in the nonrelativistic case there is no special zero mode, all modes contribute to the parity even part only. Thus, we have at zero temperature

$$Q = \frac{|eB|}{2\pi}\sum_n \theta\left(\mu - (n + \frac{1}{2})\Omega\right) = \frac{|eB|}{2\pi}\mathrm{Int}\left[\frac{\mu m}{|eB|} + \frac{1}{2}\right]. \tag{115}$$

One can see that fermion number in the nonrelativistic case is parity even ($B \to -B$ under parity). Therefore, it does not give rise to the parity-odd Chern–Simons term in action. Instead of being variational derivative of the true Chern–Simons, fermion number is the derivative of the pseudo Chern–Simons [26]

$$\langle Q\rangle = \frac{\delta}{\delta A_0} I_{pseudoCS}. \tag{116}$$

In the same manner it is possible to get fermion number with temperature introduced. For example, such calculations were done in [27], there was used another method. There the pseudo- Chern–Simons term coefficient has the form

$$\Pi_1^E = \frac{1}{\pi}\sum_n \left(\exp\beta(\lambda_n - \mu) + 1\right)^{-1} -$$
$$-\frac{1}{8\pi}\frac{\beta}{ml^2}\sum_n (2n+1)\mathrm{sech}^2\left(\frac{1}{2}\beta(\lambda_n - \mu)\right). \tag{117}$$

It is clearly seen that this expression can be rewritten in the way

$$\Pi_1^E = \frac{1}{\pi}\sum_n \Big[\frac{1}{\exp\beta(\lambda_n - \mu) + 1} -$$
$$-\frac{\beta\lambda_n}{\exp\beta(\lambda_n - \mu) + \exp(-\beta(\lambda_n - \mu)) + 2}\Big]. \tag{118}$$

After taking $\beta \to \infty$ limit one gets

$$\Pi_1^E = \frac{1}{\pi}\sum_n \theta(\mu - \lambda_n), \tag{119}$$

that coincides with the above calculations.

Another paper is [26]. There was also considered chemical potential influence on fermion number in nonrelativistic case. In this section we treat a nonrelativistic electron gas confined to a plane. We expect that some new qualitative features arise from the fact that in this case the spin degree of freedom is not enslaved by the dynamics. We continue to use a relativistic notation with $\partial_\mu = (\partial_0, \nabla,$ $\partial^\mu = (\partial_0, -\nabla)$, where ∇ is the gradient operator, and $A^\mu = (A^0, \mathbf{A})$.

Let us consider the Lagrangian

$$\mathcal{L} = \Psi^\dagger(i\partial_0 + \mu - H_{\mathrm{P}})\Psi + b\Psi^\dagger\frac{\sigma^3}{2}\Psi \tag{120}$$

which governs the dynamics of the Pauli spinor field Ψ, with Grassmann components $\psi_\uparrow$ and $\psi_\downarrow$ describing the electrons with spin-$\uparrow$ and $\downarrow$. The role of the chemical potential μ and the spin source b is the same as in the previous calculation. The Pauli Hamiltonian

$$H_{\mathrm{P}} = \frac{1}{2m}(i\nabla + e\mathbf{A})^2 - g_0\mu_{\mathrm{B}}\frac{\sigma^3}{2}B + eA_0, \tag{121}$$

with $\mu_{\mathrm{B}} = e/2m$ — the Bohr magneton and g_0 — the electron g-factor, contains a Zeeman term which couples the electron spins to the background magnetic field. Usually this term is omitted. The reason is that in realistic systems the g factor is much larger than two, the value for a free electron. In strong magnetic fields relevant to the QHE the energy levels of spin-$\downarrow$ electrons are too high and cannot be occupied; the system is spin polarized, and the electron spin is irrelevant to the problem. Setting again $A^0 = A^1 = 0,\ A^2 = Bx^1$, one finds as eigenvalues for H_P

$$E_{n,\pm} = \frac{|eB|}{m}(n + \frac{1}{2}) - \frac{eB}{m}S_\pm, \tag{122}$$

with $S_\pm = \pm\frac{1}{2}$ for spin-$\uparrow$ and spin-$\downarrow$ electrons, respectively. We note that in the nonrelativistic limit, corresponding to taking $m \to +\infty$, the relativistic Landau levels reduce to

$$E_{+n} \to \mathrm{const} + \frac{|eB|}{m}(n + \frac{1}{2}) - \frac{eB}{2m}, \tag{123}$$

where we omitted the negative energy levels which have no meaning in this limit. The main difference with (122) stems from the fact that there the spin degree of freedom is considered as an independent quantity, not enslaved by the dynamics as is the case in the relativistic problem.

