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MAGNETIC TRANSPORT ALONG ONE-DIMENSIONAL PERTURBATIONS IN THE PLANE

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We review several recent results concerning two-dimensional systems which exhibit a transport induced by a «one-dimensional» perturbation of a homogeneous magnetic field. The first concerns the «local» Iwatsuka model, where a charged particle interacts with a field which is homogeneous outside a finite strip and translationally invariant along it: we present two new sufficient conditions for absolute continuity of the spectrum and show that in most cases the number of open spectral gaps is finite. In the second model the perturbation is a periodic array of point obstacles. In this case the Landau levels remain to be infinitely degenerate eigenvalues, and between them the system has bands of absolutely continuous spectrum.

1. INTRODUCTION

The purpose of this talk is to present results of two recent papers [7, 8] investigating transport of a charged particle in the plane due to a perturbation of a homogeneous magnetic field. We give a short overview referring to the said papers for more details, proofs, references, as well as for numerical analysis of examples.

Magnetic transport and the edge states are since the eighties a standard object of solid-state physics [11–13]. Recently the subject attracted new «theoretical» interest: it was shown that current-carrying states in a halfplane or a more general domain survive a weak disorder [4, 9, 10, 14] and new sufficient conditions were found for existence of transport induced by a variation of the field alone [15].

Our aim here is to contribute to this development in two directions. First we shall present a pair of new sufficient conditions for the absolute continuity in the Iwatsuka model. Of them the second one is important being a rather weak *local* requirement; this represents a step towards the proof of a conjecture put forth in [5, Sec. 6.5] which states that *any* nonzero (translationally invariant) field variation spreads the Landau levels into a purely absolutely continuous spectrum. In addition, we shall show that the number of open gaps is finite here provided the field variation has a nonzero mean and we conjecture that this claim holds generally.

The second topic to address are the edge states: we shall show that they can exist even if there is no edge. This is illustrated by a simple model in which the magnetic transport is a purely quantum effect (in the sense that a quantum particle propagates while its classical counterpart moves on localized circular trajectories, apart of a zero-measure family of the initial conditions): a charged quantum particle in the plane exposed to a homogeneous magnetic field and interacting with a periodic array of point obstacles described by δ potentials.

2. LOCAL IWATSUKA MODEL

Consider a two-dimensional charged particle interacting with a magnetic field perpendicular to the plane. We assume that field is translationally invariant in the y-direction, nonzero and constant away of a strip of a width $2a$:

(a) the functional form of the field is $B(x,y) = B(x) = B + b(x)$, where $B > 0$ and b is bounded and piecewise continuous with $\operatorname{supp} b = [-a, a]$. With an abuse of notation, we employ the same symbol for functions on $\mathbb{R}$ and R^2 if they are independent of one variable.

We use the Landau gauge, $A_x = 0$, $A_y(x) = Bx + a(x)$, with $a(x) := \int_0^x b(t)\,\mathrm{d}t$. We also adopt the natural system of units, $2m = \hbar = c = |e| = 1$; then the Hamiltonian $H \equiv H(B,b)$ of our system is

$$H = (\mathbf{p} + \mathbf{A})^2$$

with the appropriate domain in $L^2(\mathbb{R}^2)$. Since it commutes with y-translations, it allows for a standard decomposition [12, Sec. 2] being unitarily equivalent $\int^{\oplus} H(p)\mathrm{d}p$ with the fiber space $L^2(\mathbb{R})$ and fiber operator

$$H(p) = -\partial_x^2 + (p + xB + a(x))^2. \tag{1}$$

The function a is bounded, so the spectrum of $H(p)$ is purely discrete and consists of a sequence of eigenvalues $\epsilon_n(p)$. In the absence of the perturbation b they are the Landau levels, $\{(2n+1)B : n \in \mathbb{N}_0\}$. In the perturbed case they belong to the spectrum too, at least as its accumulation points.

Lemma 2.1 *$\epsilon_n(p) \to (2n+1)B$ as $|p| \to \infty$ for any $n \in \mathbb{N}_0$.*

To proceed, let us observe first the following analyticity property.

