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An approach permitting to «project» the dynamics of a quantal many-body system on the space of a few appropriately chosen collective observables is discussed. Making use of virial theorems method one arrives at the equations of motion in the restricted space of collective variables describing the distribution of nucleons in the phase space, i.e., in the configuration and momentum spaces. A general scheme of constructing the collective Hamiltonian describing the dynamics of chosen collective variables consistent with the underlying microscopic Hamiltonian is proposed. The collective energy dissipation is accounted for by introducing a «random force» coupling the collective and intrinsic variables. The study of collective motion within the models obtained in this way can be done on the classical and quantum levels depending on the nature of problems under examination.

В данной работе обсуждается подход, позволяющий проектировать динамику квантовых многочастичных систем на пространство нескольких коллективных наблюдаемых. Используя метод вириальных теорем, мы приходим к уравнениям движения в ограниченном пространстве коллективных переменных, описывающих распределение нуклонов в фазовом пространстве, то есть в пространстве координат и импульсов. Предлагается общая схема конструирования коллективного гамильтониана, описывающего динамику выбранных коллективных переменных, так что он согласуется с соответствующим микроскопическим гамильтонианом. Диссипация коллективной энергии учитывается введением «случайной силы», связывающей коллективные и внутренние переменные. Изучение коллективного движения в рамках полученной таким образом модели может быть выполнено как на классическом, так и на квантовом уровне, в зависимости от природы исследуемых проблем.

1. INTRODUCTION

Recent experimental studies of nuclear fusion and fission reactions prompt still further development of the large amplitude collective motion theory. We briefly describe the approaches based on the quantum many-body theory which provide an effective «projection» of the microscopic dynamics on the space of a limited number of «macroscopic» collective variables. The word «projection» here must not be understood in a purely mathematical sense, where it is associated with an operator picking up a part of the wave function belonging to a chosen space of states. We are talking about methods allowing one to choose important collective variables on the basis of a wealth of available experimental data as well as on general properties of the many-body Hamiltonian in order to build a collective model. Admitting a great role of phenomenology in setting up

collective models, we pay much attention to their consistency with the underlying many-body physics.

The report is based upon formal theoretical developments of the heavy-nuclei dynamics presented in Refs. 1–5. In the quoted references the formalism is applied to nuclear fusion, and an interpretation of some new and not yet fully understood phenomena is given.

The models reported in Refs. 1–5 treat explicitly the Fermi surface deformation in the fusion reaction. Similar equations of motion, obtained using different approximations [6–10], are applied to the fission process. The authors of these two groups of references claim an important role of the dynamical effects related with the redistribution of nucleons in the momentum space in fusion and fission. They insist also on the adequacy of their models in treating various features of these processes.

However, the actual formulation of these models restricts their application field. Some limitations of the model of Refs. 1–5 are related with a mapping of the complex behavior of colliding heavy nuclei on the dynamics of a system with only one geometrical degree of freedom. Especially hampering the progress are the lack of a precise control of the energy conservation inherent in the model in Refs. 6–10 and unnecessary approximations used in solving practical problems in both approaches.

These models are formulated in terms of equations of motion for collective variables and suggest the treatment of nuclear processes in the spirit of classical mechanics. This hinders their application to a number of important problems of nuclear reaction theory, such as the study of sub-barrier effects in nuclear fusion.

The formal developments of the theory presented in the review seem to be helpful in overcoming these difficulties. The plan of presentation is as follows:

- In section 2 the notion of virial theorems and basic ideas concerning their application for studies of large-amplitude collective motion are presented.
- In section 3 a number of useful examples of virial theorems and an expression for the total energy of the nuclear system are given.
- In section 4 the necessary steps needed to pass from virial theorems to formulation of collective models are discussed. A very simple dynamical model («primitive elastoplastic model») is introduced to visualize the characteristic properties of the Fermi-system dynamics.
- In section 5 the equations of motion formulated using virial theorems are rendered a canonical form. Expressions for the collective energy and the Rayleigh dissipation function are presented.
- In section 6 consistency relations between different blocks in the equations of motion are discussed and their role in the reduction of the number of approximations is shown.
- In section 7 the dynamics of thermal fluctuations is studied using as an example the «primitive collective model».

- In section 8 the quantal formulation of the «primitive collective model» is given.
- In section 9 the possibility of estimation of the totality of functions in the dynamical collective models using the generalized Routhian functions is discussed.

2. MICROSCOPIC BASIS

The approach in Refs. 1–5 is a natural transposition on the «quantum soil» of the «virial theorems method» by Chanrasekhar [11] applied by him to macroscopic (stellar) bodies whose dynamics is governed by the classical hydrodynamics. In application to quantal many-body systems the equations of motion for macroscopic observables are found taking moments in the phase space of the kinetic equation. The latter is obtained transforming «á la Wigner» the equation for the one-particle density matrix. To describe the dissipation, a «collision term» is added to the kinetic equation.

However, not only the base of the virial theorems is changed in Refs. 1–5 as compared with the theory of Chandrasekhar: the approach discussed in the review is designed not only for the description of equilibrium figures as in Ref. 11, but is aiming at the study of dynamics of nuclear processes.

2.1. Kinetic Equation and Virial Theorems. The starting point of the theory is the kinetic equation [12,13]

$$\frac{\partial f}{\partial t} = \frac{2}{\hbar}\sin\left[\frac{1}{2}\left(\nabla_x^H \nabla_p^f - \nabla_p^H \nabla_x^f\right)\right](H_W f) + \left(\frac{\partial f}{\partial t}\right)_{\rm cls} \tag{1}$$

or the quasi-classical analogue of the former equation

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}\cdot\frac{\partial f}{\partial \mathbf{x}} - (\nabla V(\mathbf{x}))\cdot\frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial t}\right)_{\rm cls}. \tag{2}$$

Here $f(\mathbf{x},\mathbf{p},t)$ is the Wigner distribution function playing the role of the distribution function of the classical statistical mechanics and becoming identical with it in the quasi-classical approximation. The quantity $V(\mathbf{x},t)$ is the self-consistent potential which is assumed here to be local. Then, H_W is the Wigner transform of the self-consistent one-body Hamiltonian with the structure

$$H_W = \frac{(\mathbf{p})^2}{2m} + V(\mathbf{x},t).$$

The formalism presented below is designed for the study of nuclear dynamics at low and medium energies. To simplify the understanding we consider nuclei as droplets of incompressible and homogeneously distributed nuclear matter filling the volume inside the sharp surface.

The name of a virial theorem is used for any exact relation obtained integrating the kinetic equation over the phase space with some weighting function depending on the coordinates and momenta. In the quasi-classical approximation it has the form

$$\int d\mathbf{x}\, d\mathbf{p}\; a(\mathbf{x},\mathbf{p}) \left\{ D\,[f] - \left(\frac{\partial f}{\partial t}\right)_{\rm cls} \right\} = 0,$$

where $D\,[f]$ is the Liouville operator applied to the distribution function and $a(\mathbf{x},\mathbf{p})$ is the weighting function. The dynamic equations of motion are obtained using virial theorems with the following $a(\mathbf{x},\mathbf{p})$ functions:

$$a(\mathbf{x},\mathbf{p}) = \left\{ \begin{array}{llll} 1 & x_i & x_i x_j & \cdots \\ p_i & x_i p_j & x_i x_j p_k & \cdots \\ p_i p_j & x_i p_j p_k & x_i x_j p_k p_l & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array} \right.$$

Note that virial theorems obtained using the weighting functions $a(\mathbf{x},\mathbf{p})$ explicitly shown above do not distinguish the two forms of the kinetic equation: i.e., they are identical for the Wigner function and for its quasi-classical analogue. It is not the case for more complicated virial theorems [14].

The techniques of arriving at virial theorems, i.e., at relations between «integral observables» following from the kinetic equation (1) or (2) are well documented in the literature. Basic ideas are presented in Ref. 11, while specific points for treating nuclear physics problems are discussed in Ref. 14, where the method is used for studies of nuclear Multiple Giant Resonances (MGR) and stationary rotation. Still additional points may be found in Ref. 1 where the application to nuclear reactions of the method of virial theorems to nuclear reactions has been started.

A useful but not really necessary step to arrive at virial theorems consists in passing first at equations of «fluid dynamics» introducing the local collective observables like the density $\rho(\mathbf{x},t) = m\, n(\mathbf{x},t)$, velocity field $\mathbf{u}(\mathbf{x},t)$, pressure tensor $P_{ij}(\mathbf{x},t)$, etc. Such equations follow from the kinetic equation integrated over the momentum space and include the continuity relation

$$\frac{\partial \rho}{\partial t} + div(\mathbf{u}\rho) = 0 \tag{3}$$

and an infinite chain of equations describing the dynamics of nuclear matter. The two first of these equations are

$$n\,\frac{\partial u_i}{\partial t} + (\mathbf{u}\cdot\nabla)u_i + n\,\frac{\partial U}{\partial x_i} + \sum_j \frac{\partial P_{ij}}{\partial x_j} = 0, \tag{4}$$

$$\frac{\partial P_{ij}}{\partial t} + \sum_k \left(\frac{\partial u_i}{\partial x_k} P_{jk} + \frac{\partial u_j}{\partial x_k} P_{ik} + \frac{\partial u_k}{\partial x_k} P_{ij} + P_{ijk} \right) = (\mathcal{I}_{\rm cls})_{i,j}, \qquad (5)$$

where $(\mathbf{q} \equiv \mathbf{p} - m\mathbf{u})$

$$P_{ijk} = \frac{1}{m^{(1+k)}} \int d\mathbf{p}\, q_i q_j \cdots q_k\, f(\mathbf{x}, \mathbf{p}, t), \qquad (\mathcal{I}_{\rm cls})_{i,j} = \int d\mathbf{p}\, q_i q_j \left(\frac{\partial f}{\partial t} \right)_{\rm cls}.$$

2.2. Basic Assumptions. Three basic assumptions are used:

- The shape of the fusing system is considered to be well reproduced by a finite number of parameters being in a one-to-one correspondence with the collective observables like the electromagnetic moments and the total angular momentum. These observables have generally the form

$$Q_\alpha(t) = \int d\mathbf{x}\, \rho(\mathbf{x}, t) \mathcal{Q}_\alpha(\mathbf{x}),$$

  $\rho(\mathbf{x}, t) = m\, n(\mathbf{x}, t)$ being the mass density and $\mathcal{Q}_\alpha(\mathbf{x})$ is a function depending on the coordinates of one nucleon.

- The velocity field is suggested to be determined by the rate of changes of the shape and to have the form:

$$\mathbf{u}(\mathbf{x}, t) = \sum_\alpha \dot{Q}_\alpha(t) \mathbf{u}^\alpha(\mathbf{x}, Q) \qquad (6)$$

  (here and in the following we often use a single character Q to represent the totality of Q_α variables).