The induced fermion number density and spin density may be obtained in a similar calculation as in the preceding section. From the effective action,

$$S_\mathrm{eff} = -i\ \mathrm{tr}\ln(i\partial_0 - H_P + \mu + \frac{b}{2}\sigma^3), \tag{124}$$

one obtains

$$\mathcal{L}_\mathrm{eff} = \frac{|eB|}{2\pi}\sum_{n=0}^{\infty}\int\frac{dk_0}{2\pi i}\Big[\ln(k_0 - E_{n,+} + \mu + \frac{b}{2}) + \\ + \ln(k_0 - E_{n,-} + \mu - \frac{b}{2})\Big]. \tag{125}$$

The resulting value of the induced fermion number density is

$$\rho = \frac{|eB|}{2\pi}(N_+ + N_-), \tag{126}$$

with $N_\pm$ the number of filled Landau levels for spin-$\uparrow$ and spin-$\downarrow$ electrons,

$$N_\pm = \left[\frac{m\mu_\pm}{|eB|} + \frac{1}{2}\right], \tag{127}$$

and

$$\mu_\pm = \mu + \frac{eB}{m}S_\pm \tag{128}$$

their effective chemical potentials. The square brackets denote again the integer-part function. Implicit in this framework is the assumption that, just like in the relativistic case, the chemical potential lies between two Landau levels. The induced fermion number density (126) is related to a Chern–Simons term in the effective action, with a coefficient

$$\theta = \mathrm{sgn}(eB)\frac{1}{2\pi}(N_+ + N_-). \tag{129}$$

Because of the presence of the $\mathrm{sgn}(eB)$ factor, which changes sign under a parity transformation, this Chern–Simons term is invariant under such transformations. The induced spin density turns out to be independent of $N_\pm$, viz.

$$s = \frac{eB}{4\pi}. \tag{130}$$

This follows from the symmetry in the spectrum $E_{n+1,+} = E_{n,-}$ $(eB > 0)$, or $E_{n,+} = E_{n+1,-}$ $(eB < 0)$. The magnetic moment, M can be obtained from (130) by multiplying s with twice the Bohr magneton, μ_{B}. This leads to the text-book result for the magnetic spin susceptibility χ_{P}

$$\chi_{\mathrm{P}} = \frac{\partial M}{\partial B} = \frac{e^2}{4\pi m} = 2\mu_{\mathrm{B}}^2\,\nu_{2D}(0), \tag{131}$$

with $\nu_{2D}(0) = m/(2\pi)$ the density of states per spin degree of freedom in two space dimensions.

At zero field, ρ reduces to the standard fermion number density in two space dimensions $\rho \to \mu m/\pi = k_{\mathrm{F}}^2/(2\pi)$, where k_{F} denotes the Fermi momentum. A single fluxon carries according to (130) a spin $S_\otimes = \frac{1}{2}$ and, since for small fields

$$\rho \to \frac{\mu m}{\pi} + \frac{|eB|}{2\pi}, \tag{132}$$

also one unit of fermion charge. That is, in the nonrelativistic electron gas the fluxon may be thought of as a fermion in that it has both the spin and charge of a fermion. However, the close connection between spin of a fluxon and induced Chern–Simons term for arbitrary fields that we found in the relativistic case is lost. This can be traced back to the fact that in the nonrelativistic case the electron spin is an independent degree of freedom. In the next section we point out that the spin of the fluxon does not derive from the ordinary Chern–Simons term, but from the so-called mixed Chern–Simons term. Such a term is absent in the relativistic case.

To see how the spin contribution (131) to the magnetic susceptibility compares to the orbital contribution we evaluate the k_0-integral in the effective action (125) with $b = 0$ to obtain

$$\mathcal{L}_{\mathrm{eff}} = \frac{|eB|}{2\pi}\sum_{n=0}^{\infty}\sum_{\varsigma=\pm}(\mu - E_{n,\varsigma})\theta(\mu - E_{n,\varsigma}). \tag{133}$$

The summation over n is easily carried out with the result for small fields

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi} \sum_{\varsigma=\pm} \left[\mu_\varsigma^2 m - \frac{(eB)^2}{4m} \right] = \frac{\mu^2 m}{2\pi} + \frac{(eB)^2}{8\pi m} [(2\sigma)^2 - 1], \tag{134}$$

where $\sigma = \frac{1}{2}$ and $\mu_\pm$ is given by (128). The first term in the right-hand side of (134), which is independent of the magnetic field, is the free particle contribution

$$\frac{\mu^2 m}{2\pi} = -2 \int \frac{d^2k}{(2\pi)^2} \left(\frac{k^2}{2m} - \mu \right) \theta \left(\mu - \frac{k^2}{2m} \right). \tag{135}$$

The second term yields the low-field susceptibility

$$\chi = (-1)^{2\sigma+1} 2\mu_{\mathrm{B}}^2 \, \nu_{2D}(0) \left[(2\sigma)^2 - 1 \right]. \tag{136}$$

Equation (136) shows that the ratio of orbital to spin contribution to χ is different from the three-dimensional case. Also, whereas a $3D$ electron gas is paramagnetic $(\chi > 0)$ because of the dominance of the spin contribution, the $2D$ gas is not magnetic $(\chi = 0)$ at small fields since the orbital and spin contributions to χ cancel.

8.1. Mixed Chern–Simons Term. As we have seen above, in the nonrelativistic case there are no true Chern–Simons terms. Now, we will present consideration of this problem in nontrivial background field.