Lemma 2.2 *$\{H(p) : p \in \mathbb{R} \cup \{\infty\}\}$ is an analytic family of type (A). In particular, each $\epsilon_n(\cdot)$ is an analytic function.*

Let $\psi_n(\cdot, p)$ be the eigenfunctions of (1), i.e., $H(p)\psi_n(x,p) = \epsilon_n(p)\psi_n(x,p)$, and denote $f_n(x,p) := (p+xB+a(x))\psi_n(x,p)^2$. Using then a standard semiclassical technique [16], we can derive the following estimates.

Lemma 2.3 *For any p large enough there is $c(p) > 0$ such that*

$$5c(p)\,\mathrm{e}^{-p(x-x_0)} \geq f_n(x,p) \geq \frac{c(p)}{7}\,\mathrm{e}^{-3p(x-x_0)}$$

holds for all $-a \leq x_0 \leq x \leq a$.

With these preliminaries, we were able to prove in [8] the desired result under one of the following additional assumptions:

(b) $b(\cdot)$ is nonzero and does not change sign in $[-a, a]$,

(c) let $a_\ell < a_r$, where we have put $a_\ell := \sup\{x : b(x) = 0 \text{ in } (-\infty, x)\}$ and $a_r := \inf\{x : b(x) = 0 \text{ in } (x, \infty)\}$. There exist $c_0, \delta > 0$ and $m \in \mathbb{N}$ such that one of the following conditions holds:

$$|b(x)| \geq c_0(x - a_\ell)^m \qquad \text{for} \quad x \in [a_\ell, a_\ell + \delta),$$
$$|b(x)| \geq c_0(a_r - x)^m \qquad \text{for} \quad x \in (a_r - \delta, a_r].$$

Theorem 2.4 *Assume (a) and (b), or (a) and (c); then $|\epsilon_n'(p)| > 0$ for each $n \in \mathbb{N}_0$ and all $|p|$ large enough. In particular, the spectrum of H is absolutely continuous.*

We also want to know how the spectrum of H looks like as a set. It follows from direct-integral decomposition that $\sigma(H)$ consists of a union of spectral bands I_n:

$$I_n = \left[\, \inf_{p \in \mathbb{R}} \epsilon_n(p),\ \sup_{p \in \mathbb{R}} \epsilon_n(p) \,\right];$$

the question is how many gaps between them remain open. We shall distinguish two cases depending on whether the functional $A[b] := \int_{-a}^{a} b(x)\,\mathrm{d}x$ vanishes or not. In the latter situation the Bohr–Sommerfeld quantization condition yields:

Proposition 2.5 *Assume $\int_{-a}^{a} b(x)\,\mathrm{d}x \neq 0$. Let $n(E,p)$ and $n_0(E)$ be the numbers of eigenstates of $H(p)$ and H_0, respectively, with the eigenenergy smaller than E. Then for any $m \in \mathbb{N}_0$ there exist p_0 and $E(m, p_0)$ such that*

$$(n_0(E) - n(E, p_0))\ \mathrm{sgn}\, A[b] > m$$

holds for all $E > E(m, p_0)$.

Corollary 2.6 *If $A[b] \neq 0$ the number of open gaps in the spectrum of H is finite.*

If $A[b] = 0$ the situation is more complicated since the perturbed and unperturbed potentials on (1) differ only in a subset of the interval $(-a, a)$. In [8] we gave an example showing that also in this case the number of open gaps may be finite, and conjectured that this is true generally. However, to prove it one obviously needs a more sophisticated technique.