  From the continuity relation (3) it follows that the functions $\mathbf{u}^\alpha(\mathbf{x}, Q)$ may be defined so that the following relations are satisfied:

$$\int d\mathbf{x}\, \rho(\mathbf{x}, t) \mathbf{u}^\alpha(\mathbf{x}, t) \cdot \nabla \mathcal{Q}_\beta(\mathbf{x}) = \delta_{\alpha,\beta}.$$

- The collision term in the kinetic equation is taken in an approximation appropriate for «large» systems whose dimensions exceed greatly the range of the interparticle interaction:

$$\int \left(\frac{\partial f}{\partial t} \right)_{cls} d\mathbf{p} = \int \left(\frac{\partial f}{\partial t} \right)_{cls} \mathbf{p}\, d\mathbf{p} = \int \left(\frac{\partial f}{\partial t} \right)_{cls} (\mathbf{p})^2 d\mathbf{p} = 0. \qquad (7)$$

3. EXAMPLES OF VIRIAL THEOREMS

3.1. Total Energy of the System. A very important virial theorem is the one defining the total energy of the system. It is obtained integrating the kinetic equation in either of its forms with the weights $p_i u_j(\mathbf{x}, t)$, equating the indices i and j and summing the result over i.

In this way one gets

$$\frac{d}{dt}\left(T_{coll} + U\right) + \int d\mathbf{x} \sum_{i,j} u_i \frac{\partial P_{ij}}{\partial x_j} = 0. \tag{8}$$

The quantities $T_{\rm coll}$ and U in Eq. (8) are the kinetic energy of the collective flow and the potential energy defined as follows:

$$T_{\rm coll} = \frac{1}{2}\int d\mathbf{x}\ \rho(\mathbf{x})(\mathbf{u})^2 = \frac{1}{2}\sum_{\alpha,\beta} M^{\alpha,\beta}(Q)\dot{Q}_\alpha \dot{Q}_\beta, \tag{9}$$

where

$$M^{\alpha,\beta}(Q) \equiv \int d\mathbf{x}\ n(\mathbf{x})\mathbf{u}^{(\alpha)}\mathbf{u}^{(\beta)}, \tag{10}$$

$$U = \frac{1}{2}\int d\mathbf{x}\ V(\mathbf{x})n(\mathbf{x}). \tag{11}$$

The physical meaning of the quantity

$$\int d\mathbf{x} \sum_{i,j} u_i \frac{\partial P_{ij}}{\partial x_j}$$

becomes clear writing the virial theorem containing as a weight the function $q_i q_j$ and taking the trace of the matrix relation so obtained. In this way one comes to the relation (see Ref. 14)

$$\frac{dE_{\rm tot}}{dt} \equiv \frac{d}{dt}\left(T_{\rm coll} + U + \frac{1}{2}\Pi_0\right) \tag{12}$$

with $T_{\rm coll}$ and U defined by Eqs. (9), (11) and

$$\Pi_0 \equiv \sum_i \Pi_{ii} = \frac{g}{m}\int d\mathbf{x} \int d\mathbf{p}\ \mathbf{q}^2\, f(\mathbf{x}, \mathbf{p}, t)$$

(g being the degeneracy of the particle state with a given $\mathbf{p}$ value; we treat neutrons and protons as «identical» particles and take $g = 4$).

The quantity standing under the sine of the time derivative in Eq. (12) contains the kinetic energy of the collective flow and the potential energy. It remains

invariant in time. Then it is evident that this is the total energy. Note, that its conservation follows from the properties of the collision term formulated in Eq. (7). Apparently, these properties are compatible with various mechanisms of dissipation: with those leading to the «one-body» and «two-body» frictions, in particular.

The part of the energy given by $\Pi_0/2$ in Eq. (12) cannot be considered as a contribution to the potential energy. It represents the intrinsic kinetic energy of nucleons constituting the system and contains the energy of the «zero-point» motion of nucleons when the system is in the ground state. It contains also the energy of statistical excitation into which all the excitation energy is transformed after the equilibration. The sum $T_{\rm coll}+U$ is not conserved even when the temperature remains constant. Indeed, for the incompressible matter (when $div(\mathbf{u})=0$) from Eq. (5) it follows

$$\frac{d\Pi_0}{dt} = -2\sum_{i,j}\int d\mathbf{x}\, P_{ij}\frac{\partial u_i}{\partial x_j}. \tag{13}$$

Thus, this part of the energy contains a component having the nature of a collective energy. One sees that the changes of Π_0 are essentially $\dot{Q}$-dependent. The collective velocity is explicitly present here. When $\mathbf{u}(\mathbf{x},t)$ vanishes $\dot{\Pi}_0$ vanishes also. The anisotropy of the pressure tensor is a necessary condition for making the time derivative of Π_0 different from zero. Thus, this part of the collective energy is due to coherent changes in the distribution of nucleons in the momentum space, i.e., it is associated with deformations of the Fermi-surface. As was shown in the literature (see Ref. 14) such deformations play a dominant role in the formation of the isoscalar MGR. The studies of nuclear fusion reactions (see Refs. 2, 4) show that they are also important here.

An explicit treatment of every contribution to the collective energy distinguishes the approach in Refs. 1–5 from most of other approaches. This is true, in particular, for the relations between this approach and the one presented in Refs. 6–9 and also for the conceptually close to that approach pursued in Ref. 15 where it is applied for the study of nuclear giant resonances. In these approaches an approximate relation is borrowed from the condensed matter physics [16, 17] to relate $\dot{\Pi}_0$ with the shape variables and their time derivatives. This allows one to transform Eq. (12) into a closed set of dynamic equations for the shape variables. We point out that the introduction of approximations into the energy functional creates many problems in application to the study of nuclear reactions. One particular difficulty created by it is in the determination of the time dependence of the statistical excitation energy.

3.2. Virial Theorems to Describe the Head-On Collisions of Identical Nuclei. Consider the collision of two identical nuclei. In this case the system has the left-right symmetry in the direction of motion and the axial symmetry in

the perpendicular plane. The simplest physical observable carrying information on the shape of such a system is, of course, the quadrupole moment. If the shapes admitted for the system are restricted to a one-parametric family of axially symmetrical figures, the quadrupole moment may be put in a one-to-one correspondence with the value of the shape parameter. So the exact relations involving the mass (or/and charge) quadrupole moment yield information on the dynamics of the shape parameter whatever complicated is the shape. Here the total number of nucleons (protons) in the system of fusing or fissioning nucleus is named by the character A (Z). The fusion (fission) direction coincides with the z-axis of the reference frame with the origin placed at the center of mass.

To study the motion of colliding nuclei before or after the collision, as well as the neck formation, one may be interested in introducing in the theory of an «elongation» variable carrying additional information on the geometry of the system. It can be defined, e.g., as the distance between the centres of mass of nuclei. Then the geometric variables describing the system are:

$$Q(t) = \int d\mathbf{x}\, \rho(\mathbf{x},t)(2z^2 - x^2 - y^2), \quad L = \frac{4}{A}\int_{z\geq 0} d\mathbf{x}\, n(\mathbf{x},t)\, z.$$

Virial theorems involving L and Q are*:

$$\frac{1}{2}\frac{d^2Q}{dt^2} = \kappa^{ll}(L,Q)\dot{L}^2 + 2\kappa^{l,q}(L,Q)\dot{L}\dot{Q} + \kappa^{qq}(L,Q)\dot{Q}^2 - W(L,Q) + \Pi(t) \quad (14)$$

$$M\frac{d^2L}{dt^2} = \mathcal{F}^{(L)}. \quad (15)$$

The functions in Eq. (14) are:

$$\kappa^{\alpha\beta}(L,Q) = m\int d\mathbf{x}\; n(\mathbf{x})\left(3u_z^\alpha u_z^\beta - \mathbf{u}^\alpha\mathbf{u}^\beta\right) \quad (16)$$

(α, β standing for l or q) and

$$W(L,Q) = \int d\mathbf{x}\; n(\mathbf{x})\left(3z\frac{\partial V(\mathbf{x})}{\partial z} - \mathbf{x}\cdot\nabla V(\mathbf{x})\right). \quad (17)$$

In Eq. (15) there appears the «elongation driving force»

$$\mathcal{F}^{(L)} = \mathcal{F}_1^{(L)} + \mathcal{F}_2^{(L)}$$

*To these virial theorems contributes only the first term of the operator sine function in the kinetic equation (1), the one which is retained in the quasi-classical approximation in Eq. (2). This makes the two versions of the kinetic equation not distinguishable at this level.

with

$$\mathcal{F}_1 = -4\int_{z\geq 0} d\mathbf{x}\, n(\mathbf{x},t)\frac{\partial\, V(\mathbf{x},t)}{\partial z} = 4n_0\int_S dS\, V(\mathbf{x}_s,t), \tag{18}$$

$$\mathcal{F}_2^{(L)} = -4\int_{z\geq 0} d\mathbf{x}\sum_j \frac{\partial\, P(\mathbf{x},t)_{ij}}{\partial x_j} = 4\int_S dS\, P(\mathbf{x}_s,t)_{zz}. \tag{19}$$

(Virial theorems containing the time derivatives of L are obtained integrating Eq. (1) or (2) over the space coordinates with $z \geq 0$.)

The $(\lambda,\mu) = (2,0)$ component of the tensor $\Pi_{i,j}$ (pressure $(P_{i,j})$ integrated over the volume), which is present in Eq. (14), plays a key role in our study.

This quantity satisfies the first order differential in time equation

$$\frac{d}{dt}\Pi + 2\int d\mathbf{x}\, \left(\frac{\partial u_z}{\partial z}\right)(2P_\parallel + P_\perp) = I_{\rm cls} \tag{20}$$

with $P_\parallel(\mathbf{x},t) = P_{zz}(\mathbf{x},t)$ and $P_\perp(\mathbf{x},t) = P_{xx}(\mathbf{x},t) = P_{yy}(\mathbf{x},t)$.

Equation (20) describes the coupling between the changes in the nuclear shape and in the distribution of nucleons in the momentum space. In the right-hand side of this equation there appears the quantity $I_{\rm cls} = \int d\mathbf{x}\, \mathcal{I}_{\rm cls}$ responsible for the dissipative phenomena.

The surface integrals in Eqs. (18), (19) cover the part of the plane $z = 0$ limited by the intersection with the surface surrounding the nuclear matter. The quantity $\mathcal{F}_1^{(L)}$ does not vanish neither in the compact configurations of coalesced nuclei nor during their approach (or recoil) because of the interaction between the nucleons occupying the parts of the space with positive and negative z values.

The second component of the elongation driving force $\mathcal{F}_2^{(L)}$ vanishes when the nuclei are separated. In coalesced configurations the two components cancel each other to a large extend when the system approaches its state of equilibrium. In the general case the quantity $\mathcal{F}_2^{(L)}$ must be regarded as an additional collective variable and its dynamics must be learned from appropriate integral relations.

