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VORTEX INSTABILITY OF THE CONTINUOUS MEDIUM MOVEMENTS

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Hydrodynamics equations are rewritten for the velocity deformation tensor $D_{ik} = (\partial v_i/\partial x_k + \partial v_k/\partial x_i)/2$ and for vorticity ($\Omega = \mathrm{curl}\ \mathbf{v}$) as observation variables. It ensues from the new written equations that there takes place a pendulum-like process of the energetic conversion between D_{ik} and Ω_i and intermittence of the two structures: tube-like and sheet-like («pancake»-like). It is formulated at the end the analogy of the Bernulli theorem for the vorticity case: the pressure in the area with developed vorticity becomes lower in the average.

It is well known that Halileo invariance dictates the existence of the term $(\mathbf{v}\nabla)\mathbf{v}$ in the continuous medium equations [1]. And vice versa, term $(\mathbf{v}\nabla)\mathbf{v}$ guarantees the Halileo invariance of the movements equations [1,2]. On the other hand, term $(\mathbf{v}\nabla)\mathbf{v}$ is connected closely with the medium vortex behavior due to the identity $(\mathbf{v}\nabla)\mathbf{v} = (1/2)\nabla v^2 - [\mathbf{v}\Omega]$, ($\Omega = \mathrm{curl}\ \mathbf{v}$). It means that the medium vortex behavior and Halileo invariance are mutually interconnected, i.e., the vorticity of the continuous medium is the geometrical property to some extent. Let us study the vorticity and its behavior for the hydrodynamics case.

It is convenient to use the generalized Helmholz equation [2] to study the vortex solutions of the Euler or Navie–Stokes hydrodynamic equations:

$$\frac{\partial \Omega_i}{\partial t} + (\mathbf{v}\nabla)\Omega_i = \frac{1}{2}(\partial v_i/\partial x_k + \partial v_k/\partial x_i)\Omega_k - \nu\ \mathrm{curl}\ \mathrm{curl}\ \Omega, \qquad \mathrm{div}\ \mathbf{v} = 0. \tag{1}$$

If we can find some solution for the equations system (1) we can then find the pressure p, as we have an expression for ∇p. The system (1) is a compatibility condition for the initial hydrodynamical equations then.

The system (1) shows that the deformation and vorticity of the fluid are interconnected [3,4]. It is easier to see this if we multiply the system (1) by Ω:

$$\frac{\partial \Omega^2}{\partial t} = 2\Omega_i D_{ik}\Omega_k - \mathrm{div}(\Omega^2\mathbf{v} - 2\nu\ [\Omega\ \mathrm{curl}\ \Omega]) - 2\nu\ (\mathrm{curl}\ \Omega)^2, \qquad \mathrm{div}\ \mathbf{v} = 0. \tag{2}$$

If the vector Ω is parallel to the eigenvector of the velocity deformation matrix $D_{ik} = (\partial v_i/\partial x_k + \partial v_k/\partial x_i)/2$ corresponding to a positive (negative) eigenvalue, then the first term on the right-hand side of (2) will be positive (negative). Then the local vorticity $|\Omega|$ of the fluid will increase (decrease) if the deformation rate is sufficiently large (large eigenvalues) in order that the first term on the right-hand side of (2) dominates the second and third ones together. A positive eigenvalue exists always due to the assumption of fluid incompressibility: div $\mathbf{v} = 0$, i.e., sum of all eigenvalues is equal to zero. Consequently every enough intensive fluid flow selforganizes so that the vorticity $|\Omega|$ increases due to the deformation motion, determined by the symmetrical tensor D_{ik}, and consequently the spontaneous symmetry distortion takes place as a consequence of the vortex instability process when antisymmetrical part of the tensor $\partial v_i/\partial x_k$ (the vorticity) responds to the symmetrical one behavior. So, the first term on the right hand side of (1) or (2) describes the generation (distortion) of the vorticity $|\Omega|$ [3–5].

The second (divergent) term on the right-hand side of (2) describes the transfer of vorticity in (out of) the point under consideration from (into) the adjoined volume. If we consider the confined volume, then the integral of the second divergent term can be converted into a surface integral after integrating (2) over the volume. Due to zero boundary conditions for $\mathbf{v}$: $\mathbf{v}|_s = 0$ only surface integral remains: $2\nu \oint [\Omega \text{ curl } \Omega]_n dS_n$ and it will affect the integral growth of the vorticity as a result of the vorticity transferring from the volume boundary.