3. ARRAY OF POINT PERTURBATION

Let us turn now to the second model mentioned in the introduction. The Hamiltonian can be formally written as

$$H_{\alpha,\ell} = (-i\partial_x + By)^2 - \partial_y^2 + \sum_j \tilde{\alpha}\delta(x - x_0 - j\ell)\,, \tag{2}$$

where $\ell > 0$ is the array spacing. To introduce the interaction term in a rigorous way, we follow the usual definition [1] which employs the boundary conditions

$$L_1(\psi, \vec{a}_j) + 2\pi\alpha L_0(\psi, \vec{a}_j) = 0\,, \quad j = 0, \pm1, \pm2, \ldots$$

with $\vec{a}_j := (x_0 + j\ell, 0)$, where L_k are the generalized boundary values

$$L_0(\psi, \vec{a}) := \lim_{|\vec{x}-\vec{a}|\to 0} \frac{\psi(\vec{x})}{\ln|\vec{x}-\vec{a}|}\,,\; L_1(\psi, \vec{a}) := \lim_{|\vec{x}-\vec{a}|\to 0} \Big[\psi(\vec{x}) - L_0(\psi, \vec{a})\, \ln|\vec{x}-\vec{a}|\Big]\,,$$

and α is the (rescaled) coupling constant; the free (Landau) Hamiltonian corresponds to $\alpha = \infty$. Using the periodicity, we can write the Bloch decomposition in the x direction, $H_{\alpha,\ell} = \frac{\ell}{2\pi} \int_{|\theta\ell|\le\pi}^{\oplus} H_{\alpha,\ell}(\theta)\, d\theta$, where the fiber operator $H_{\alpha,\ell}(\theta)$ is of the form (2) on the strip $0 \le x \le \ell$ with the boundary conditions

$$\partial_x^i \psi(\ell-, y) = e^{i\theta\ell} \partial_x^i \psi(0+, y)\,, \quad i = 0, 1\,,$$

and its Green's function is given by means of the Krein formula

$$\begin{aligned}(H_{\alpha,\ell}(\theta) - z)^{-1}(\vec{x}, \vec{x}') &= G_0(\vec{x}, \vec{x}'; \theta, z) \\ &\quad + (\alpha - \xi(\vec{a}_0; \theta, z))^{-1} G_0(\vec{x}, \vec{a}_0; \theta, z) G_0(\vec{a}_0, \vec{x}'; \theta, z)\,,\end{aligned}$$

where

$$\xi(\vec{a}; \theta, z) := \lim_{|\vec{x}-\vec{a}|\to 0} \left(G_0(\vec{a}, \vec{x}; \theta, z) - \frac{1}{2\pi} \ln|\vec{x}-\vec{a}| \right)$$

and G_0 is the free Green's function,

$$G_0(\vec{x}, \vec{x}'; \theta, z) = -\sum_{m=-\infty}^{\infty} \frac{u_m^\theta(y_<) v_m^\theta(y_>)}{W(u_m^\theta, v_m^\theta)}\, \eta_m^\theta(x) \overline{\eta_m^\theta}(x')\,,$$

where $\eta_m^\theta(x) = \frac{1}{\sqrt{\ell}}\,\mathrm{e}^{i(2\pi m+\theta\ell)x/\ell}$, m runs through integers, $y_<, y_>$ is the smaller and the larger value, respectively, of y, y', and u_m^θ, v_m^θ are solutions to the equation

$$-u''(y) + \left(By + \frac{2\pi m}{\ell} + \theta\right)^2 u(y) = zu(y)$$

such that u_m^θ is L^2 at $-\infty$ and v_m^θ is L^2 at $+\infty$; in the denominator we have their Wronskian. We have $u_m^\theta(y) = u\left(y + \frac{2\pi m+\theta\ell}{B\ell}\right)$ and the analogous relation for v_m^θ, where

$$\left\{ \begin{array}{c} u \\ v \end{array} \right\}(y) = \sqrt{\pi}\, e^{-By^2/2}\left[\frac{M\left(\frac{B-z}{4B}, \frac{1}{2}; By^2\right)}{\Gamma\left(\frac{3B-z}{4B}\right)} \pm 2\sqrt{B}y\,\frac{M\left(\frac{3B-z}{4B}, \frac{3}{2}; By^2\right)}{\Gamma\left(\frac{B-z}{4B}\right)}\right].$$