3.3. Virial Theorems for Peripheral Collisions. In peripheral collisions the angular momentum is not equal to zero. Then the nuclear system looses its axial symmetry. The most important integral variables in this case are

- Inertia tensor

$$Q_{ij}(t) = \int d\mathbf{x}\, \rho(\mathbf{x},t) x_i x_j. \tag{21}$$

- Kinetic energy tensor

$$K_{ij} = \int d\mathbf{x}\, u_i u_j \rho(\mathbf{x}). \tag{22}$$

- Pressure tensor integrated over the nuclear volume

$$\Pi_{ij} = \int d\mathbf{x}\; P_{ij}. \tag{23}$$

- Tensors related with the potential

$$W_{ij} = \int d\mathbf{x}\; x_j \frac{\partial V}{\partial x_i} n(\mathbf{x}). \tag{24}$$

- Relaxation integrals

$$I_{i,j} = \int d\mathbf{x} (\mathcal{I}_{\rm cls})_{i,j}. \tag{25}$$

Following [11] we consider two frames of reference with a common origin in the centre of mass of the system: an inertial frame, (X_1, X_2, X_3), and a moving frame, (x_1, x_2, x_3). Let $x_i = \sum_j T_{ij} X_j$ be the linear transformation relating the coordinates, (X_1, X_2, X_3) and (x_1, x_2, x_3). The following vector

$$\Omega_i = \frac{1}{2} \sum_{j,k,m} \varepsilon_{ijk} \left(\frac{dT}{dt} \right)_{jm} T^{+}_{mk} \tag{26}$$

represents a general time-dependent rotation of the (x_1, x_2, x_3) frame with respect to the inertial frame.

The second order virial equations in a rotational frame are [18]

$$\begin{aligned} \frac{d^2}{dt^2} Q_{ij} \;&+\; \sum_k \Omega_k (\Omega_i Q_{jk} + \Omega_j Q_{ik}) - 2\Omega^2 Q_{ij} \\ &+\; 2 \sum_{s,k} \Omega_s \int d\mathbf{r}\; \rho u_k (\varepsilon_{isk} x_j + \varepsilon_{jsk} x_i) \\ &+\; 2W_{ij} - 2K_{ij} - 2\Pi_{ij} + \sum_{s,k} \frac{d\Omega_s}{dt} (\varepsilon_{isk} Q_{kj} + \varepsilon_{jsk} Q_{ki}) = 0, \end{aligned} \tag{27}$$

$$\begin{aligned} \frac{dL_k}{dt} \;&+\; \sum_{i,j,m} \varepsilon_{kji} \Omega_i \Omega_m Q_{jm} - 2 \sum_s \Omega_s \int d\mathbf{r}\; \rho u_k x_s \\ &-\; \sum_s \frac{d\Omega_s}{dt} Q_{ks} + \frac{d}{dt} (\Omega_k \sum_j Q_{jj}) = 0, \end{aligned} \tag{28}$$

where the dynamical part of the angular momentum is given by

$$L_k \equiv \sum_{i,j} \varepsilon_{kij} \int_V d\mathbf{r}\; \rho x_i u_j, \tag{29}$$

$$\frac{d}{dt}\Pi_{ij} \quad + \quad F_{ij} + 2\sum_{s,k}\Omega_s(\varepsilon_{isk}\Pi_{kj} + \varepsilon_{jsk}\Pi_{ki}) = I_{i,j},$$
$$F_{ij} \quad \equiv \quad \sum_k \int d\mathbf{r}\,\left(P_{ik}\frac{\partial u_j}{\partial x_k} + P_{jk}\frac{\partial u_i}{\partial x_k}\right). \tag{30}$$

The choice of Ω may be done in different ways. For formal studies presented below, equations of motion in the inertial frame are preferable. In applications to physical problems a moving frame may be chosen. A particularly useful definition of the moving frame is such that its coordinate axes coincide with the principal axes of inertia tensor ($Q_{ij} = Q_{ii}\delta_{ij}$). Assuming that $\Omega = \{\Omega(t), 0, 0\}$ one finds equations for components of the inertia tensor

$$\begin{aligned}
\frac{d^2}{dt^2}Q_{11} + 2(W_{11} - K_{11} - \Pi_{11}) \quad &= 0, \\
\frac{d^2}{dt^2}Q_{22} + 2(W_{22} - K_{22} - \Pi_{22}) \quad - \quad &2Q_{22}\Omega^2 - 2\Omega L = 0, \\
\frac{d^2}{dt^2}Q_{33} + 2(W_{33} - K_{33} - \Pi_{33}) \quad - \quad &2Q_{33}\Omega^2 - 2\Omega L = 0.
\end{aligned} \tag{31}$$

Equations (28),(29) for the dynamical part of the angular momentum are:

$$\frac{d}{dt}\left(L_1 + \Omega(Q_{22} + Q_{33})\right) = 0, \qquad \frac{d}{dt}L_2 = \frac{d}{dt}L_3 = 0. \tag{32}$$

The equations for the pressure tensor integrated over the volume are:

$$\begin{aligned}
\frac{d}{dt}\Pi_{11} + F_{11} = I_{1,1}, \qquad & \frac{d}{dt}\Pi_{22} + F_{22} - 4\Omega\Pi_{23} = I_{2,2}, \\
\frac{d}{dt}\Pi_{33} + F_{33} + 4\Omega\Pi_{23} = I_{3,3}, \qquad & \frac{d}{dt}\Pi_{12} + F_{12} - 2\Omega\Pi_{13} = I_{1,2}, \\
\frac{d}{dt}\Pi_{23} + F_{23} + 2\Omega(\Pi_{22} - \Pi_{33}) = I_{2,3}, \qquad & \frac{d}{dt}\Pi_{13} + F_{13} + 2\Omega\Pi_{12} = I_{1,3}.
\end{aligned} \tag{33}$$

The nondiagonal terms of Eqs. (27) become:

$$K_{12} + \Pi_{12} = 0, \quad K_{13} + \Pi_{13} = 0, \quad K_{23} + \Pi_{23} - \frac{d}{dt}\left(\Omega\frac{Q_{22} - Q_{33}}{2}\right) = 0. \tag{34}$$

4. FROM VIRIAL THEOREMS TO COLLECTIVE MODELS

According to the terminology of Chandrasekhar, the relations presented above belong to virial theorems of the second order: they are obtained integrating kinetic equation with second order polynomials in coordinates (and/or momenta). The

number of useful virial theorems may be substantially increased. Applications of virial theorems of higher orders to the nuclear structure problems are known in the literature. Virial theorems of the third order are used in Ref. 19, 20, 21 in the study of negative parity collective excitations. Forth order virial theorems have been considered in Ref. 22 for the description of collective states of positive parity in spherical nuclei with multipolarities up to $\lambda = 4$. Virial theorems allowing the study of both isoscalar and isovector collective motions have been considered in Ref. 23.

Some of the integral characteristics entering into virial theorems like the multipole mass or/and charge moments can be taken for collective variables describing the distribution of the nuclear matter in the configuration space (hereafter called Q-variables). A part of virial theorems (e.g., in Eq. (14)) resemble classical equations of motion for such «geometrical variables». These relations contain blocks having to do with the self-consistent potential and with inertia properties of nuclear systems. To formulate a model of nuclear dynamics one must express such blocks in terms of Q variables and their time derivatives. In practice this leads to a restriction of shapes allowed for the system during its evolution to families described by a limited set of parameters. One must also introduce some ideas concerning the evaluation of nuclear inertia properties. In virial theorems enters also the pressure tensor integrated over the volume (Π). Its first time derivative is given by Eq. (20) and contains terms $\int d\mathbf{x}\,(\partial u_i/\partial x_k)\,P_{jk}$. It means that the information on the velocity field must be sufficiently detailed to account for phenomena related with the dynamics of the pressure.

Relations represented by virial theorems are as exact as the underlying microscopic equations of motion (1), (2). However, in the general case they do not form a closed set of equations which could be considered as a model of collective motion. Even in cases when chosen virial theorems form a closed set of equations, their solutions do not represent generally particular exact solutions of the underlying many-body problem. The errors are coming basically from two sources: from the choice of admitted shapes of the system and from assumptions on the velocity field and on the pressure tensor.

The rare known cases when virial theorems yield exact solutions of many-body problems correspond to elliptically shaped bodies composed of particles interacting in a very special way. In macroscopic theory an exact solution to the equations of classical hydrodynamics describing the evolution of self gravitating liquid bodies is given by virial theorems of the second order considered before (see Ref. 11 and the references therein). In quantum many-body theory the same virial theorems give an exact solution to the collective motion in the semi-classical limit when the interparticle interaction force is proportional to the distance between the particles (Ref. 24).

In their application to nuclear physics, virial theorems may serve only as a mean to arrive at an approximate description of nuclear properties. In order

to do so one must first of all find a reasonable approximation for the potential tensors $W_{i,j}(Q)$ and for the potential energy $V(Q)$ depending on the collective variables Q. An abundant information on this problem may be found in the literature (see, e.g., Ref. 25). Much less is known on inertia parameters $M(Q)$ although much work is done on this account. The properties of F_{fs} blocks remain virtually unexplored notwithstanding their important role in the dynamics of nuclear reactions.

It is because in most of models dealing with nuclear dynamics, the pressure is considered as a «function of state» in the meaning of classical condensed matter theory, i.e., as being determined by the distribution of the density, the matter currents and the temperature. However, the form of virial theorems (containing the time derivative of such integral characteristics as, e.g., in Eq. (20)) excludes the possibility to consider such integral characteristics as functions determined by the nuclear state, and for this reason we consider the integral characteristics containing the pressure tensor as independent collective variables (hereafter Π variables).

Their role is distinctly different in fast and slow processes, as it is shown in Refs. 1,2: they contribute to the elasticity of the system in respect to fast variations of the shape and describe dissipation of collective energy when the system is left to equilibrate. To give an idea of phenomena related with the evolution of the integrated pressure tensor we consider a «primitive elastoplastic system»*. This model corresponds to equations of motion which are obtained from equations (14), (20) in the limit of small amplitude ellipsoidal vibrations around the spherical equilibrium shape [4]:

$$\begin{aligned} \ddot{Q} + \alpha\, Q &= 2\Pi, \\ \dot{\Pi} + \beta\, \dot{Q} &= -\gamma\Pi. \end{aligned} \tag{35}$$

The right-hand side of the first of Eqs. (35) plays the role of a part of the force affecting the physical quantity $Q(t)$. The second of these equations may be transformed to an integral form giving

$$\Pi(t) = -\frac{\beta}{2}\int_{t_0}^{t} dt' \exp\left(-\frac{t-t'}{\tau}\right)\dot{Q}(t') + \Pi(t=0)\exp\left(-\frac{t}{\tau}\right) \tag{36}$$

and showing that this part of the force is determined by the evolution of the system during the preceding period of time of the order of the mean relaxation time τ. One may say that the system «decides» what to do in the next moment,

*The words «elastoplastic behaviour» have been introduced in nuclear physics by W.Nörenberg [26] to describe specific nuclear properties which were known from the time of Maxwell as typical for very viscous liquids [27]. Note that in condensed matter physics the word «plasto-elasticity» is used with a completely different meaning, being attributed to materials loosing their property of solids and behaving as a viscous liquid when subjected to a high pressure [28].

remembering what has happened to it previously during a finite period of time $t \sim \tau$. For this reason this system could be called also as a system with a «memory», and this memory is expressed by the integral form of the equation for the quantity $\Pi(t)$ and by an integro-differential form of the equation for the «physical observable» $Q(t)$:

$$\frac{1}{2}\ddot{Q} + \frac{\alpha}{2}Q + \frac{\beta}{2}\int_0^t dt' \exp\left(-\frac{t-t'}{\tau}\right)\dot{Q}(t') = 0 \tag{37}$$

(here we have assumed that the constant $\Pi(t=0)$ appearing in Eq. (36) is equal to zero).