So, abstracting from an irreversible attenuation of the vorticity, described by the third term on the right-hand side of (2), we can represent the following scenario of the growth of the integral of vorticity in the volume studied: the vorticity in the volume increases due to the vorticity transfer from the boundary and due to the actual increasing of the vorticity directed along an eigenvector of the matrix D_{ik} corresponding to a positive eigenvalue. The illustration of both the growth mechanisms is presented in [3, 4] for some examples of the hydrodynamical equations exact solutions.

Here we would like to consider a simple exact solution of the hydrodynamical equations and to follow the actual vorticity growth phenomenon without the influence of the transfer process. Let us consider the following three-dimensional solution [6]:

$$\begin{aligned} v_x &= a_{11}(x - x_0) - a_{12}(y - y_0) + a_{13}(z - z_0), \\ v_y &= a_{12}(x - x_0) + a_{22}(y - y_0) - a_{23}(z - z_0), \\ v_z &= -a_{13}(x - x_0) + a_{23}(y - y_0) + a_{33}(z - z_0). \end{aligned} \tag{3}$$

Here $a_{ik}(t)$ are functions of time. The vorticity vector $\Omega = \text{curl } \mathbf{v}$ depends only upon the time: $\Omega = 2(a_{23}, a_{13}, a_{12})$, i.e., $a_{ik} \quad (i \neq k)$ are components of the axial vorticity vector curl $\mathbf{v}$ relative to the principal axes and the matrix D_{ik} has a diagonal form: $D_{ik} = 0$ at $i \neq k$, and $D_{11} = a_{11}$, $D_{22} = a_{22}$, $D_{33} = a_{33}$. Consequently a_{11}, a_{22}, a_{33} are eigenvalues and represent the principal extension–contraction of the fluid flow. Due to the assumption of incompressibility:

$a_{11} + a_{22} + a_{33} = 0$. The expression (3) for v_i is the solution of (1) if a_{ik} satisfies the system of the ordinary differential equations:

$$\mathrm{d}a_{23}/\mathrm{d}t = a_{11}a_{23}, \qquad \mathrm{d}a_{13}/\mathrm{d}t = a_{22}a_{13}, \qquad \mathrm{d}a_{12}/\mathrm{d}t = a_{33}a_{12}. \tag{4}$$

It is interesting to note that the time behavior of the vorticity component along any axis depends upon the extension (contraction) along the same axis. All three components satisfy the condition $a_{12}a_{13}a_{23} = \mathrm{const}$ that follows from (4) and div $\mathbf{v} = 0$. Consequently our solution depends on two arbitrary functions of time. It is possible to combine the results (3),(4) as follows:

$$\begin{aligned} v_x &= (\tau_1'/\tau_1)(x-x_0) - \tau_3(y-y_0) + \tau_2(z-z_0) \\ v_y &= \tau_3(x-x_0) + (\tau_2'/\tau_2)(y-y_0) - \tau_1(z-z_0) \qquad \tau_1(t)\tau_2(t)\tau_3(t) = \mathrm{const}, \\ v_z &= -\tau_2(x-x_0) + \tau_1(y-y_0) + (\tau_3'/\tau_3)(z-z_0) \end{aligned} \tag{5}$$

where τ_i' denotes derivative of the time function $\tau_i(t)$: $\mathrm{d}\tau_i/\mathrm{d}t = \tau_i'$.

The solutions (3), (4) describe the increasing of the vorticity vector components directed along eigenvectors (one or two) corresponding to positive eigenvalues and decreasing in the other direction. It means that the vorticity vector turns with time towards the extension directions, whereas along contraction directions the corresponding components are diminishing to zero. The solutions (3) and (4), or (5) demonstrate the important intercorrelation of the deformational and vorticity parts of the flow: the vorticity responds to the deformational structure. The hydrodynamical equations connect two, generally speaking, independent parts of the tensor $\partial v_i/\partial x_k$, i.e., symmetrical (six components) part and antisymmetrical (three components) one. Moreover the presented solution is nonstationary: the vorticity depends on time exponentially even for the constant principal compression–extension. It is important that the solution (5) is essentially three-dimensional despite its simplicity.