In this section we investigate the origin of the induced spin density (130) we found in the nonrelativistic electron gas. To this end we slightly generalize the theory (120) and consider the Lagrangian

$$\mathcal{L} = \Psi^\dagger \left[i\partial_0 - eA_0 + \mu - \frac{1}{2m}(i\nabla + e\mathbf{A})^2 \right] \Psi + \frac{e}{m} B^a \Psi^\dagger \frac{\sigma^a}{2} \Psi. \tag{137}$$

It differs from (120) in that the spin source term is omitted, and in that the magnetic field in the Zeeman term is allowed to point in any direction in some internal space labelled by latin indices $a, b, c = 1, 2, 3$. As a result also the spin will have components in this space. It is convenient to consider a magnetic field whose direction in the internal space varies in space-time. We set

$$B^a(x) = B n^a(x), \tag{138}$$

with n^a a unit vector in the internal space. The gauge potential A_μ appearing in the first term of (137) still gives $\epsilon_{ij}\partial_i A^j = B$. Equation (138) allows us to make the decomposition

$$\Psi(x) = S(x)\chi(x) \; ; \quad S^\dagger S = 1, \tag{139}$$

with $S(x)$ a local SU(2) matrix which satisfies

$$\sigma \cdot \mathbf{n}(x) = S(x) \sigma^3 S^\dagger(x). \tag{140}$$

In terms of these new variables the Lagrangian (137) becomes

$$\mathcal{L} = \chi^\dagger \left[i\partial_0 - eA_0 - V_0 + \mu - \frac{1}{2m}(i\nabla + e\mathbf{A} + \mathbf{V})^2 \right] \chi + \frac{eB}{2m}\chi^\dagger \sigma^3 \chi, \quad (141)$$

where the 2×2 matrix $V_\mu = -iS^\dagger(\partial_\mu S)$ represents an element of the SU(2) algebra, which can be written in terms of (twice) the generators σ^a as

$$V_\mu = V_\mu^a \sigma^a. \quad (142)$$

In this way the theory takes formally the form of a gauge theory with gauge potential V_μ^a. In terms of the new fields the spin density operator,

$$j_0^a = \Psi^\dagger \frac{\sigma^a}{2} \Psi, \quad (143)$$

becomes [29]

$$j_0^a = R_{ab}\chi^\dagger \frac{\sigma^b}{2}\chi = -\frac{1}{2}R_{ab}\frac{\partial \mathcal{L}}{\partial V_0^b}. \quad (144)$$

In deriving the first equation we employed the identity

$$S^\dagger(\theta)\sigma^a S(\theta) = R_{ab}(\theta)\sigma^b, \quad (145)$$

which relates the SU(2) matrices in the $j = \frac{1}{2}$ representation, $S(\theta) = \exp(\frac{i}{2}\theta \cdot \sigma)$, to those in the adjoint representation $(j = 1)$, $R(\theta) = \exp(i\theta \cdot \mathbf{J}^{\mathrm{adj}})$. The matrix elements of the generators in the latter representation are $\left(J_a^{\mathrm{adj}}\right)_{bc} = -i\epsilon_{abc}$.

The projection of the spin density j_0^a onto the spin quantization axis, i.e. the direction n^a of the applied magnetic field [29],

$$\mathbf{n} \cdot \mathbf{j}_0 = -\frac{1}{2}\frac{\partial \mathcal{L}}{\partial V_0^3}, \quad (146)$$

only involves the spin gauge field V_μ^3. So when calculating the induced spin density $s = \langle \mathbf{n} \cdot \mathbf{j}_0 \rangle$ we may set the fields V_μ^1 and V_μ^2 equal to zero and consider the simpler theory

$$\mathcal{L} = \sum_{\varsigma=\pm} \chi_\varsigma^\dagger \left[i\partial_0 - eA_0^\varsigma + \mu_\varsigma - \frac{1}{2m}(i\nabla + e\mathbf{A}^\varsigma)^2 \right] \chi_\varsigma, \quad (147)$$

where the effective chemical potentials for the spin-$\uparrow$ and spin-$\downarrow$ electrons are given in (128) and $eA_\mu^\pm = eA_\mu \pm V_\mu^3$. Both components $\chi_\uparrow$ and $\chi_\downarrow$ induce a Chern–Simons term, so that in total we have

$$\begin{aligned} \mathcal{L}_{\mathrm{CS}} &= \frac{e^2}{2}\epsilon^{\mu\nu\lambda}(\theta_+ A_\mu^+ \partial_\nu A_\lambda^+ + \theta_- A_\mu^- \partial_\nu A_\lambda^-) \quad (148) \\ &= \frac{\theta_+ + \theta_-}{2}\epsilon^{\mu\nu\lambda}(e^2 A_\mu \partial_\nu A_\lambda + V_\mu^3 \partial_\nu V_\lambda^3) + e(\theta_+ - \theta_-)\epsilon^{\mu\nu\lambda} V_\mu^3 \partial_\nu A_\lambda, \end{aligned}$$

where the last term involving two different vector potentials is a mixed Chern–Simons term. The coefficients are given by

$$\theta_{\pm} = \frac{1}{2\pi}\mathrm{sgn}(eB)N_{\pm}, \tag{149}$$

assuming that $|eB| > \frac{1}{2}|\epsilon_{ij}\partial_i V_j^3|$, so that the sign of eB is not changed by spin gauge contributions. The integers $N_{\pm}$ are the number of filled Landau levels for spin-$\uparrow$ and spin-$\downarrow$ electrons given by (127). Since $N_+ - N_- = \mathrm{sgn}(eB)$, we obtain for the induced spin density s precisely the result (130) we found in the preceding section,

$$s = \langle \mathbf{n}\cdot\mathbf{j}_0\rangle = -\frac{1}{2}\left.\frac{\partial\mathcal{L}_{\text{eff}}}{\partial V_0^3}\right|_{V_\mu^3=0} = \frac{eB}{4\pi}. \tag{150}$$

The present derivation clearly shows that the induced spin in the nonrelativistic electron gas originates not from the standard Chern–Simons term, but from the mixed Chern–Simons term involving the electromagnetic and spin gauge potential.