Thus, the parameter τ determines the memory scale. In the processes going slowly in this scale (in adiabatic processes) the system described by Eqs. (35) follows in its evolution an equation of motion of a vibrator (when $\alpha > 0$) with the frequency $\sqrt{\alpha}$ dampened by the friction force $\chi\dot{Q}(t)$, where $\chi = \tau\beta/2$:

$$\frac{1}{2}\ddot{Q} + \chi\dot{Q} + \frac{\alpha}{2}Q \sim 0. \tag{38}$$

When $\chi^2 \geq \alpha$ vibrations are over-dampened, and the body is plastic. The fast (diabatic) processes in the same system proceed as if the system were an elastic body:

$$\ddot{Q} + (\alpha + \beta)Q \sim 0.$$

In this regime the frequency is renormalized and is equal to $\Omega = \sqrt{\alpha + \beta}$.

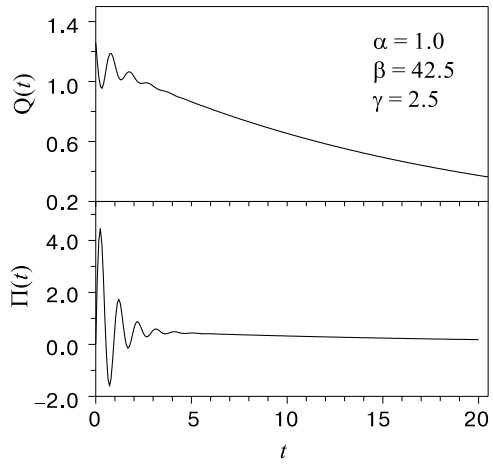

Fig. 1. Time dependence of variables Q (above) and Π (below) in the primitive elastoplasic model

Subjected to fast perturbations the elastoplastic system reacts as an elastic body, while in the slow motion regime, it behaves as being plastic. The elastoplastic properties are well pronounced when $\beta/\alpha \gg 1$. The evolution of the primitive elastoplastic system having at $t = 0$ a nonvanishing deformation $Q(t = 0) = Q_0$ and zero value of Π is demonstrated in Fig. 1 in the time scale in which $\alpha = 1$. Here the time dependence of Q and Π variables is shown for β/α and γ values which are typical for nuclear reactions. In fact, Eqs. (35) describe the bulk properties of the Giant Quadrupole Resonance in atomic nuclei [14]: the centroid of quadrupole electric excitations* ($E_{GQR} = \sqrt{2}\hbar v_F/R$), and the width of GQR ($\Gamma_{GQR} = \hbar/\tau$). The parameters chosen for calculation are estimated using virial theorems in which Q and Π are quadrupole moments in the configuration and momentum spaces respectively, and making simple assumptions on the mean potential (Coulomb and surface potentials in a spherical nucleus with homogeneous distribution of the matter within a sharp surface). They correspond to a nuclear system composed of two fused ^{58}Ni:

$$\frac{\beta}{\alpha} = 42.5, \qquad \frac{\gamma}{\alpha} = 2.5.$$

Elastic properties are well pronounced at the beginning of the process giving way to a plastic behaviour at later times. Regarding the nonzero starting values of Q and Π variables as a result of an instantaneous external perturbation, one may say that an external force applied to such a system produces, at the beginning, phenomena typical of elastic bodies. The final part of the evolution process corresponds to an exponential decrease in the absolute value of the variable Q. Hence, the way of coming to the equilibrium state of this system reveals its elastoplastic properties.

5. COLLECTIVE ENERGY AND THE RAYLEIGH DISSIPATION FUNCTION

Approximations which are necessary to transform virial theorems into a coupled set of differential equations for the collective variables Q and Π may lead to serious drawbacks in the theory. Probably the most important of them is a possible loss of an exact conservation of the total energy of the system. Here we want to incorporate into the theory the accurate description of the energy and, in particular, of its statistical component. In our current study of nuclear fusion this allows us to relate the input in the equations of motion for collective variables with the microscopic Hamiltonian.

*In the expression which follows $v_F = \sqrt{2m\epsilon_F}$, while v_F and ϵ_F are respectively the velocity and the energy of a nucleon on the Fermi surface; $R = r_0 A^{1/3}$ is the nuclear radius.

From our previous studies it follows that the evolution of a nuclear system depends in a sensitive way on the quantities

$$F_{i,j} = \sum_k \int d\mathbf{x} \frac{\partial u_i}{\partial x_k} P_{jk}$$

appearing in virial theorems containing the time derivative of Π variables (see Eqs. (20) and (30)). The collective flow contributes to the pressure terms quadratic in collective velocities $\dot{Q}$ when calculated in the simplest but quite reasonable approximation. Generalizing slightly the formulae established in Refs. 1, 2 we write

$$F_{i,j} = \frac{d}{dt} \left(\sum_{\beta,\gamma} \mu_{i,j}^{\beta,\gamma}(Q) \dot{Q}_\beta \dot{Q}_\gamma \right) + \sum_\alpha \dot{Q}_\alpha F_{i,j}^\alpha(Q). \tag{39}$$

Let us suppose that the other quantities in the virial theorems are known as functions on the Q and $\dot{Q}$ variables. Then, the virial theorems form a closed set of differential equations describing the collective motion, and we want to find an expression for the collective energy corresponding to it.

In Fermi systems the kinetic energy of individual particles contributes to (i) the kinetic energy of collective flow, (ii) the statistical excitation energy and (iii) the energy of coherent motions which do not reveal themselves neither in the redistribution of the matter in the space nor in the temperature depending processes (particle evaporation, for example). The third part is called the Fermi-surface deformation energy. Although the exact definition of these components is well known on the microscopic level, the needed accuracy of estimation of each of them in a nuclear system containing $A \sim 100 - 200$ nucleons presents many difficulties. The problem of such an estimation becomes especially stringent in the study of nuclear fusion whose occurrence depends on the transformation of the collective energy of colliding nuclei into the energy of statistical excitation prior to the recoil of the projectile or some another competing process.

Obviously, the definition of the collective and statistical energy in the approaches with «restricted» dynamics cannot be completely model-independent. However, some general conditions must be imposed on it to make an approach satisfactory:

- The division of the energy into different components must be made in accordance with the total energy conservation rule. Applied to models under study, this condition signifies that Eqs. (8)–(12) must be satisfied*.

*Note that equation (8) may be used instead of the virial theorems to formulate a model of nuclear dynamics as is done, e.g., in Refs. 6–10 where an approximation for the integrated pressure

- The changes in the collective energy should be related with the dissipative properties of the system. The model must be formulated demanding the conservation of the collective energy in the absence of elements making the dynamics irreversible. Thus, we define the collective energy conditioning its conservation in the case when the quantity $I_{\rm rel}$ in Eq. (20) and analogous quantities $I_{i,j}$ in Eq. (33) are equal to zero.

- The second law of thermodynamics must be fulfilled: the collective energy cannot increase during the process without an external perturbation. This condition demands an improvement of the «mean-relaxation-time» approximation often used in macroscopic models.

- A rather good knowledge of static properties of the nuclear mean field gained in the extensive studies of nuclear structure and reactions must be given justice.

5.1. Canonical Form of Equations of Motion. Here we show how to determine the collective energy of the system whose dynamics follows Eqs. (14) and (20) for N «geometrical» variables $Q_\alpha(t)$ and M variables of a «dynamical» nature $\Pi_p(t)$:

$$\frac{1}{2}\ddot{Q}_\alpha - \sum_{\beta,\gamma}\kappa_\alpha^{\beta,\gamma}(Q)\dot{Q}_\beta\dot{Q}_\gamma + W_\alpha(Q) - \sum_q \Gamma_\alpha^q(Q)\Pi_q(t) = 0, \tag{40}$$

$$\frac{d}{dt}\left(\Pi_p + \sum_{\beta,\gamma}\mu_p^{\beta,\gamma}(Q)\dot{Q}_\beta\dot{Q}_\gamma\right) + \sum_\alpha \dot{Q}_\alpha F_p^\alpha(Q) = I_{\rm rel}^p, \tag{41}$$

where the following convention is used: $f(Q) \equiv f(Q_{\alpha_1}, Q_{\alpha_2}...Q_{\alpha_N})$.

One finds the collective energy by giving the equations of motion a canonical Lagrange–Rayleigh form [29]. The Lagrange ($\mathcal{L}$) and Rayleigh ($\mathcal{R}$) functions which we are looking for are such that $N+M$ equations

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{Q}_i} - \frac{\partial\mathcal{L}}{\partial Q_i} = -\frac{\partial\mathcal{R}}{\partial\dot{Q}_i} \tag{42}$$

coincide with Eqs. (40) and (41). $N+M$ equations (42) contain N coordinates Q_α ($i = 1\cdots N$) and M additional coordinates Q_i ($i = N+1\cdots N+M$) related with the variables Π_p and defined later on in Eq. (57). A special case of rendering a canonical form to such equations with $M = N = 1$ is considered in [5].

is introduced. This approximation is borrowed from the condensed matter physics [16] where the conservation of the total energy plays no role. The absence of the exact control of the energy in such models limits their applicability in solutions of nuclear physics problems.

Here and in the following part of this section the notation Q_i for the totality of coordinates will be kept in the relations where all coordinates enter symmetrically. In other cases we shall use Greek symbols $\alpha, \beta, \cdots$ for indices of geometrical coordinates and Latin symbols $p, q, \cdots$ for the rest of coordinates. We suppose that the «adiabatic» Lagrangian $\mathcal{L}_{\rm ad}$ exists and is known. It is understood as the Lagrangian leading to equations of motion for geometrical variables when the Fermi-surface deformations are absent.