An explicit computation then leads to the formula

$$\begin{aligned} G_0(\vec{x}, \vec{x}'; \theta, z) = &-\frac{2^{(z/2B)-(3/2)}}{\sqrt{\pi B}\ell}\,\Gamma\left(\frac{B-z}{2B}\right)\mathrm{e}^{i\theta(x-x')} \\ &\times \sum_{m=-\infty}^{\infty} u\left(y_< + \frac{2\pi m+\theta\ell}{B\ell}\right) v\left(y_> + \frac{2\pi m+\theta\ell}{B\ell}\right)\mathrm{e}^{2\pi i m(x-x')/\ell}\,. \end{aligned}$$

As expected the function has singularities which are independent of θ and coincide with the Landau levels, $z_n = B(2n+1),\ n = 0, 1, 2, \dots$. Using an argument modified from [3, 6] one can check that these points are preserved in the spectrum of the «full» fiber operator $H_{\alpha,\ell}(\theta)$. On the other hand, $H_{\alpha,\ell}(\theta)$ has also eigenvalues away of z_n which we denote as $\epsilon_n(\theta) \equiv \epsilon_n^{(\alpha,\ell)}(\theta)$; they are given by the implicit equation

$$\alpha = \xi(\vec{a}_0; \theta, \epsilon) \tag{3}$$

and the corresponding eigenfunctions are

$$\psi_n^{(\alpha,\ell)}(\vec{x}; \theta) = G_0(\vec{x}, \vec{a}_0; \theta, \epsilon_n(\theta))\,. \tag{4}$$

The regularized Green's function appearing in (3) can be computed to be

$$\xi(\vec{x}; \theta, z) = \sum_{m=-\infty}^{\infty}\left\{\frac{1-\delta_{m,0}}{4\pi|m|} - \frac{2^{-2\zeta-1}}{\sqrt{\pi B}\ell}\,\Gamma(2\zeta)\,(uv)\left(y + \frac{2\pi m+\theta\ell}{B\ell}\right)\right\}, \tag{5}$$

where $\zeta := \frac{B-z}{4B}$. Spectral bands of the model are given by the ranges of the functions $\epsilon_n(\cdot)$. Solutions of the condition (3) do not cross the Landau levels, because $\xi(\vec{a}_0; \theta, \cdot)$ is increasing in the intervals $(-\infty, B)$ and $(B(2n-1), B(2n+1))$ and diverges at the endpoints. It is easy to see that $\xi(\vec{x}; \cdot, z)$ is real-analytic, hence the spectral bands will be absolutely continuous if the function is non-constant in the whole Brillouin zone $[-\pi/\ell, \pi/\ell)$. Using the explicit expression (5) together with properties of the Fourier transformation, we have arrived in [7] at the following conclusion:

Theorem 3.1 *For any real α the spectrum of $H_{\alpha,\ell}$ consists of the Landau levels $B(2n+1),\ n = 0, 1, 2, \ldots,$ and absolutely continuous spectral bands situated between adjacent Landau levels and below B.*

Let us mention for comparison that a chain of point scatterers in a three-dimensional space with a homogeneous magnetic field was discussed recently in [2]. Due to the higher dimensionality, the spectrum is purely a.c. in that case and has at most finitely many gaps.

The band function for different values of the parameters are computed in [7]. When α runs from $+\infty$ to $-\infty$ a band splits from each Landau level and moves down being finally absorbed by the neighbouring LL (with the exception of the lowest one). To characterize the transport associated with the bands, one can also use the probability current, $\vec{\jmath}_n(\vec{x};\theta) = 2\,\mathrm{Im}\left(\bar{\psi}_n^{(\alpha,\ell)}(\vec{\nabla} - i\vec{A})\psi_n^{(\alpha,\ell)}\right)(\vec{x};\theta)$, which is in general nonzero because the Bloch functions (4) are complex-valued. The current pattern changes with θ oscillating between a symmetric «two-way» picture and the situations where one direction clearly prevails; examples are worked out in [7]. They show in particular that the probability current may exhibit vortices in some regions.

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