The first term in (148) is a standard Chern–Simons term, the combination $\theta_+ + \theta_-$ precisely reproduces the result (129) and is related to the induced fermion number density (126).

9. TRACE IDENTITY

As was shown in [2–4] the trace identities connect the Chern–Simons term and chiral anomaly. These identities may be derived for Hamiltonians of the form

$$H = \begin{bmatrix} m & D \\ D^+ & -m \end{bmatrix}. \tag{151}$$

Here m is a constant; D, a differential operator of the form $D = iP_i\partial_i + Q(x)$; and D^+, the hermitian conjugate of D. The P_i are constant matrices that satisfy $P_i^+P_j + P_j^+P_i = 2\delta_{ij}$ and $P_iP_j^+ + P_jP_i^+ = 2\delta_{ij}$ and $Q(x)$ includes all background fields. It is assumed that these background fields are static, so

$$H = H_0 + m\Gamma^c = i\Gamma_i\partial_i + K(x) + m\Gamma^c, \tag{152}$$

where

$$\Gamma_i = \begin{bmatrix} 0 & P_i \\ P_i^+ & 0 \end{bmatrix}, \Gamma_c = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, K(x) = \begin{bmatrix} 0 & Q(x) \\ Q^+(x) & 0 \end{bmatrix}, \tag{153}$$

here Γ matrices satisfy the Euclidian Dirac algebra and the operator H_0 anticommutes with Γ^c. As a consequence $H^2 = H_0^2 + m^2 \geq m^2$ and all eigenvalues of H

$$H\Psi = H\begin{bmatrix} u \\ v \end{bmatrix} = \lambda\begin{bmatrix} u \\ v \end{bmatrix} = \lambda\Psi \tag{154}$$

satisfy $\lambda^2 \geq m^2$. Using (151) we obtain the first-order equations

$$D^+ u = (\lambda + m) v, \quad Dv = (\lambda - m) u \tag{155}$$

and by iterating, we find

$$DD^+ u = (\lambda^2 - m^2) u, \tag{156}$$

$$D^+ D v = (\lambda^2 - m^2) v. \tag{157}$$

If u is a solution of (156) with eigenvalue $\lambda^2 - m^2 = \chi \neq 0$, then $D^+ u$ is a solution of (157) with the same eigenvalue χ. However, if u is a zero mode of D^+, in general it is not a zero mode of D. Every solution of (156) or (157) yields two solutions of (154) if $\lambda \neq \pm m$ and one if $\lambda = \pm m$, and consequently the Dirac problem (154) is equivalent to (156), (157).

The fermion number operator has the form (for discussion on fermion number definition see beginning of section 8)

$$N = \frac{1}{2} \int dx \left[\Psi^+(x), \Psi(x) \right]. \tag{158}$$

At the time $t = 0$ the second quantized fermion field operator can be expanded as

$$\Psi(x) = \sum_n b_n \omega_n(x) + \sum_n d_n^+ \phi_n(x) + \int dk \left(b_k \omega_k(x) + b_k^+ \phi_k(x) \right), \tag{159}$$

where $\omega_n(x)$ and $\phi_n(x)$ are the positive and negative energy bound state solutions of the eigenvalue equation

$$H \psi_n = \lambda_n \psi_n, \tag{160}$$

and $\omega_k(x)$ and $\phi_k(x)$ are the positive and negative energy continuum solutions. Thus, the fermion number operator can be rewritten as follows

$$N = N_0 - \frac{1}{2} \eta_H, \tag{161}$$

where we have defined

$$N_0 = \sum_n \left[b_n^+ b_n - d_n^+ d_n \right] + \int dk \left[b_k^+ b_k - d_k^+ d_k \right],$$
$$\eta_H = \sum_k \ \mathrm{sgn}(\lambda_k). \tag{162}$$

The summation extends over both the discrete and continuum portions of the spectrum, and if a continuum spectrum is present, we rather write as

$$\eta_H = \int d\lambda \rho_H(\lambda)\,\mathrm{sgn}(\lambda). \tag{163}$$

Here $\rho_H(\lambda)$ is the spectral density function of the Hamiltonian H, and we may express it in terms of its even and odd parts:

$$\rho_H(\lambda) = \frac{1}{2}\left[\rho_H(\lambda) + \rho_H(-\lambda)\right] + \frac{1}{2}\left[\rho_H(\lambda) - \rho_H(-\lambda)\right] = \tau_H(\lambda) + \sigma_H(\lambda). \tag{164}$$

If we substitute it in (163) we obtain

$$\eta_H = \int d\lambda \sigma_H(\lambda)\,\mathrm{sgn}(\lambda) \tag{165}$$

since only the odd part of $\rho_H(\lambda)$ can contribute to η_H. So, η_H yields the difference in the number of positive and negative energy eigenstates of the Hamiltonian H, and thus it is a measure of its spectral asymmetry. However, the sum is not absolutely convergent and it needs to be regulated: the Atiyah–Patodi–Singer η invariant of the Hamiltonian H is defined by

$$\eta_H(s) = \sum_k \mathrm{sgn}(\lambda_k)|\lambda|^{-s} = \int d\lambda \sigma_H(\lambda)\,\mathrm{sgn}(\lambda)|\lambda|^{-s}. \tag{166}$$