We assume that the adiabatic Lagrangian

$$\mathcal{L}_{\rm ad} = T_{\rm coll}(\dot{Q}, Q) - U(Q), \tag{43}$$

where $T_{\rm coll}(\dot{Q}, Q)$ and $U(Q)$ are defined as in Eqs. (9–11), is such that

$$\frac{d}{dt}\frac{\partial \mathcal{L}_{\rm ad}}{\partial \dot{Q}_\alpha} - \frac{\partial \mathcal{L}_{\rm ad}}{\partial Q_\alpha} = 2\sum_{q,\delta} M^{\alpha,\delta}\Gamma^q_\delta \left(\Pi_q + \sum_{\beta,\gamma} \mu^{\beta,\gamma}_q \dot{Q}_\beta \dot{Q}_\gamma \right). \tag{44}$$

Adiabatic Lagrangian exists when the following relations are satisfied:

- between the potential energy $U(Q)$ in the expression for the total energy of the system and $W_\alpha(Q)$ functions in the dynamical equations (40):

$$\sum_\beta (M^{-1})^{\alpha,\beta}\frac{\partial U}{\partial Q_\beta} = -2W_\alpha; \tag{45}$$

- between the inertia functions $M^{\alpha,\beta}$, $\kappa^{\beta,\gamma}_\alpha$ and $\mu^{\beta,\gamma}_q$:

$$\sum_\delta (M^{-1})^{\alpha,\delta}\frac{\partial M^{\delta\beta}}{\partial Q_\gamma} = -4(\sum_q \Gamma^q_\alpha \mu^{\beta,\gamma}_q - \kappa^{\beta,\gamma}_\alpha). \tag{46}$$

Let us find an expression for the time derivative of the «adiabatic energy»

$$E_{\rm ad} = \sum_\alpha \dot{Q}_\alpha \frac{\partial \mathcal{L}_{\rm ad}}{\partial \dot{Q}_\alpha} - \mathcal{L}_{\rm ad}. \tag{47}$$

Using Eq. (44) it is easy to see that

$$\frac{dE_{\rm ad}}{dt} = 2\sum_\alpha \dot{Q}_\alpha \sum_{q,\delta} M^{\alpha,\delta}\Gamma^q_\delta \left(\Pi_q + \sum_{\beta,\gamma} \mu^{\beta,\gamma}_q \dot{Q}_\beta \dot{Q}_\gamma \right). \tag{48}$$

Consider now the functions $U^q_{fs}(Q)$ defined by the relation

$$\frac{\partial U^q_{fs}(Q)}{\partial Q_\alpha} = 2\sum_{q,\delta} M^{\alpha,\delta}\Gamma^q_\delta. \tag{49}$$

Such functions exist if

$$\frac{\partial}{\partial Q_\beta}\sum_{q,\delta} M^{\alpha,\delta}\Gamma^q_\delta = \frac{\partial}{\partial Q_\alpha}\sum_{q,\delta} M^{\beta,\delta}\Gamma^q_\delta. \tag{50}$$

When these relations hold, Eq. (48) may be written as

$$\frac{d}{dt}\left[E_{\rm ad} - \sum_q U^q_{fs}\left(\Pi_q + \sum_{\beta,\gamma}\mu^{\beta,\gamma}_q \dot{Q}_\beta \dot{Q}_\gamma\right) + \mathcal{U}\right] = \sum_q U^q_{fs} I^q_{\rm rel}, \tag{51}$$

with $\mathcal{U}$ satisfying

$$\frac{\partial \mathcal{U}}{\partial Q_\alpha} = \sum_q U^q_{fs} F^q_\alpha. \tag{52}$$

The long passage from Eq. (43) up to Eq. (52) leads to the conclusion that the function

$$\mathcal{I}_0 = E_{\rm ad} - \sum_q U^q_{fs}\left(\Pi_q + \sum_{\alpha,\beta}\mu^{\alpha,\beta}_q \dot{Q}_\alpha \dot{Q}_\beta\right) + \mathcal{U} \tag{53}$$

is an «adiabatic invariant». In other words, this is a function which remains invariant in the case when $I^p_{\rm rel} = 0$ while the variables change according to the equations of motion. Containing the energy of adiabatically slow motion ($E_{\rm ad}$) the quantity $\mathcal{I}_0$ must constitute a part of the total collective energy of the system. The rest of it must also be adiabatically invariant, that is to be a sum of adiabatic invariants. The equations of motion (41) allow one to establish M such invariants

$$\mathcal{I}_p = \left(\Pi_p + \sum_{\alpha,\beta}\mu^{\alpha,\beta}_p(Q)\dot{Q}_\alpha \dot{Q}_\beta\right) + W_p(Q)\,, \qquad p = 1, 2, ...M, \tag{54}$$

with $W_p(Q)$ functions satisfying differential equations

$$\frac{\partial W_p}{\partial Q_\alpha} = F^\alpha_p(Q). \tag{55}$$

Indeed, $\dot{\mathcal{I}}_p \equiv I^p_{\rm rel}$ and vanish in the absence of relaxation effects.

The equations of motion are linear in Π_p variables, and thus one expects that the energy is a quadratic function in them. Assuming that the dynamical contribution to the energy is positive-definite, we can dismiss the contributions to $E_{\rm coll}$ linear in $\mathcal{I}_p$. Then the total collective energy is given by:

$$E_{\rm coll} = \mathcal{I}_0 + \frac{1}{2}\sum_{p,q} C_{p,q}\mathcal{I}_p\mathcal{I}_q \tag{56}$$

with a symmetrical and positive-definite matrix $C_{p,q}$.

The difference in the structure of Eq. (40) containing the second time derivative of geometrical coordinates and of Eq. (41) for the variables Π_p is evident. Indeed, the latter are the first order differential (in time) equations. Their number may be even or odd depending on the choice of geometrical variables. The variables Π_p describing the distribution of nucleons in the momentum space have a common transformation property under the time inversion: they are time-even. So, independently of their number they cannot be grouped in pairs and considered as representatives for some conjugated generalized coordinates and momenta. As in Ref. 5 we precise now the definition of M coordinates associated with the quantities Π_p. We suggest that these coordinates are cyclic and write

$$\begin{aligned} Q_{\alpha_i} &= Q_i\,, \qquad 1 \le i \le N \\ \Pi_p &= \dot{Q}_{i=N+p} - \sum_{\alpha,\beta} \mu_p^{\alpha,\beta} \dot{Q}_\alpha \dot{Q}_\beta - f_p(Q)\,, \;\; N+1 \le i \le N+M. \end{aligned} \tag{57}$$

With this definition of the collective coordinates $Q_{i=N+p} \equiv Q_p$ the collective energy is:

$$E_{\rm coll} = E_{\rm ad} + \frac{1}{2} \sum_{p,q} C_{p,q} \dot{Q}_p \dot{Q}_q + \delta U(Q) + \sum_p \dot{Q}_p \xi_p(Q), \tag{58}$$

where

$$\delta U(Q) = \mathcal{U}(Q) - \sum_p U_{fs}^p(Q)\, f_p(Q) + \frac{1}{2} \sum_{p,q} C_{p,q}\, f_p(Q)\, f_q(Q)\,, \tag{59}$$

$$\xi_p(Q) = \sum_q C_{p,q} f_q(Q) - U_{fs}^p(Q). \tag{60}$$

The choice of $f_p(Q)$ functions in Eq. (57) is completely arbitrary, and we may do it in such a way that the last term in Eq. (58) disappears, demanding that $\xi_p = 0$. Then

$$\sum_q C_{p,q}\, f_q(Q) = U_{fs}^p(Q) \tag{61}$$

and $E_{\rm coll}$ takes the form of a sum of kinetic and potential energies. Then one is ready to introduce the Lagrangian function

$$\mathcal{L} = \mathcal{L}_{\rm ad} + \frac{1}{2} \sum_{p,q} C_{p,q} \dot{Q}_p \dot{Q}_q - \delta U(Q) + \sum_p \dot{Q}_p \Lambda_p(Q), \tag{62}$$

containing the difference of these two components and also the term linear in generalized velocities* $\dot{Q}_p$.

Straightforward calculation of derivatives in Eq. (42) and the comparison of the equations of motion obtained in this way with the dynamical equations of motion (Eqs. (14), (20)) yield the following relations:

$$\frac{\partial \Lambda_p}{\partial Q_\alpha} + 2\sum_\beta M_{\alpha,\beta}\Gamma^p_\beta(Q) = \sum_q C_{p,q}F^\alpha_q, \tag{63}$$

$$\frac{\partial \mathcal{R}}{\partial \dot{Q}_\alpha} = 0, \qquad \frac{\partial \mathcal{R}}{\partial \dot{Q}_p} = -\sum_q C_{p,q} I^q_{\rm rel}. \tag{64}$$

Equation (63) establishes a relation between $C_{p,q}$ and Λ_p coefficients. We shall see in section 5.2 how the coefficients $C_{p,q}$ can be found to fix the Lagrangian and the energy.

The time derivative of collective energy is given by the rules of the classical mechanics

$$\begin{aligned} \frac{dE_{\rm coll}}{dt} &= -\sum_i \dot{Q}_i \frac{\partial \mathcal{R}}{\partial \dot{Q}_i} \\ &= -\sum_p \dot{Q}_p \sum_q C_{p,q}\, I^q_{\rm rel}. \end{aligned} \tag{65}$$

The condition according to which $\dot{E}_{\rm coll} \leq 0$ imposes a restriction on the relaxation integrals $I^p_{\rm rel}$. This condition is satisfied when

$$\mathcal{R} = -\frac{1}{2}\sum_{p,q} \nu_{p,q}\dot{Q}_p\dot{Q}_q \tag{66}$$

with a nonnegative matrix $\nu_{p,q}$. This condition is satisfied when $I^p_{\rm rel}$ are linear combinations of generalized velocities $\dot{Q}_p$:

$$I^q_{\rm rel} = \sum_{p,p'} C^{-1}_{q,p}\nu_{p,p'}\dot{Q}_{p'}, \tag{67}$$

where $C^{-1}_{q,p}$ is the inverse matrix with respect to $C_{q,p}$.

*It is easy to verify that the expression for the energy in Eq. (58) may be obtained from the Lagrangian (62) using the standard relation of the classical mechanics [29]

$$E_{\rm coll} = \sum_i \dot{Q}_i \frac{\partial \mathcal{L}}{\partial \dot{Q}_i} - \mathcal{L}.$$

Being linear in the quantities Π_p, the relaxation integrals, defined like this, resemble those of the mean-relaxation-time approximation. Note, however, that this definition of $I^p_{\rm rel}$ introduces into the relaxation terms an explicit dependence on the geometrical coordinates and their velocities.

The collective energy represents only a part of the total energy of the system which may be written as

$$E = E_{\rm coll} + E_{\rm stat}. \tag{68}$$

The second term in the above equation is associated with the energy of statistical excitation ($E_{\rm stat} = aT^2$). From the conservation of the total energy it follows then:

$$\frac{dE_{\rm stat}}{dt} = \frac{d\mathcal{R}}{dt}. \tag{69}$$

The coordinates Q_p, being the primitives of $\dot{Q}_p$, are time-odd. This makes them analogous in some way to the vector potential in the electrodynamics. One important property of the vector potential is that this quantity is not associated with any physical observable. This property is shared by Q_p coordinates. As in the case with the vector–potential, these coordinates are introduced to facilitate the passage from the dynamical equations of motion for physical observables to Lagrangian formulation of the theory. As we shall see in more detail later, the analogy continues in the quantum formulation of the theory. Understood as pairs of quantum mechanically conjugated operators, the quantum operators of the generalized coordinate $\hat{Q}_p$ and the corresponding momentum $\hat{P}_p$ may be defined as operators producing «shifts» in the reciprocal spaces [30]. Then, in $\hat{P}_p$ representation the quantum $\hat{Q}_p$ operator may be defined as

$$\hat{Q}_p = \imath\hbar\frac{\partial}{\partial P_p}.$$

One anticipates that in the Q_p representation $\hat{Q}_p$ determines the phase of the quantum wave function (see section 8 for more information on this subject).

Contrary to Q_p, the variables $\dot{Q}_p$ are by definition physical observables. Here, as in Refs. 1, 2, they are introduced on the basis of the TDHF-approximation to the many-body quantum theory. We refer to Ref. 24 where a variable of such type is introduced in the many-body quantum equation without any allusion to the independent motion of nucleons.

