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THE GRAVITY THEORY ON A BACKGROUND OF THE LOBACHEVSKY GEOMETRY

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In this paper four models are being discussed, concerning the gravitational field of a star at rest and the equations of motion of the companion planet. The first model has been created by Newton; and the second model, by Lobachevsky. The third model has been initiated by Einstein, further developed by Schwarzschild and completed by Fock. The fourth model has been created by the author of this paper. In the second and in the fourth models the Lobachevsky geometry with the characteristic constant k is introduced in the background space. The constant k is the absolute measure of the length in the background space. In the third and in the fourth models the Lobachevsky geometry with the characteristic constant c is introduced in the velocity space. The constant c is the absolute measure of the rapidity in the velocity space. It equals the light velocity. In the first and in the third models the gravitational field of the star obeys the Einstein's equations. In the second and in the fourth models the gravitational field of the star obeys new equations, proposed by the author. The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

1. INTRODUCTION

Much can be understood, considering the gravitational theory on the background of the Lobachevsky geometry. For example, it can be understood why, despite all the achievements of relativistic theory of gravitation, some shortcomings in this theory can also be found. It can be understood also how one can remove these shortcomings.

As is known, Einstein has set up all the achievements of relativistic theory of gravitation, replacing in Newton's model the gravitational potential U with the gravitational metrics $g_{mn}dx^m dx^n$, and replacing the gravitational connection, given in the nonrelativistic case by the symbol $\mathrm{grad}\, U$, by the relativistic gravitational connection, expressed by the Christoffell's symbol for the tensor g_{mn}.

It is true that because of such a replacement the energy density of the gravitational field has turned out to depend on the choice of coordinate map, and the reason for this is the loss of the background connection. Without noticing this loss, the gravitationalists declared that the energy of the gravitational field is *non-localized*, thus damaging the relativistic theory of gravitation. From here come all the shortcomings in the theory.

Seemingly, such a loss has been performed under milder circumstances, because in Newton's model the background connection is primitive. But here is an intricate and subtle danger: in some coordinate maps all components of primitive connection equal zero, while in other coordinate maps the components of the same connection are not equal to zero. Therefore, it is more reliable to deal with a nonprimitive connection: the aggregate of its components does not equal zero in all the coordinate maps.

The Lobachevsky model [1], [2] helps to restore the background connection in the relativistic theory of gravitation. In this model the background connection is nonprimitive, but one can again return in the framework of Einstein's theory, keeping the restored background connection. For the purpose one has to set up to infinity the characteristic length for the Lobachevsky geometry. The restored connection in this limit will be, as in Newton's model, the primitive one.

As a result of the introduction of the Lobachevsky geometry in the background connection, during the last years some difficult questions in gravity theory become more clear. For example, the problem about the choice of harmonical coordinates has been clarified. The situation is analogous to the one, which Bogoliubov [3] has solved in statistical mechanics by applying the method of quasi-average quantities. The role, which in Bogoliubov's method is played by the magnetic field, in the current case is transferred to the length measure.

I have found the following method for the restored background connection [4].

Let us denote by Γ^a_{mn} the gravitational connection, by $\breve{\Gamma}^a_{mn}$ the background connection and by $P^a_{mn} = \breve{\Gamma}^a_{mn} - \Gamma^a_{mn}$ their affine deformation tensor.

And let us denote by R_{mn} the gravitational Ricci tensor, by $\breve{R}_{mn}$ the background Ricci tensor and by $S_{mn} = \breve{R}_{mn} - R_{mn}$ their difference.

According to the method, on that place, where (in the pseudoscalar Lagrangian and in the energy-momentum pseudotensor) a geometrical object with components $(-\Gamma^a_{mn})$ stands, (according to Manoff [5], it is called a covariant affine connection) we must put the tensor P^a_{mn}, and also on the place, where (in the Einstein equations of gravity) the tensor $(-R_{mn})$ stands, we must put the tensor S_{mn}.

In the new equations of the gravitational field

$$S_{mn} - \frac{1}{2}\, S\, g_{mn} = -\frac{8\pi\gamma}{c^4}\, M_{mn}, \qquad S = g^{mn} S_{mn},$$

the background connection is given, but the gravitational connection is to be found.

In the region, where $M_{mn} = 0$, the new equations of gravitational field take the form $S_{mn} = 0$.

The trivial solution $\Gamma^a_{mn} = \breve{\Gamma}^a_{mn}$ means that the background connection is the gravitational connection in its trivial form. In this case there is no gravitational field.