For a large class of Hamiltonians the residue at $s = 0$ vanishes, and we assume that $s = 0$ is a regular point of $\eta_H(s)$, so we can define

$$\eta_H = \lim_{s\to 0} \eta_H(s) = \sum_k \mathrm{sgn}(\lambda_k) \equiv \int d\lambda \sigma_H(\lambda)\,\mathrm{sgn}(\lambda). \tag{167}$$

We shall now show how the spectral density $\rho_H(\lambda)$ of the Hamiltonian (151) can be represented in terms of the spectral densities $\rho_{DD^+}(\chi)$ and $\rho_{D^+D}(\chi)$ of the operators DD^+ and D^+D, respectively. For this we first consider the following Stieltjes transformation of the even part of $\rho_H(\lambda)$:

$$\int_{-\infty}^{\infty} d\lambda \rho_H(\lambda)\frac{1}{\lambda^2 + z^2} = 2\int_{|m|}^{\infty} d\lambda \tau_H(\lambda)\frac{1}{\lambda^2 + z^2}. \tag{168}$$

Here z^2 is an arbitrary complex number which does not belong to the spectrum of H. Introducing the coordinate representation we obtain

$$\begin{aligned}
2\int_{|m|}^{\infty} d\lambda \tau_H(\lambda)\frac{1}{\lambda^2 + z^2} &= \int dx\,\mathrm{tr}\langle x|\frac{1}{H^2 + z^2}|x\rangle \\
= \int dx \left(\mathrm{tr}\langle x|\frac{1}{DD^+ + m^2 + z^2}|x\rangle + \mathrm{tr}\langle x|\frac{1}{D^+D + m^2 + z^2}|x\rangle \right) \\
= \int d\chi \left(\rho_{DD^+}(\chi) + \rho_{D^+D}(\chi)\right)\frac{1}{\chi + m^2 + z^2} &\equiv F(m^2 + z^2).
\end{aligned} \tag{169}$$

Comparing (168) with (169) we conclude that

$$\tau_H(\lambda) = |\lambda| \left(\rho_{DD^+}(\lambda^2 - m^2) + \rho_{D^+D}(\lambda^2 - m^2)\right). \tag{170}$$

Similarly, we find a representation for the odd part of $\rho_H(\lambda)$ by considering

$$2\int_{|m|}^{\infty} d\lambda \sigma_H(\lambda) \frac{\lambda}{\lambda^2 + z^2} = \int dx \ \mathrm{tr}\langle x| \frac{H}{H^2 + z^2} |x\rangle$$
$$= \int dx \left(\mathrm{tr}\langle x| \frac{m}{DD^+ + m^2 + z^2} |x\rangle - \mathrm{tr}\langle x| \frac{m}{D^+D + m^2 + z^2} |x\rangle \right)$$
$$= \int d\chi \left(\rho_{DD^+}(\chi) - \rho_{D^+D}(\chi)\right) \frac{m}{\chi + m^2 + z^2} \equiv mG(m^2 + z^2). \tag{171}$$

We conclude that the odd spectral density, which contributes to (167), is given by

$$\sigma_H(\lambda) = m \ \mathrm{sgn}(\lambda) \left[\rho_{DD^+}(\lambda^2 - m^2) - \rho_{D^+D}(\lambda^2 - m^2)\right]. \tag{172}$$

Since the fermion number is essentially a Mellin transformation of the odd spectral density, we find the following spectral representation for the fermion number

$$N = -m \int_0^{\infty} d\lambda \left[\rho_{DD^+}(\lambda^2 - m^2) - \rho_{D^+D}(\lambda^2 - m^2)\right] =$$
$$= -\frac{1}{2} \int_0^{\infty} d\chi \left[\rho_{DD^+}(\chi) - \rho_{D^+D}(\chi)\right] \frac{m}{\sqrt{\chi + m^2}} =$$
$$= -\frac{m}{\pi} \int_0^{\infty} d\omega G(m^2 + \omega^2), \tag{173}$$

where $G(m^2 + \omega^2)$ is defined by (171). We shall now show how the axial anomaly is connected with fermion number. First notice that

$$\mathrm{tr}\langle x| \frac{m}{DD^+ + m^2 + \omega^2} |y\rangle - \mathrm{tr}\langle x| \frac{m}{D^+D + m^2 + \omega^2} |y\rangle =$$
$$= \mathrm{tr}\langle x| \Gamma^c \frac{m}{H^2 + \omega^2} |y\rangle. \tag{174}$$

Further, we get

$$\mathrm{tr}\langle x| \Gamma^c \frac{m}{H^2 + \omega^2} |y\rangle = i \frac{m}{\sigma} \ \mathrm{tr}\langle x| \Gamma^c \frac{1}{H_0 + i\sigma} |y\rangle, \tag{175}$$

where $\sigma = \sqrt{m^2 + \omega^2}$. Let's now consider

$$\mathrm{tr}\langle x| i\Gamma^i \partial_i \Gamma^c \frac{1}{H_0 + i\sigma} - \Gamma^c \frac{1}{H_0 + i\sigma} i\Gamma^i \partial_i |y\rangle =$$
$$= 2i\sigma \ \mathrm{tr}\langle x| \Gamma^c \frac{1}{H_0 + i\sigma} |y\rangle + \ \mathrm{tr}\left([K(y) - K(x)] \langle x| \Gamma^c \frac{1}{H_0 + i\sigma} |y\rangle \right). \tag{176}$$

Combining (175) and (176) we then obtain the following trace identity

$$i\frac{m}{\sigma}\ \mathrm{tr}\langle x|\Gamma^c\frac{1}{H_0+i\sigma}|y\rangle = \frac{m}{2\sigma^2}[\partial_x+\partial_y]\ \mathrm{tr}\langle x|i\Gamma^i\Gamma^c\frac{1}{H_0+i\sigma}|y\rangle + \\ +\frac{m}{2\sigma^2}\ \mathrm{tr}\left([K(y)-K(x)]\langle x|\Gamma^c\frac{1}{H_0+i\sigma}|y\rangle\right). \quad (177)$$

Notice that (177) has the structure of the standard axial anomaly equation for the Dirac operator $H_0+i\sigma$. Thus, when we take $x\to y$ limit we need to discuss two cases:

First, if the space dimension D is odd the second term on the right-hand side of (177) vanishes, since there are no axial anomaly in this case, and the only contribution to fermion number gives the first term (boundary term).