5.2. Case of Fusion after the Head-On Collision. As an example we consider here equations of motion describing the fusion after the head-on collision of two identical nuclei (see subsection (3.2)). We simplify them further, omitting the «elongation variable» L from the equations, and obtain:

$$\frac{1}{2}\frac{d^2Q}{dt^2} = \kappa(Q)(\dot{Q})^2 - W(Q) + \Pi(t), \tag{70}$$

$$\frac{d}{dt}(\Pi + \kappa'(Q)(\dot{Q})^2) + \dot{Q}F_{fs}(Q) = I_{\rm cls}. \tag{71}$$

These equations of motion have been used in Ref. 2 for nuclear fusion studies. The canonical form of Eqs. (70) and (71) is obtained in Ref. 5 suggesting that $\kappa' = 0$. The slightly more general case with nonzero function κ' is reduced to the problem solved in the quoted reference introducing the quantities

$$\tilde{\Pi} = \Pi + \kappa'(Q)\dot{Q}^2, \qquad \tilde{\kappa}(Q) = \kappa(Q) - \kappa'(Q). \tag{72}$$

Then the following expressions follow from the above formalism:

- For the Lagrangian

$$L(\dot{Q}, Q, \dot{Z}) = \frac{M(Q)}{2}\dot{Q}^2 + \frac{C}{2}\dot{Z}^2 - \mathcal{U}_{\rm tot}(Q) + \dot{Z}\mu(Q) \tag{73}$$

  with the Z coordinate defined by the relation

$$\Pi = \dot{Z} - \kappa'(Q)\dot{Q}^2 + f(Q);$$

- For the Rayleigh function

$$\mathcal{R} = \frac{C}{2\tau}(\dot{Z})^2; \tag{74}$$

- For the collective energy

$$E_{\rm coll} = \frac{M(Q)}{2}\dot{Q}^2 + \mathcal{U}_{\rm tot}(Q) + \frac{C}{2}\dot{Z}^2; \tag{75}$$

The «effective mass» function and the potential are:

$$M(Q) = M_0 \exp\left(-4\int_0^Q dQ'\tilde{\kappa}(Q')\right), \tag{76}$$

$$\mathcal{U}_{\rm tot}(Q) = \mathcal{U}(Q) + 2\int_0^Q dQ' M(Q') W_{fs}(Q') - \frac{1}{2C}\mu(Q)^2. \tag{77}$$

In these equations M_0 and C are some constants whose definition will be given in section 6 after further examination of the model. The other functions on Q are defined as follows

$$\mathcal{U}(Q) = 2\int_0^Q dQ' W(Q') M(Q'), \tag{78}$$

$$W_{fs}(Q) = \int_0^Q dQ' F_{fs}(Q'),$$

$$\mu(Q) = 2\int_0^Q dQ' M(Q'),$$

$$f(Q) = \frac{\mu(Q)}{C} - W_{fs}(Q). \tag{79}$$

Note that the Z variable is cyclic and thus the corresponding momentum is conserved in the absence of dissipation. It could be seen from the direct calculations that with this definition the Lagrange-Rayleigh equations for $Q_1 = Q$, $Q_2 = Z$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i} - \frac{\partial L}{\partial Q_i} = -\frac{\partial \Sigma}{\partial \dot{Q}_i}, \tag{80}$$

coincide with Eqs. (70), (71).

6. CONSISTENCY RELATIONS

The numerous postulates introduced in the previous section appear for two reasons:

- The dynamical model considered above may be regarded as a projection of the «true» many-body dynamics on the limited space of some collective variables. In the general case, the «projected» dynamics cannot be described by means of «standard» classical mechanics with the Hamiltonian and Lagrangian functions depending quadratically on the generalized velocities. Only in some very special cases (see Ref. 24) one may hope to come in this way to an exact description of some features of the many-body dynamics. These special cases must correspond to certain relations between the blocks appearing in the «projected» equations of motion. At least some of our postulates may be considered as depicting the conditions at which the canonical form may be given to the equations of motion.

- The equations of motion (see Eqs. (14), (20)) plus the expression for the collective and statistical energies constitute an overcomplete set of equations. The conditions for the existence of a physically acceptable solution of this set of equations is expressed in a form of some relations also.

Relations established before allow one to reduce the number of *ad hoc* parameters and functions on Q variables in the equations of motion. To see clearly their meaning we come back to the model described at the end of section 5.2. An expression for the collective energy in Eq. (75) contains two functions $F_{fs}(Q)$, $\kappa'(Q)$ which are not specified in a precise way. The necessary information on these quantities may be inferred comparing the expressions for the energy in Eqs. (12), (9) and (11) with Eq. (75) and using the consistency relations discussed before. Then one obtains:

$$\tilde{\kappa}(Q) = -\frac{1}{4}\frac{d}{dQ}\ln M(Q), \tag{81}$$

where the collective mass parameter $M(Q)$ is defined as

$$M(Q) = 2m \int d\mathbf{x} (\mathbf{u}^q(\mathbf{x}, Q))^2 n(\mathbf{x}). \tag{82}$$

Equating the potential energy terms in the two expressions for the energy $\mathcal{U}_{\rm tot}(Q)$ in Eq. (75) and $U(Q)$ in Eq. (11)) one obtains:

$$\frac{dU}{dQ} = 2M(Q) \left[W(Q) + \int_0^Q dQ' \left(F_{fs}(Q') - \frac{2}{C} M(Q') \right) \right]. \tag{83}$$

The direct calculation of $\kappa(Q)$ in Eq. (46) together with Eq. (72) determines the function $\kappa'(Q)$ in the second of equations of motion. Equation (83) treated in the same way bears information on the function $F_{fs}(Q)$ and yields

$$\begin{aligned} F_{fs}(Q) &= \frac{2M(Q)}{C} + \frac{d\Delta W(Q)}{dQ}, \\ \Delta W(Q) &= \frac{1}{2M(Q)} \frac{dU(Q)}{dQ} - W(Q). \end{aligned} \tag{84}$$

Thus, the consistency relations determine all the ingredients in the equations of motion up to the yet unknown constants M_0 and C. The latter may be found demanding that the model describes the well established properties of the GQR: E_{GQR} and the strength of the E_2 transitions from the GQR region. These two quantities may be put in a one-to-one correspondence with M_0 and C giving

$$M_0 = \frac{5}{24 m A^{5/3} {r_0}^2}, \qquad C = \frac{5}{12 m A {v_F}^2}.$$

Coming back to the primitive elastoplastic system one obtains:

- The generalized coordinate $Z(t)$

$$\Pi(t) = \dot{Z}(t); \tag{85}$$

- The Lagrangian function

$$L = \frac{M}{2} \left(\dot{Q}^2 - \alpha Q^2 \right) + 2M \left(\frac{1}{\beta} \dot{Z}^2 + \dot{Z} Q \right); \tag{86}$$

- The dissipation function

$$\Sigma = 2 \frac{M}{\tau \beta} \dot{Z}^2; \tag{87}$$

- The collective energy

$$E_{\rm coll} = \frac{M}{2}\left(\dot{Q}^2 + \alpha\, Q^2 + \frac{4}{\beta}\,;\Pi^2\right), \tag{88}$$

- And finally the expression for the time derivative of $E_{\rm coll}$:

$$\frac{d}{dt}H_{\rm coll} = -4\frac{M}{\tau\beta}\Pi^2. \tag{89}$$

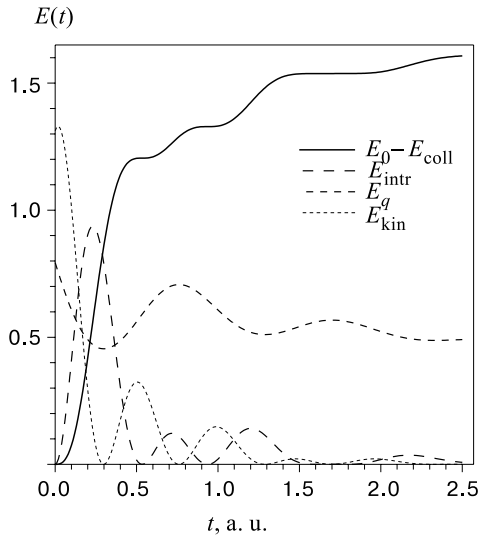

Fig. 2. Partition of the total energy into different channels corresponding to the process shown in Fig. 1. Arbitrary units are chosen both for the time and for the energy. 1 — thermal energy $E_0 - E_{\rm coll}$; 2 — intrinsic collective energy $E_{\rm intr} = 2M_Q\Pi^2/\beta$; 3 — potential energy $E_q = M_Q\alpha Q^2/2$; 4 — kinetic energy of collective flow $E_{\rm kin} = M_Q\dot{Q}^2/2$

The first term in Eq. (88) may be interpreted as the kinetic energy of the collective flow. Hence, the parameter M in the Hamiltonian must be equal to the effective mass parameter of the collective flow under consideration. The second term $E_{\rm pot}(Q) = \alpha\, M\, Q^2/2$ is the deformation potential energy.

Appealing to physical arguments we shall look at the last term in Eq. (88) as at the part of collective energy issuing from the zero-point motion of fermions constituting the system. It may be regarded then as the intrinsic kinetic energy.

The contributions to the collective energy are shown in Fig. 2. The calculations correspond to the motion depicted in Fig. 1. One sees a very strong

participation of the intrinsic kinetic energy at the beginning of the process when the motion has an elastic character.

In elastoplastic systems, the collective energy constitutes only a part of the total excitation energy, the rest being the energy of statistical excitation of the system. One sees that $E_{\rm coll}(t)$ decreases monotonically as it should do when the decrease in the collective energy is due to its transformation into the energy of statistical excitation.

The energy conservation law implies that the decrease of the collective energy is accompanied by the heating of the system. To establish the rate of the temperature changes one must know the relation between the intrinsic energy and the temperature itself. In moderately excited fermionic systems this relation is typically given by:

$$E_{intr}(Q,T) = E_{intr}(Q) + aT^2. \tag{90}$$

The solid line in Fig. 3 shows the temperature found from Eq. (90) for a sufficiently long period of time during which the system looses the deformation almost completely. At the beginning, the raise of the temperature is quite fast. However, the decrease in the potential energy takes all its importance in the energy balance only after the installation of the plastic regime when the kinetic energy disappears. Correspondingly, the temperature reaches its maximal value at sufficiently large time.

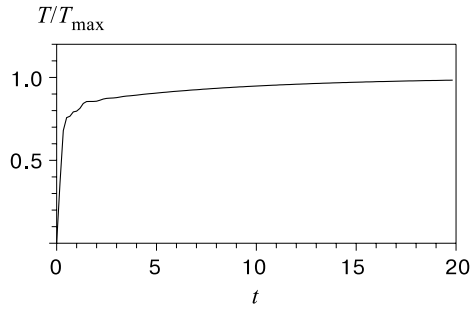

Fig. 3. Time dependence of the temperature $T(t)$ for the process shown in Fig. 1 in units of $T_{\rm max} = \sqrt{E_{\rm exc}/a}$

7. ON THE TREATMENT OF FLUCTUATIONS

The description of the many-body dynamics in the spirit of models discussed before may be called «macroscopic» as, for example, one calls the description in terms of the friction force of a massive (Brownian) particle propagating through a

liquid [31]. When the size of the Brownian particle decreases, the quality of such a description diminishes. To learn more about the motion of a «small» system, one must pass from the «macroscopic» to «microscopic» description. Knowing the Hamiltonian, it can be done following the standard rules of statistical mechanics (Refs. 32–34).