The background connection is defined by the equations of motion for a free particle.

The gravitational connection is defined by the equations of motion for a particle in a gravitational field, when there are no any other forces.

The condition of harmonicity for the background connection in respect of the gravitational field has a form $\Phi^a = 0$, where

$$\Phi^a = g^{mn} P^a_{mn}.$$

I have shown in my works [6], that any physical theory is founded on the concept of velocity space and that the geometry of this space is the Euclidean or the Lobachevsky one. In the first case the theory is named nonrelativistic and in the second case it is named relativistic. It is strange, of course, but it has been named in this way.

In the first case there is no characteristic measure of velocity. In the second case there is such a measure. It equals the light velocity c. The constant c is the analogue of the length measure k. The nonrelativistic case we shall denote by $c = \infty$. The relativistic case we shall denote by $c < \infty$.

The gravitational metrics may be transformed to the following sum

$$g_{mn} dx^m dx^n = f^1 f^1 + f^2 f^2 + f^3 f^3 - c^2 \; f^4 f^4,$$

where f^m are linear differential forms.

If the gravitational field is absent, we put

$$g_{mn} dx^m dx^n = \breve{g}_{mn} dx^m dx^n = h_{\mu\nu} dx^\mu dx^\nu - c^2 \; dt dt,$$

where $h_{\mu\nu}$ do not depend on $x^4 = t$.

The quadratic form $h_{\mu\nu} dx^\mu dx^\nu$ is either the metrics of the Euclidean space in the Newton's model (the case $k = \infty$), or the metrics of the Lobachevsky space in the Lobachevsky model (the case $k < \infty$).

The components of Christoffell's connection for the metrics $h_{\mu\nu} dx^\mu dx^\nu$ we shall denote by $h^\alpha_{\mu\nu}$.

The Ricci tensor $r_{\mu\nu}$ for the connection $h^\alpha_{\mu\nu}$ equals $r_{\mu\nu} = -k^{-2} \; h_{\mu\nu}$ in the case $k < 0$ and it equals zero ($r_{\mu\nu} = 0$) in the case $k = \infty$.

It is interesting, that the background connection $\breve{\Gamma}^a_{mn}$ does not depend on the light velocity c. Consequently it refers to the Absolute Geometry of Bolyai in velocity space. Indeed, the equations of geodesical lines in the case of the metrics $h_{\mu\nu} dx^\mu dx^\nu - c^2 \; dt dt$, may be written as

$$\frac{d^2 x^\alpha}{d\tau^2} + h^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0, \quad \frac{d^2 t}{d\tau^2} = 0.$$

But in such a form the equations of motion for a particle may be written if Lagrangian equals

$$\frac{1}{2}\,h_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\ .$$

Consequently, both in the relativistic case and in the nonrelativistic case

$$\breve{\Gamma}^\alpha_{\mu\nu}=h^\alpha_{\mu\nu},\quad \breve{\Gamma}^\alpha_{\mu 4}=0,\quad \breve{\Gamma}^\alpha_{4\nu}=0,\quad \breve{\Gamma}^\alpha_{44}=0,\quad \breve{\Gamma}^4_{mn}=0.$$

Accordingly, both in the relativistic and in the nonrelativistic case the background Ricci tensor equals

$$\breve{R}_{\mu\nu}=r_{\mu\nu},\quad \breve{R}_{4n}=0,\quad \breve{R}_{m4}=0.$$

Further we consider a star at rest with its planet. As coordinates x^1, $x^2 < x^3$ we choose the distance ρ from the star, the polar angle θ and the azimuth ϕ on a sphere $\rho=\mathrm{const}$; the notation $x^4=t$ we will preserve. With such a restriction we must solve the equations

$$S_{mn}=0.$$

2. THE NEWTON'S MODEL: THE CASE $(k=\infty,\ c=\infty)$

According to Newton, the equations of motion for a planet are:

$$\frac{d^2\rho}{d\tau^2}-\rho\frac{d\theta}{d\tau}\frac{d\theta}{d\tau}-\rho\sin^2\theta\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}+\frac{\gamma M}{\rho^2}\frac{dt}{d\tau}\frac{dt}{d\tau}=0,$$

$$\frac{d^2\theta}{d\tau^2}+\frac{2}{\rho}\frac{d\rho}{d\tau}\frac{d\theta}{d\tau}-\sin\theta\cos\theta\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}=0,$$

$$\frac{d^2\phi}{d\tau^2}+\frac{2}{\rho}\frac{d\rho}{d\tau}\frac{d\phi}{d\tau}-2\cot\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}=0,\qquad \frac{d^2t}{d\tau^2}=0.$$

Here γ is the Newton's constant, M is a mass of the star.