Second, if the space dimension D is even, the second term on the right-hand side of (177) gives the axial anomaly, and taking in mind (173) we get for fermion number

$$N = -\frac{1}{2\pi}\int_0^\infty d\omega\frac{m}{m^2+\omega^2}\left(2T_D+\int dS^i\ \mathrm{tr}\langle x|i\Gamma^i\Gamma^c\frac{1}{H_0+i\sigma}|x\rangle\right), \quad (178)$$

here T_D is the Pontryagin index of the background gauge fields that arises from the space integral of anomaly term. There is also boundary term, which vanishes for a trivial gauge background.

Let us consider 2-dimensional case, for the Hamiltonian

$$H_0 = -i\sigma^2\frac{d}{dx}+\sigma^1\phi(x). \quad (179)$$

Since this Hamiltonian can be interpreted as a one-dimensional Dirac operator, and since there are no anomalies in one dimension, the fermion number is given by

$$N = -\frac{1}{2\pi}\int_0^\infty d\omega\frac{m}{m^2+\omega^2}\int_{-\infty}^{+\infty}dx\frac{d}{dx}\ \mathrm{tr}\langle x|i\Gamma^1\Gamma^c\frac{1}{H_0+i\sigma}|x\rangle = \\ = \frac{1}{2\pi}\int_0^\infty d\omega\frac{m}{m^2+\omega^2}\Big[\ \mathrm{tr}\langle\infty|\sigma^1\frac{1}{H_0+i\sigma}|\infty\rangle - \\ -\ \mathrm{tr}\langle-\infty|\sigma^1\frac{1}{H_0+i\sigma}|-\infty\rangle\Big], \quad (180)$$

where we have used the representation $\Gamma^1=\sigma^2$ and $\Gamma^c=\sigma^3$ of the Dirac algebra. We assume that the soliton field $\phi(x)$ has the asymptotes $\phi(\pm\infty)=\hat{\phi}_\pm$. Taking in mind that

$$\mathrm{tr}\langle\pm\infty|\sigma^1\frac{1}{H_0+i\sigma}|\pm\infty\rangle =$$

$$= \operatorname{tr}\langle\pm\infty|\frac{1}{-\sigma^3\partial-\phi(x)-i\sigma^2\sigma^3\sqrt{m^2+\omega^2}}|\pm\infty\rangle =$$

$$2\operatorname{tr}\langle\pm\infty|\frac{\phi(x)}{\partial^2-\phi^2(x)-m^2-\omega^2}|\pm\infty\rangle =$$

$$= -2\hat{\phi}_\pm\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\frac{1}{k^2+\hat{\phi}_\pm^2+m^2+\omega^2}, \tag{181}$$

we find for fermion number

$$N = -\frac{1}{2\pi}\left[\arctan\left[\frac{\hat{\phi}_+}{m}\right]-\arctan\left[\frac{\hat{\phi}_-}{m}\right]\right]. \tag{182}$$

This result [4, 31] is obtained without the use of any specific soliton profile. However, at the presence of nonzero density (μ), fermion number depends not only on asymptotic properties of soliton profile, but also it depends on the local properties of the soliton profile such as the width of the soliton [32].

The result for fermion number at the presence of density and temperature can be found in [32]. There was studied the Hamiltonian [33]

$$H_0 = \sigma^2\frac{d}{dx}+\sigma^1\phi(x)+\sigma^3\epsilon. \tag{183}$$

This Hamiltonian has the following positive- and negative-energy continuum solutions and a bound state solution (assuming a soliton profile which has only one bound state)

$$\begin{aligned}\psi_{k\alpha} &= \left[\begin{array}{l}[(\alpha E+\epsilon)/2\alpha E]^{1/2}u_k\\ [2\alpha E(\alpha E+\epsilon)]^{1/2}(\partial_x+\phi)u_k\end{array}\right],\\ \psi_s &= N_0\left[\begin{array}{l}\exp[-\int^x dx'\phi(x')]\\ 0\end{array}\right],\end{aligned} \tag{184}$$

where N_0 is a normalization factor, $\alpha=\pm1$ distinguishes positive and negative energy solutions. In the ground state the soliton charge is defined as

$$Q = \int_{-\infty}^{\infty}dx\sum\left[\rho_i^s(x)-\rho_i^0(x)\right], \tag{185}$$

where $\rho_i^s(x)$ and $\rho_i^0(x)$ are the fermion number density at a point x in the presence and absence of the soliton, due to occupied state i.