This proceeds by adding to relaxation terms $I_{\rm cls}$ a random force whose strength is fixed by the condition that, at the equilibrium, «the golden rule» of statistical excitation energy distribution were satisfied. This leads to Langevin equations for collective variables which may be effectively resolved numerically. Another way of studying the effects of fluctuations consists in passing from the Langevin equations to the formalism of the distribution function in the collective space $\Phi(Q_i, \dot{Q}_i, t)$ giving the probability to find the system with fixed values of collective variables and their first time derivatives. The general problems related with numerical estimations of fluctuation effects using the Langevin equations and the kinetic equation for $\Phi(Q_i, \dot{Q}_i, t)$ are discussed in detail in the review papers [10], [35] where an abundant bibliography on the subject is also given*

The specificity of the approach considered here lies in the presence of collective degrees of freedom associated with deformations of Fermi-surface. It amounts mostly to an enlargement of the number of collective variables, leading to quite spectacular phenomena. The application of these techniques to the primitive elastoplastic system is described in Refs. 4,5. In this case the kinetic equation for the distribution function $\Phi(Q, \dot{Q}, \Pi; t)$ is

$$\frac{\partial\Phi}{\partial t} + \dot{Q}\frac{\partial\Phi}{\partial Q} + (2\Pi - \alpha Q)\frac{\partial\Phi}{\partial\dot{Q}} - \frac{\partial}{\partial\Pi}\left(\left(\frac{\beta}{2}\dot{Q} + \frac{\Pi}{\tau}\right)\Phi\right) - \frac{kT\beta}{4\tau M}\frac{\partial^2\Phi}{\partial\Pi^2} = 0, \quad (91)$$

where k is the time Boltzmann constant.

The kinetic equation (91) allows one to find the standard square deviations matrix

$$Y = \begin{pmatrix} \overline{x_1x_1} & \overline{x_1x_2} & \overline{x_1x_3} \\ \overline{x_1x_2} & \overline{x_2x_2} & \overline{x_2x_3} \\ \overline{x_1x_3} & \overline{x_2x_3} & \overline{x_3x_3} \end{pmatrix}, \quad (92)$$

where the bar stands for the mean value calculated with the distribution function Φ and $x_1 = Q - \overline{Q(t)}$, $x_2 = \dot{Q} - \overline{\dot{Q}(t)}$, $\quad x_3 = \Pi - \overline{\Pi(t)}$.

The results of numerical calculations of Y are shown in Fig. 4. They correspond to the same evolution of the mean values of collective coordinates as in the previous figures. At large time, when the Q-variable approaches to zero, the off-diagonal matrix elements of the standard deviation matrix vanish and the diagonal ones attain such an amplitude that the mean energy per each degree of

*An important information on the subject is given in Ref. 36.

freedom becomes equal to kT as it should be at the equilibrium. At smaller time, fluctuations in different variables are correlated, so that the off-diagonal matrix elements of the mean square deviations matrix are not equal to zero. The amplitude of fluctuations is smaller than it would be in the state of equilibrium at the momentary temperature value. It is especially pronounced in the time dependence of $\overline{x_1{}^2}$ (curve 1), i.e., of the quantity related with the shape fluctuations.

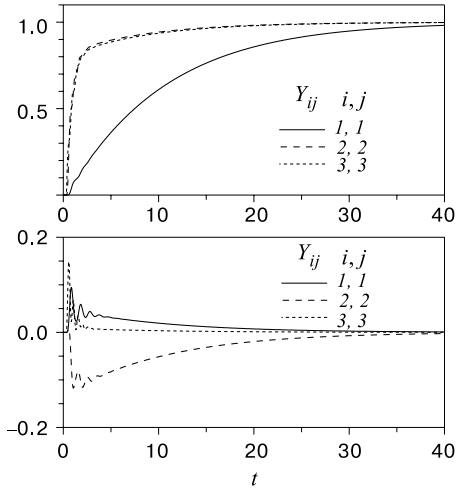

Fig. 4. Time dependence of the elements of the square deviation matrix Y_{ij}. Top: diagonal m.e., bottom: off diagonal m.e.

The time for arriving at the equilibrium shape and for the installation of fluctuations is determined by the characteristic time $t_1 \sim \tau(\beta/2\alpha)$ which is much larger than the relaxation time τ. However, the rate of changes of fluctuation amplitude of different quantities is drastically different. This could be seen by comparing the time dependence of the quantity $\overline{x_1{}^2}$ with that of $\overline{x_3{}^2}$ (curve 3). The latter grows with a very small retardation in respect to the temperature. The former increases much more slowly: $\overline{x_1{}^2(t)} = \overline{x_1{}^2(t=\infty)}\left(1 - \exp\left[-\frac{2\alpha t}{\beta\tau}\right]\right)$. The retardation in the installation of fluctuations of the shape signifies that their amplitude depends not only on the momentary value of the temperature but also at the «prehistory» of the temperature evolution.

8. QUANTIZATION OF THE PRIMITIVE COLLECTIVE MODEL [37]

To make the theory presented before applicable for the description of quantal effects like, e.g., the sub-barrier fusion, a quantal formulation of the model is

necessary. To fix ideas of this procedure we limit ourselves to the primitive elastoplastic system and consider the quantities Q_i ($Q_1 = Q$ and $Q_2 = Z$) and the conjugated momenta $P_i = \partial L/\partial Q_i$ as quantum mechanical operators.

The momentum operators are:

$$P_Q = M\,\dot{Q}, \qquad P_Z = \frac{4\,M}{\beta}\dot{Z} + 2M\,Q.$$

There are two representations of interest. In the first representation Q and Z stand for eigenvalues of $\hat{Q}$ and $\hat{Z}$ while $\hat{P}_Q = -i\hbar\,\partial/\partial Q$ and $\hat{P}_Z = -i\hbar\,\partial/\partial Z$. In the second representation P_Z stands for the eigenvalue of $\hat{P}_Z$ while $Z = i\hbar\,\partial/\partial P_Z$. In the second representation the collective Hamiltonian $H_{\rm coll} = \sum_i \dot{Q}_i P_i - L$ becomes

$$\hat{H}_{\rm coll} = \frac{1}{2M}\hat{P}_Q^2 + \frac{M\alpha}{2}Q^2 + \frac{\beta}{8M}\left(P_Z - 2MQ\right)^2.$$

The presence in the theory of the cyclic coordinate Z does not introduce any problem in finding the eigenstates and eigenvalues of $\hat{H}_{\rm coll}$.

However, for the following, the first representation is more appropriate. We introduce now the total Hamiltonian of the system:

$$\hat{H}_{\rm tot} = \hat{H}_{\rm coll} + \hat{\mathcal{H}}_1, \tag{93}$$

where $\hat{\mathcal{H}}_1$ is a virtually unknown part of the Hamiltonian depending on collective (Q, Z) and intrinsic (ξ) variables.

Next natural step is to introduce the density matrix

$$\rho(Q, Z; Q', Z'; t) = \int d\xi \Psi^*(Q', Z'; \xi)\Psi(Q, Z; \xi) \tag{94}$$

satisfying the equation

$$i\hbar\frac{\partial\rho}{\partial t} - \left[\hat{H}_{\rm coll}, \rho\right] - \int d\xi \Psi^*(Q', Z'; \xi)\hat{\mathcal{H}}_1\Psi(Q, Z; \xi) = 0, \tag{95}$$

where $\left[\hat{a}, \hat{b}\right]$ is the commutator of operators $\hat{a}$ and $\hat{b}$.

The final preparatory step consists in an introduction of elements necessary for the statistical treatment of intrinsic degrees of freedom. We consider the intrinsic Hamiltonian $\mathcal{H}_1$ as an element of a stochastic ensemble. Averaging the density matrix over the statistical ensemble of intrinsic Hamiltonians with appropriately chosen properties we obtain the new density matrix ($\overline{\rho}$). Being interested only in this quantity, we shall omit in the following the bar over $\overline{\rho}$.

The techniques of quantum statistical mechanics (see Refs. 38 and 39) allow one to specify the properties of the ensemble and to bring the last term on the

l.h.s. of Eq. (95) to the form of an effective potential $\hat{W}$ connecting Z and Q with Z' and Q':

$$\overline{\int d\xi \Psi^* \mathcal{H}_1 \Psi} = W(Q, Z, Q^{'}, Z')\ \rho(Q, Z; Q', Z'; t). \tag{96}$$

Then Eq. (95) becomes

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} - \left[\hat{H}_{\rm coll}, \hat{\rho}\right] - \hat{W}\,\hat{\rho} = 0. \tag{97}$$

The physical observables cannot depend on the Z coordinate. Thus $\hat{\rho}$ and $\hat{W}$ matrices depend only on $z = Z - Z'$:

$$\hat{\rho} = \rho(Q, Q'; z; t), \qquad \hat{W} = W(Q, Q'; z, \partial/\partial z).$$

Let us introduce the Wigner transformation of the density matrix:

$$\tilde{\Phi}(\overline{Q}, P_Q, P_Z, t) = \int \frac{ds\,dz}{(2\pi\hbar)^2} \exp\left[-i\frac{P_Q s + P_Z z}{\hbar}\right] \rho(\overline{Q} + \frac{s}{2}, \overline{Q} - \frac{s}{2}; z; t), \tag{98}$$

where $\overline{Q} = (Q + Q')/2,\ s = Q - Q'$.

Using the well-known formula for the Wigner transform of the commutator of two hermitian operators, one brings Eq. (97) to the form

$$\frac{\partial \tilde{\Phi}}{\partial t} = -\frac{P_Q}{M}\frac{\partial \tilde{\Phi}}{\partial \overline{Q}} + \left(\alpha\, M\, \overline{Q} - \frac{\beta}{2} P_Z\right) \frac{\partial \tilde{\Phi}}{\partial P_Z} + \\ + \frac{1}{i\hbar} \int ds\, dz\ \exp\left[-i\frac{P_Q s + P_Z z}{\hbar}\right] \hat{W}\,\hat{\rho}. \tag{99}$$

We suggest that ρ is a smooth function of $\overline{Q}$ and that its dependence on s and z is of a sharp bell-shaped type. Using a power expansion for $\hat{W}$:

$$W(\overline{Q}, z, \partial/\partial z; t) = i\hbar \sum_{n,m} \left(\frac{iz}{\hbar}\right)^n \left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right)^m W_{n,m}(\overline{Q}),$$

one obtains

$$\frac{1}{i\hbar} \int \frac{ds\,dz}{(2\pi\hbar)^2} \exp\left[-\frac{i(P_Q s + P_Z z)}{\hbar}\right] \hat{W}\hat{\rho} = \sum_{n,m} \frac{\partial^n}{\partial P_Z^n}\ P_Z^m W_{n,m}(\overline{Q}) \tilde{\Phi}. \tag{100}$$

It is easy to see now that the $\hat{W}$ function, taken in the form,

$$\frac{1}{i\hbar}\hat{W} = \frac{1}{\tau}\left\{\frac{iz}{\hbar}\left(\frac{\hbar}{i}\frac{\partial}{\partial z} - 2M\,\overline{Q}\right) - \frac{4M\,k\,T\ z^2}{\hbar^2 \beta}\right\} \tag{101}$$

makes Eq. (95) equivalent to the kinetic equation (91) and becomes identical with it after passing from variables (P_Q, P_Z) to ($\dot{Q}$, Π). Thus, in the considered simple case, the solution of the quantum problem is given by the inverse Wigner transform of an appropriate solution of the classical kinetic equation.