From here we find the gravitational connection Γ^a_{mn} in the Newton's case. If $M=0$, it coincides with the background connection. In this case all components of the affine deformation tensor equal zero except

$$P^1_{44}=-\,\frac{\gamma M}{\rho^2},$$

which equals the force, with which the star attracts the planet's unit mass.

It is remarkable that the Newton's gravitational connection with an arbitrary constant γM is an exact solution of Einstein equations

$$R_{mn}=0.$$

3. THE LOBACHEVSKY MODEL: THE CASE $(k < \infty,\ c = \infty)$

In the Lobachevsky geometry the length of a circle of radius ρ equals $2\pi r$, and the area of a sphere of the same radius equals $4\pi r^2$, where $r = k \sinh \frac{\rho}{k}$. Because of it Lobachevsky has shown [1, c. 159] that in the new model the force, with which the star attracts the planet's unit mass should equal

$$P^1_{44} = -\frac{\gamma M}{r^2}.$$

The rest components of tensor P^a_{mn} must be equal to zero. The force of attraction in the Lobachevsky model has potential [2], which equals

$$U = \frac{\gamma M}{k}\left(1 - \coth \frac{\rho}{k}\right).$$

In order to find the background connection in the Lobachevsky model we must write down the equations of motion for a particle in the case when the Lagrangian equals

$$\frac{1}{2}\frac{d\rho}{dt}\frac{d\rho}{dt} + \frac{1}{2}\, r^2\, \frac{d\theta}{dt}\frac{d\theta}{dt} + \frac{1}{2}\, r^2\, \sin^2\theta\, \frac{d\phi}{dt}\frac{d\phi}{dt}.$$

From these equations we receive

$$\begin{aligned} \breve{\Gamma}^1_{22} &= -k \sinh \frac{\rho}{k}\, \cosh \frac{\rho}{k}, \quad && \breve{\Gamma}^1_{33} = \breve{\Gamma}^1_{22} \sin^2\theta, \\ \breve{\Gamma}^2_{12} &= k^{-1} \coth \frac{\rho}{k} = \breve{\Gamma}^2_{21}, \quad && \breve{\Gamma}^2_{33} = -\sin\theta\cos\theta, \\ \breve{\Gamma}^3_{13} &= k^{-1} \coth \frac{\rho}{k} = \breve{\Gamma}^3_{31}, \quad && \breve{\Gamma}^3_{23} = \cot\theta = \breve{\Gamma}^3_{32}, \end{aligned}$$

the remaining components $\breve{\Gamma}^a_{mn}$ being equal to zero.

The background Ricci tensor in the coordinates ρ, θ, ϕ, t is a diagonal one. Its diagonal elements are

$$\breve{R}_{11} = -2k^{-2},$$

$$\breve{R}_{22} = -2k^{-2} r^2,$$

$$\breve{R}_{33} = -2k^{-2} r^2 \sin^2\theta,$$

$$\breve{R}_{44} = 0.$$

The Lobachevsky gravitational connection with an arbitrary constant γM is an exact solution of the equations $R_{\mu\nu} = -2k^{-2} h_{\mu\nu}, R_{m4} = R_{4m} = 0$.

4. THE EINSTEIN–SCHWARZSCHILD–FOCK MODEL: THE CASE $(k = \infty,\ c < \infty)$

Einstein began the construction of this model, and it was continued by Schwarzschild, and completed by Fock, who insisted on the application of the harmonicity condition. The gravitational metrics in this case equals

$$\left(\frac{\rho+\alpha}{\rho-\alpha}\right) d\rho^2 + (\rho+\alpha)^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \left(\frac{\rho-\alpha}{\rho+\alpha}\right) c^2 dt^2,$$

where $\alpha = \gamma M c^{-2}$ is the gravitational radius of mass M. This metrics satisfies the equation $R_{mn} = 0$. (See [7, c.263]).

In the case of the static spherical symmetric metrics

$$g_{mn} dx^m dx^n = F^2 d\rho^2 + H^2(d\theta^2 + \sin^2\theta d\phi^2) - V^2 dt^2$$

the components $\Phi^2,\ \Phi^3,\ \Phi^4$ of anharmonicity vector equal zero.