It is possible to consider the solution in the form of (3) as a decomposition of every hydrodynamical solution in the vicinity of any point x_0, y_0, z_0 and the rotation of coordinate system up to principal axis. Then we can state: at every point the small vorticity disturbance grows in accordance with (4), as there is always one or two positive from the set a_{11}, a_{22}, a_{33} due to $a_{11} + a_{22} + a_{33} = 0$ (incompressibility). This means that the deformational flow of incompressible ideal fluids is unstable everywhere relative to the vorticity disturbances if the external force is represented by a gradient of some function.

The essential nonstationarity of the vortex behavior can be an important moment for the turbulence problem solution in case of continuous medium movements. But it is necessarily to study not only the vortex dependance of deformation but the influence of the vorticity on the deformation of the flow. Corresponding equations for D_{ik} can be obtained from Navie–Stoks equations:

$$\frac{\partial D_{ik}}{\partial t} + (\mathbf{v}\nabla)D_{ik} = -D_{in}D_{nk} + \frac{1}{4}\Omega^2\delta_{ik} - \frac{1}{4}\Omega_i\Omega_k - \frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_k} + \nu\frac{\partial}{\partial x_n}(\frac{\partial D_{in}}{\partial x_k} + \frac{\partial D_{kn}}{\partial x_i}) + \frac{\partial^2 \phi}{\partial x_i \partial x_k}, \qquad D_{kk} = 0, \tag{6}$$

where ϕ is a potential of the external force $\nabla\phi$. The equations (1) and (6) define mutual influence of the deformation D_{ik} and vortex Ω_i. In accordance with the Halileo principle we observe the gradients of velocity (but not velocity itself), i.e., nine values $\partial v_i/\partial x_k$. Tensor $\partial v_i/\partial x_k$ consists of symmetrical part D_{ik} (six components) and antisymmetrical part Ω_i (three components): $\partial v_i/\partial x_k = (\partial v_i/\partial x_k + \partial v_k/\partial x_i)/2 + (\partial v_i/\partial x_k - \partial v_k/\partial x_i)/2$. In the end, the formal conservation laws (1), (6) connect both parts D_{ik} and Ω_i so, that observable variables D_{ik} and Ω_i are not independent. For the better understanding of the hydrodynamical flow main characteristics it is convenient to have equation for $D^2 = D_{ik}D_{ik}$ multiplying (6) by D_{ik}:

$$4\frac{\partial D^2}{\partial t} = -2\Omega_i D_{ik}\Omega_k - 8D^3 - 16\nu(\frac{\partial D_{ik}}{\partial x_k})(\frac{\partial D_{in}}{\partial x_n}) + \frac{8}{\rho}\frac{\partial}{\partial x_i}[(\frac{\partial v_i}{\partial x_k})\frac{\partial(\rho\phi - p)}{\partial x_k}) + \frac{\rho}{2}v_i D^2 + 2\rho\nu D_{ik}(\frac{\partial D_{kn}}{\partial x_n})], \tag{7}$$

where $D^3 = D_{in}D_{nk}D_{ki}$. The first terms on the right-hands in (2) and (7) have the same absolute value but different signs. Consequently they are describing the channel of the energy exchange between vortex and deformation. The described above phenomenon of the local vortex instability everywhere is a stage of the general pendulum-like exchange process. **The nonlinear pendulum-like process of the energy exchange between vortex and deformations means substantial nonstationarity of the ideal incompressible fluid with chaotic elements of the behavior.** The role of the term D^3 could be clarified after volume integrating of the equation (7) and taking into account zero boundary conditions. D^2 and D^3 are invariants (scalars) and we can therefore express D^2 and D^3 via principal values of the fluid flow extension–contraction. Then:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\overline{\lambda_1^2} + \overline{\lambda_2^2} + \overline{\lambda_3^2} + \overline{\omega^2}/4) = -6\overline{\lambda_1\lambda_2\lambda_3} - 4\nu\left[\overline{\left(\frac{\partial D_{ik}}{\partial x_k}\right)\left(\frac{\partial D_{in}}{\partial x_n}\right)} + \frac{\overline{(\mathrm{curl}\Omega)^2}}{8}\right], \tag{8}$$

where the line above symbols means volume integrating. The value $\overline{\lambda_1\lambda_2\lambda_3}$ depends on the two characteristic elements population of the deform–vortex medium behavior. First element has one positive eigenvalue and two negative ones ($\lambda_1\lambda_2\lambda_3 > 0$) and gets the name tube-like vortex structure, and the second one under the name sheet-like structure or «pancake» has one negative eigenvalue and two positive ones ($\lambda_1\lambda_2\lambda_3 < 0$). Abstracting from dissipation we can conclude: the behavior of the continuous media is organized so that both characteristic elements should alternate or intermit each other in space and time in such a way to convert $\overline{\lambda_1\lambda_2\lambda_3}$ into zero for the stationary case (for example the case of the isotropic turbulence). Consequently, the approach with observable variables D_{ik} and Ω_i reveals the fundamental intermittence low: **under the intermittence condition there takes place the conservation of the sum of the deformation square and vortex square.** This new intermittence low is very important for the theory of the developed, stationary, homogeneous turbulence. It brings the elements of the organized chaos together with the pendulum-like process into the continuous medium behavior.