9. WAY BACK TO THE MANY-BODY THEORY: GENERALIZED ROUTHIANS

The consistency relations reduce the number of unknown functions in the equations of motion but leave much space for speculations on the potential and inertial properties of the system. Here we discuss, in a rather schematic way, the possibility of determining some of them on the basis of the underlying many-body theory [40]. As an example we consider the model of fusion after the head-on collision of identical nuclei (see section 5.2) neglecting the possible influence of the statistical excitation on the collective motion.

Then, the collective dynamics corresponds to the Lagrangian $L(\dot{Q}, Q, \dot{Z})$ in Eq. (73), containing the functions $M(Q)$, $U(Q)$ and $\mu(Q)$, in which Q is the quadrupole moment of the system. Here we want to show that these functions may be found using a variational approach to solutions of the many-body Schrödinger equation.

Consider the variational equation of the Time Dependent Hartree-Fock theory

$$\delta \int dt \left\langle \Psi \mid \imath\hbar \frac{\partial}{\partial t} - \hat{H} \mid \Psi \right\rangle = 0, \tag{102}$$

where $\hat{H}$ is the many-body Hamiltonian of the system. One may suggest that the time dependence of the wave function is coming from the collective motion. Consistently with hypotheses made in the derivation of the model, one may write the wave function in the quasi-classical approximation as in Ref. 30

$$\mid \Psi(t) >= \exp\left(\frac{-\imath}{\hbar} \int_0^t L(Q_i(t'), \dot{Q}_i(t'))dt' \right) \mid \Phi >$$

with the action integral containing the Lagrange function (73). Then for the time derivative of the many-body function one has

$$\imath\hbar \frac{\partial}{\partial t} \mid \Psi(t) >= L \mid \Psi(t) > . \tag{103}$$

The variable quantity in Eq. (102) is the many-body function $\mid \Phi >$. Its changes produce variations of the collective coordinate Q, of the generalized velocities $\dot{Q}_i$ ($\dot{Q}_1 = \dot{Q},\ \dot{Q}_2 = \Pi - f(Q)$) and hence of the Lagrangian function. Let us write:

$$\delta Q = \delta < \Phi \mid \hat{q} \mid \Phi >, \quad \delta\dot{Q}_i = \delta < \Phi \mid \dot{\hat{q}}_i \mid \Phi > \quad (i = 1, 2). \tag{104}$$

For the variation of the Lagrangian we write

$$\delta L = \frac{\partial L}{\partial Q}\delta Q + \sum_i \frac{\partial L}{\partial \dot{Q}_i}\delta\dot{Q}_i. \tag{105}$$

Then Eq. (102) takes the form of the variational equation of the cranking type

$$\delta < \Phi \mid \left(\hat{H} - \Lambda_Q \hat{q} - \Lambda_{P_Q} \hat{\pi}_Q - \Lambda_{P_Z} \hat{\pi}_Z \right) \mid \Phi >= 0, \tag{106}$$

where

$$\hat{q} = m(3z^2 - \mathbf{r}^2), \quad \hat{\pi}_Q = 2\frac{M(Q)}{n(\mathbf{r})}(3z\hat{p}_z - \mathbf{r} \cdot \mathbf{p}),$$

$$\hat{\pi}_Z = \frac{2M_0}{F_{fs}^0}(3(\hat{p'}_z)^2 - (\hat{\mathbf{p}'})^2) + \mu(\hat{q}) \qquad (\hat{\mathbf{p}'} = \hat{\mathbf{p}} - m\mathbf{u}(\mathbf{r}, t)). \tag{107}$$

Equation (106) tells that the collective motion must be adjusted to the intrinsic structure in such a way that the «generalized Routhian» operator

$$\hat{R} = \hat{H} - \Lambda_Q \hat{q} - \Lambda_{P_Q} \hat{\pi}_Q - \Lambda_{P_Z} \hat{\pi}_Z, \tag{108}$$

containing the Lagrange multipliers Λ_Q, Λ_{P_Q} and Λ_{P_Z} and the operators $\hat{q}_i$ and $\hat{\pi}_i$, remains stationary in respect to variations of the many-body function $\mid \Phi >$. The constraining operators in Eq. (108) have positive and negative parities as in the case considered in Ref. 41.

Equation (108) must be supplemented by the self-consistency conditions relating the Lagrange multipliers with the partial derivatives of the collective Lagrangian function and with the variational derivatives of the energy functional:

$$\Lambda_Q = \frac{\partial L}{\partial Q}, \qquad \Lambda_{P_Q} = \frac{\partial L}{\partial P_Q}, \qquad \Lambda_{P_Z} = \frac{\partial L}{\partial P_Z}. \tag{109}$$

We see now that the cranking-type equation (106) with the self-consistency relations (109) for ingredients of the collective Hamiltonian and the equations of motion for macroscopic variables (Eqs. (14), (20)), constitute a macroscopic model of collective dynamics consistent with the microscopic Hamiltonian of the system.

9.1. Linear Response to the Velocity-Dependent Constraints. Let us show results, obtained from the variational procedure outlined before, using the linear response approximation. We introduce the many-body function $\mid \Phi_{\rm stat} \rangle$ describing the stationary state of a system. This function satisfies the variational equation

$$\delta \left\langle \Phi_{\rm stat} \mid (\hat{H} - \Lambda_Q^{(0)} \hat{q}) \mid \Phi_{\rm stat} \right\rangle = 0 \tag{110}$$

and the self-consistency relation

$$\Lambda_Q^{(0)} = \frac{\partial \mathcal{E}_{\rm stat}}{\partial Q} = -\frac{dU}{dQ}. \tag{111}$$

Writing

$$\mid \Phi >=\mid \Phi_{\rm stat} > + \mid \Phi' >, \quad \Lambda_Q = \Lambda_Q^{(0)} + \delta\Lambda_Q$$

we obtain in the lowest order in the velocity-dependent terms

$$\mid \Phi' >= \frac{1}{(\hat{H} - \mathcal{E}_{\rm stat})}(\delta\Lambda_Q \hat{q} + \Lambda_{P_Q}\hat{\pi}_Q + \Lambda_{P_Z}\hat{\pi}_Z) \mid \Phi_{\rm stat} > \tag{112}$$

and thus

$$\begin{aligned} \delta Q &= \delta\Lambda_Q\, \mu_{Q,Q} + \Lambda_{P_Q}\, \mu_{Q,P_Q} + \Lambda_{P_Z}\, \mu_{Q,P_Z}, \\ P_Q &= \delta\Lambda_Q\, \mu_{P_Q,Q} + \Lambda_{P_Q}\, \mu_{P_Q,P_Q} + \Lambda_{P_Z}\, \mu_{P_Q,P_Z}, \\ P_Z &= \delta\Lambda_Q\, \mu_{P_Z,Q} + \Lambda_{P_Q}\, \mu_{P_Z,P_Q} + \Lambda_{P_Z}\, \mu_{P_Z,P_Z}, \end{aligned} \tag{113}$$

where

$$\mu_{i,j} =< \Phi_{\rm stat} \mid \hat{o}_i \frac{1}{(\hat{H} - \mathcal{E}_{\rm stat})}.\hat{o}_j \mid \Phi_{\rm stat} > + \text{c.c.}. \tag{114}$$

(here c.c. means the complex conjugated expression appearing when $i \neq j$).

Using the self-consistency conditions, these equations may be rewritten in the form

$$\begin{aligned} \delta Q &= \delta\frac{\partial L}{\partial Q}\mu_{Q,Q} + \frac{\partial L}{\partial P_Q}\mu_{Q,P_Q} + \frac{\partial L}{\partial P_Z}\mu_{Q,P_Z}, \\ P_Q &= \delta\frac{\partial L}{\partial Q}\mu_{P_Q,Q} + \frac{\partial L}{\partial P_Q}\mu_{P_Q,P_Q} + \frac{\partial L}{\partial P_Z}\mu_{P_Q,P_Z}, \\ P_Z &= \delta\frac{\partial L}{\partial Q}\mu_{P_Z,Q} + \frac{\partial L}{\partial P_Q}\mu_{P_Z,P_Q} + \frac{\partial L}{\partial P_Z}\mu_{P_Z,P_Z}. \end{aligned} \tag{115}$$

Equations (115) impose relations between the functions appearing in the Lagrangian and matrix elements $\mu_{i,j}$. The conditions for the self-consistency of the variational procedure with the collective dynamics become:

$$M(Q) = \mu_{P_Q,P_Q},$$

$$(F_{fs}^0)^2 = \frac{2M_0}{\mu_{P_Z,P_Z}}, \qquad F_{fs}(Q) = \frac{M(Q)}{M_0}F_{fs}^0. \tag{116}$$

Of much interest is the last relation in Eq. (116) which is similar to the one obtained examining the consistency relation between the ingredients of the collective Hamiltonian (see Eq. (79) in section 5.2). The difference between the two forms of this relations may be understood as coming from the application of the linear response approximation.

We leave for future studies a more careful examination of possibilities offered by this particular formulation of the variational principle.

10. CONCLUDING REMARKS

The aim of this report is to show in which way the large-amplitude nuclear collective dynamics may by treated theoretically by reasonably simple mathematical means without going too far from what is known about the quantum many-body nuclear Hamiltonian. So, we want to see the collective motion in a way sufficiently simple and realistic at the same time. For the sake of simplicity one must limit the number of variables describing the dynamics, that is to make a step towards the geometrical models of nuclear reactions. In order to remain close to the microscopic basis one must judiciously choose collective coordinates retained within the models. In the review, we advocate for the «virial theorems» providing some good guidelines in this problem.

The general line in this review paper is in establishing various consistency relations in the theory of the large-amplitude collective motion. By the name of consistency we mean at this moment all kinds of relations guaranteeing the common physical ground for studies of different aspects of nuclear structure and reactions. The theory must provide a coherent description of the slow and fast processes accompanying the reactions between heavy nuclei. The changes in the statistical excitation energy must be balanced with the loss of the collective energy. The description of reactions at much higher energy than the barrier and of subbarrier effects must be achieved starting from the same physical picture.

All such aspects of the theory are tackled in this review paper. Many of them are described rather schematically, using in different places a simple realization of the general scheme: the «primitive elastoplastic model». The main reason for that was to avoid technical complications in the presentation.

The most developed section is, probably, the one in which we show how the equations of motion of a certain type can be rendered a canonical form. It is because we consider the definition of a collective Hamiltonian as a very important element of the theory. The estimation of the temperature, the study of fluctuations, the possibility to requantize the collective model, all these steps are largely facilitated once the collective Hamiltonian is known.

We included in the review the presentation of an approach generalizing the cranking model which may be used eventually to calculate all the ingredients of collective dynamical models once the collective coordinates are fixed.

A part of the ideas and techniques described above have already been used in studies of nuclear fusion. The exposition of results obtained in this application of the approach will be done in a separate publication.

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