In regard to the radial component Φ^1, it depends on the choice of the background connection. In the considered case it equals

$$\Phi^1 = \frac{1}{VFH^2}\ \left[\frac{d}{d\rho}(F^{-1}\,V\,H^2) - 2\,F\,V\rho\right]\ .$$

As a consequence of the Fock harmonicity condition

$$\frac{d}{d\rho}(F^{-1}\,V\,H^2) - 2\,F\,V\,\rho = 0\ ,$$

the radial component Φ^1 equals zero. But the Fock condition does not follow from the Einstein's equations.

5. THE GENERAL CASE $(k < \infty,\ c < \infty)$

I had considered the general case on the occasion of the 200th anniversary of Lobachevsky's birthday. In [4] I have given the following solution of the new equations of gravity:

$$R_{11} = -\frac{2}{k^2}\ ,\ R_{22} = -2\,\sinh^2\frac{\rho}{k},\ R_{33} = -2\sinh^2\frac{\rho}{k}\sin^2\theta,\ R_{44} = 0,$$

$$R_{mn} = 0\ , \quad \text{if} \quad m \neq n\ .$$

According to [4], in the case $(k < \infty,\ c < \infty)$ the gravitational metrics equals

$$k^2 e^{-2\beta}[\Xi^{-1}\, d\xi^2 + \sinh^2(\xi+\beta)(d\theta^2 + \sin^2\theta\, d\phi^2)] - c^2 e^{2\beta}\Xi\, dt^2,$$

where

$$\Xi = \frac{\sinh(\xi - \beta)}{\sinh(\xi + \beta)}, \quad \xi = \frac{\rho}{k}\,, \quad \frac{1}{2}\sinh 2\beta = \frac{\gamma M}{kc^2}$$

(See [8] for details).

In this case the background connection is a harmonic one unconditionally. Indeed, like in the previous case we have Φ^2, Φ^3, Φ^4 being equal to zero. But from new equations of gravity the theorem follows

$$\Phi^m \breve{R}_{mn} + \frac{1}{2}\, g^{am}(\breve{\nabla}_a \breve{R}_{mn} + \breve{\nabla}_m \breve{R}_{an} - \breve{\nabla}_n \breve{R}_{am}) = 0\,.$$

According to this theorem, in the given case we have $\Phi^1(-2k^{-2}) = 0$. If $k < \infty$, from the last equality follows the equality $\Phi^1 = 0$.

It is interesting that in the given case

$$\Phi^1 = \frac{1}{V\,F\,H^2}\,\left[\frac{d}{d\rho}(F^{-1}\,V\,H^2) - F\,V\,k\sinh\frac{2\rho}{k}\right]\,.$$

Consequently

$$\frac{d}{d\rho}(F^{-1}V\,H^2) - FVk\sinh\frac{2\rho}{k} = 0\,.$$

In the limit $k \to \infty$ this equality makes a transition to the Fock condition.

REFERENCES

1. **Lobachevsky N.I.** — New Foundations of Geometry with a Complete Theory of Parallels (1835-1838). In: Complete Collected Works, v.2 [in Russian]. Gostekhizdat, Moscow-Leningrad, 1949, p.159.
2. **Chernikov N.A.** — Introduction of Lobachevsky Geometry into the Theory of Gravitation. Fis. Elem. Chastits At. Yadra, 1992, v.23[5], p.1155-1191.
3. **Bogoliubov N.N.** — Quasi-Average Quantities in the Problems of Statistical Mechanics. In: Selected Works, 1971, v.3 [in Russian]. Naukova Dumka, Kiev, p.174.
4. **Chernikov N.A.** — Lobachevsky Geometry and the Modern Theory of Gravitation. Russian Mathematics (Iz. VUZ), 1994, v.38, No.2, pp.50-64.
5. **Manoff S.** — Spaces with Contravariant and Covariant Affine Connections and Metrics. Fis. Elem. Chastits At. Yadra., 1999, v.30[5], p.1211-1267.
6. **Chernikov N.A.** — Lobachevsky Geometry and Relativistic Mechanics. Fis. Elem. Chastits At. Yadra, 1973, v.4[3], p.772-810.
7. **Fock V.A.** — The Theory of Space, Time and Gravitation [in Russian]. M.: Gostekhizdat, 1955.
8. **Chernikov N.A.** — The Gravitational Field of a Star in the Lobachevsky Space. Gravitation & Cosmology, 1996, v.2, No.3 (7), pp. 179-188. Russian Gravitational Society.