It is useful to connect pressure p with D^2 and Ω^2. We can get such connection putting in (6) $i = k$, summing up and taking into account $D_{kk} = 0$. Then:

$$(\Omega^2/2) - D^2 = \Delta p. \tag{9}$$

Equation (9) is Laplace equation with volume sources presented by difference between vortex square and deformation square. Volume integration of (9) gives:

$$\overline{\Omega^2}/2 - \overline{D^2} = \oint \nabla_i p dS_i, \tag{10}$$

i.e., the average value of the pressure gradient on the surface, surrounding the considered volume, is equal to the difference of the integral vortex square and the integral deformation square. Then we can state: **the pressure averaged value in the volume becomes lower relative to the pressure averaged value on the boundary surrounding volume if the vorticity is enough intensive relative to deformation so that $\overline{\Omega^2}/2 > \overline{D^2}$.** This statement is an analogy of the Bernulli theorem.

It is interesting to give as an example two important applied consequences of the studied instabilities. First, there is a mathematical analogy between an equation for vorticity Ω and an equation for magnetic field in magnetohydrodynamics. The result concerning local vortex instability, everywhere we have studied above, can be reformulated in the magnetohydrodynamics as follow: magnetohydrodynamic flows are locally unstable everywhere. Such conclusion could follow from paper [9], too. Consequently it is possible to say that **the stationary magnetic restraint of plasma is impossible.** Second, it is supposed, that the movement of the interstellar or intergalactic medium is described by hydrodynamical equations [10]. Consequently we could appropriately apply our results to the study of the such medium behaviors. We can see the vortex structure everywhere in the Universe: for example spiral galaxies. And the streamlines are, generally speaking, spirals as is possible to see from characteristic equations $\mathrm{d}x_i/\mathrm{d}t = v_i$ with special choice of parameters a_{ik} in (3), for example. The questions appear in connection with the incompressibility condition, which is not valid, generally speaking, for cosmical gas medium. But the study of the incompressible behavior of the gas medium will be correct if the relative movements in this medium have velocities smaller than sound speed. And such situation takes place [10] for the problem of the galaxies or galactic system vortex instability. Let us consider now the very important and very intriguing cosmological problem of dark matter using our result concerning the total instability of the hydrodynamical flow.

The dark-matter problem we speak about originates under the interpretation of the objects motion on the boundary of large-scale gravitating formations such as galaxies or galactic systems [10,11]. The velocity of this objects is considered as an orbital motion due to the attractive force influence of the such formations gravity mass. It means that we assume stationary condition. Then the attracting mass, evaluated at observed velocities, is ten or more times larger than the visible mass [10, 11]. In this case people are speaking about dark matter, i.e., about latent, invisible, nonbaryon mass that is not observed yet. But if we consider the velocity of the boundary objects as a result of the vortex instability development due to the deformation of the gravitating formation, then the dark matter problem may be dismissed and considered as artifact of the stationary hypothesis. For example, the round motion of the air particles in tornado is a result of the vortex instability development but not due to the existence of the gravitating mass in the middle of tornado. Indeed the Navie–Stokes equations describe the deformational flow of the cosmic cloud due to the gravitational influence of the neighboring mass through $\nabla\phi$ in the right-hand side of the equations. The compatibility system (1) has the nonstationary vortex solutions as we have shown, i.e., a vorticity will develop due to the deformational collective flow of the cosmic medium. It means that the gravitational influence on the vortex behavior takes place through the process of deformation (the gravitational force $\nabla\phi$ disappears in the compatibility equations (1)). Consequently the large-scale movements of the interstellar or intergalactic medium are principally nonstationary when we can neglect the medium viscosity. So the dark matter paradox means indeed the nonstationarity and vortex instability of the large-scale cosmic formation flow under the indirect influence of the gravity. Such resolution of the paradox could give an evaluation of the instability rate.

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