The generalization of (185) to finite μ and T is straightforward since we have a noninteracting sea of fermions

$$Q(\mu,T) = \int_{-\infty}^{\infty}dx\sum\left[\rho_i^s(x)-\rho_i^0(x)\right]n(\epsilon_i-\mu), \tag{186}$$

where $n(\epsilon-\mu)=[\exp[\beta(\epsilon-\mu)]+1]^{-1}$ is the Fermi distribution function. Thus, substitution of ρ yields

$$Q=\int_{-\infty}^{\infty}dx\sum_{\alpha=\pm 1}\int_{-\infty}^{\infty}\frac{dk}{2\pi}(|u_k^s|^2-|u_k^0|^2)n(\alpha E-\mu)$$
$$+\sum_{\alpha=\pm 1}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left[\frac{(\partial_x|u_k^s|^2+2|u_k|^2\phi)|_{x=-\infty}^{x=+\infty}}{4\alpha E(\alpha E+\epsilon)}\right]n(\alpha E-\mu)+n(\epsilon-\mu). \quad (187)$$

The square bracket in the second term of the above expression can be simplified further using $(\partial_x|u_k^s|^2+2|u_k|^2\phi)|_{x=-\infty}^{x=+\infty}=2$. For $T=0$ and $\mu=0$ the first term is easily evaluated using the completeness properties of u_k. But for finite μ we have to choose a soliton profile. So, we take $\phi(x)=\phi_0\ \mathrm{th}(\phi_0 x)$, for which the eigenfunctions $u_k^s(x)$ are known exactly [33] to be

$$u_k(x)=-\exp(ikx)\left[\frac{\mathrm{th}\phi_0 x-(ik/\phi_0)}{1+(ik/\phi_0)}\right]. \quad (188)$$

Substitution of the u_k in (187) yields

$$Q(\mu,T)=-2\phi_0\sum_{\alpha=\pm 1}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{n(\alpha E-\mu)}{k^2+\phi_0^2}+$$
$$+2\phi_0\sum_{\alpha=\pm 1}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{n(\alpha E-\mu)}{2\alpha E(\alpha E+\epsilon)}+n(\epsilon-\mu). \quad (189)$$

In particular, this integrals can be evaluated exactly for zero temperature and finite μ to get [32]

$$Q(\mu,0)=-\,\mathrm{sgn}(\mu)Q_0(\epsilon)-\theta(\mu)G(k_F,\epsilon)+\theta(-\mu)G(k_F,-\epsilon) \quad (190)$$

for $|\mu|>m$, where

$$Q_0(\epsilon)=-\frac{1}{\pi}\tan^{-1}\left[\frac{\phi_0}{\epsilon}\right],$$
$$G(k_F,\epsilon)=\frac{w}{\pi}\tan^{-1}\left[\frac{\phi_0\tan[\frac{1}{2}\tan^{-1}(k_F/m)]}{m+\epsilon}\right],$$
$$k_F=(\mu^2-m^2)^{1/2}\quad ,\quad m=(\phi_0^2+\epsilon^2)^{1/2}.$$

As we have seen above, the boundary term is dependent on soliton profile at finite density. Thus, the generalization of the trace identities on finite density and temperature is hardly possible because of nontopological part of it.

At finite temperature and zero density trace identity still holds and one has for fermion number [2, 4]

$$\langle N\rangle_\beta = -\frac{1}{2\beta}\sum_{-\infty}^{+\infty}\frac{m}{m^2+\omega_n^2}$$
$$\left(\int dx(\text{anomaly}) + \int dx\partial_i \ \mathrm{tr}\langle x|i\Gamma_i\Gamma^c\frac{1}{H_0+i\sqrt{m^2+\omega_n^2}}\rangle\right). \qquad (191)$$

Now we'll prove that chiral anomaly doesn't depend on temperature at any even dimension. The second term at left-hand side of (191) is a surface term, which doesn't contribute to topological part of the trace identity [2, 4]. So, we won't consider nontopological part of the trace identity, i.e., nontopological part of fermion density and surface term. Thus for topological part trace identity takes the form

$$\langle N\rangle_\beta^{\text{topological}} = -\frac{1}{2\beta}\sum_{-\infty}^{+\infty}\frac{m}{m^2+\omega_n^2}\left(\int dx(\text{anomaly})\right). \qquad (192)$$

The result for left-hand side of Eq. (192) we know in arbitrary odd dimension. Really, from (106) we have

$$\langle N\rangle_\beta^{\text{CS}} = \langle N\rangle_\beta^{\text{topological}} = \frac{\delta I_{eff}^{\text{C.S}}}{g\delta A_0}. \qquad (193)$$

By using the fact that

$$\frac{1}{2\beta}\sum_{n=-\infty}^{+\infty}\frac{m}{\omega_n^2+m^2} = \frac{1}{4}\frac{\mathrm{sh}(\beta m)}{1+\ \mathrm{ch}(\beta m)}, \qquad (194)$$

one can see that the only possibility to reconcile left and right sides of Eq. (192) is to put temperature independence of anomaly. Thus, we proof that axial anomaly doesn't depend on temperature in any even-dimensional theory.

Moreover, now we can generalize trace identity on arbitrary finite density. Really, taking in mind (106) and (193) one can see

$$\langle N\rangle_{\beta,\mu}^{\text{CS}} = -\frac{1}{4}\ \mathrm{th}(\beta m)\frac{1}{1+\ \mathrm{ch}(\beta\mu)/\ \mathrm{ch}(\beta m)}\int dx\,(\text{anomaly}), \qquad (195)$$

where $\langle N\rangle_{\beta,\mu}^{\text{CS}}$ — odd part of fermion number in D-dimensional theory at finite density and temperature, (anomaly) — axial anomaly in $(D-1)$-dimensional theory. On the other hand, as we have seen above, the anomaly doesn't depend on μ in 2- and 4-dimensions and doesn't depend on T in any even-dimensional theory. Our comprehension of the problem allows us to generalize these on arbitrary even dimension. Indeed, anomaly is the result of ultraviolet regularization, while μ and T don't effect on ultraviolet behavior of a theory. Taking in mind

(195) and that at finite density

$$\frac{1}{2\beta}\sum_{n=-\infty}^{+\infty}\frac{m}{\omega_n^2+m^2}=\frac{1}{4}\,\mathrm{th}(\beta m)\frac{1}{1+\,\mathrm{ch}(\beta\mu)/\,\mathrm{ch}(\beta m)} \tag{196}$$

we can identify $\langle N\rangle_{\beta,\mu}^{\text{topological}}$ and $\langle N\rangle_{\beta,\mu}^{\text{CS}}$. So, we get generalized on finite density trace identity

$$\langle N\rangle_{\beta,\mu}^{\text{CS}}=\langle N\rangle_{\beta,\mu}^{\text{topological}}=-\frac{1}{2\beta}\sum_{-\infty}^{+\infty}\frac{m}{m^2+\omega_n^2}\left(\int dx(\text{anomaly})\right). \tag{197}$$

Let us take, for example, 3-dimensions. We know that chiral anomaly in 2-dimensions has the form

$$\int d^2x\frac{e}{4\pi^2}\epsilon^{ij}F_{ij}, \tag{198}$$

substituting this in (197) we'll get for fermion number

$$\langle N\rangle_{\beta,\mu}^{\text{CS}}=\frac{e}{16\pi^2}\,\mathrm{th}(\beta m)\frac{1}{1+\,\mathrm{ch}(\beta\mu)/\,\mathrm{ch}(\beta m)}\int d^2x\epsilon^{ij}F_{ij}. \tag{199}$$

Covariantizing fermion number we get for the Chern–Simons term in action

$$I_{eff}^{\text{CS}}=\,\mathrm{th}(\beta m)\frac{1}{1+\,\mathrm{ch}(\beta\mu)/\,\mathrm{ch}(\beta m)}\frac{g^2}{16\pi}\int_x e^{\mu\nu\alpha}\,\mathrm{tr}\,(A_\mu F_{\nu\alpha}). \tag{200}$$

Really, we've seen that only zero modes contribute to P-odd part in contrast to P-even part which is contributed by all modes. Therefore, index theorem and trace identities hold only for parity odd part of fermion number. Thus, the main result of this section is Eq. (197) which connects the Chern–Simons term and chiral anomaly in arbitrary-dimensional theory at finite density and temperature.

10. CONCLUSIONS

Thus, there is obtained finite temperature and density influence on the Chern–Simons term generation in any odd-dimensional theory both for Abelian, and for non-Abelian case. It is very interesting that $\mu^2=m^2$ is the crucial point for Chern–Simons at zero temperature. Indeed, it is clearly seen from (106) that when $\mu^2<m^2$, μ influence disappears and we get the usual Chern–Simons term $I_{eff}^{\text{CS}}=\pi W[A]$. On the other hand, when $\mu^2>m^2$, the Chern–Simons term disappears because of nonzero density of background fermions. The coefficient at the Chern–Simons term is the same in any odd dimension. It must be stressed that at $m = 0$ even negligible density or temperature, which always take place in any physical processes, lead to vanishing of the parity anomaly.

It is shown that the chiral anomaly is not influenced by medium effects such as chemical potential and temperature in any even-dimensional theory. Moreover, even if we introduce conservation of chiral charge on quantum level, the chiral anomaly arises and isn't affected.

The appearance of the Chern–Simons number in even-dimensional theories is discussed under two types of constraints. So, it is shown both for conserved charge, i.e., finite density of the background fermions, and for conserved axial charge what corresponds to conservation of the left(right)-handed fermions asymmetry in the background.

The topological part of the trace identity is generalized on finite density. Thus, the connection between the Chern–Simons term and chiral anomaly at finite density and temperature is obtained in arbitrary dimensional theory.

In conclusion we would like to touch the problem, which has attracted recently a wide interest [34, 35]. This is gauge invariance of the effective action under large gauge transformations. Really, the Chern–Simons term coefficient has to be *"topologically quantized"* for gauge invariance of the effective action under large gauge transformations. But as we have seen above (56), (57), (58), even in nonperturbative calculations of Chern–Simons in even dimensions (due to existence of the chiral anomaly), it gets chemical potential (temperature) as a coefficient, which is not an integer function. This fact is hardly understandable. One can treat these that density or temperature just break invariance under large gauge transformations, leaving action invariant under local ones. On the other hand, one can hope that the whole effective action will be gauge invariant [34,35]. But, for example, essentially nonperturbative and simple calculations in one dimension [34] do not give understandable contradiction between fermion number and effective action. That is the fermion number here is not a functional derivative of the effective action, what is very strange.

The amazing fact is that at zero temperature and finite density the Chern–Simons term coefficient does not break gauge invariance. Indeed, theta function gives us 0 or 1 as the coefficient, and we have two topological domains $\mu^2 > m^2$ and $\mu^2 < m^2$ connected by large gauge transformations.

Thus, this area is yet an open field for research.

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