BETHE-ANSATZ EQUATIONS FOR QUANTUM HEISENBERG CHAINS WITH ELLIPTIC EXCHANGE

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The eigenvectors of the Hamiltonian $\mathcal{H}_N$ of N-sites quantum spin chains with elliptic exchange are connected with the double Bloch meromorphic solutions of the quantum continuous elliptic Calogero-Moser problem. This fact allows one to find the eigenvectors via the solutions to the system of highly transcendental equations of Bethe-ansatz type which is presented in explicit form.

It is known [1] that for a one-parameter set $\mathcal{H}_N$ of linear combinations of $N(N-1)/2$ elementary transpositions $\{P_{jk}\}$, $\mathcal{H}_N = \frac{J}{2}\sum_{1 \leq j \neq k}^{N} \wp(j-k)P_{jk}$ at arbitrary natural $N \geq 3$, one can construct a variety $\{I_m\}$ $(3 \leq m \leq N)$ of operators which commute with $\mathcal{H}_N$. Being applied to $SU(2)$ spin representations of the permutation group, this proves the integrability of 1D periodic spin chains with elliptic short-range interaction and the Hamiltonian

$$\mathcal{H}^{(s)} = \frac{J}{4}\sum_{1 \leq j \neq k \leq N} h(j-k)(\vec{\sigma}_j\vec{\sigma}_k - 1), \tag{1}$$

where

$$h(j) = \left(\frac{\omega}{\pi}\sin\frac{\pi}{\omega}\right)^2\left[\wp_N(j) + \frac{2}{\omega}\zeta_N\left(\frac{\omega}{2}\right)\right], \tag{2}$$

where $\wp_N(x)$, $\zeta_N(x)$ are the Weierstrass functions defined on the torus $T_N = \mathbf{C}/\mathbf{Z}N + \mathbf{Z}\omega$, $\omega = i\alpha$, $\alpha \in \mathbf{R}_+$ is a free parameter.

The symmetry of two limiting cases of this one-parameter model, i.e., the Bethe lattice with nearest-neighbor interaction [2] $(\alpha \to 0)$ and long-range $\left(\frac{N}{\pi}\sin\frac{\pi j}{N}\right)^{-2}$ exchange [3] $(\alpha \to \infty)$, is now well understood, and regular procedures of finding eigenvectors are described in the literature [4–7]. At present, a number of impressive results are known for both these models. In particular, they include the additivity of the spectrum under proper choice of «rapidity» variables [2,3], the description of underlying symmetry [4,5], construction of thermodynamics in the limit $N \to \infty$ [9,10]. However, all that still cannot be applied to the general elliptic case.

In the paper [8], I have shown that there is a remarkable connection between the eigenvectors of the Hamiltonian of the above model with M down

spins and double Bloch meromorphic solutions to the quantum continuous elliptic Calogero-Moser problem at the special value of the coupling constant, i.e., the eigenfunctions of the *differential* operator

$$H = -\frac{1}{2}\sum_{j=1}^{M}\frac{\partial^2}{\partial x_j^2} + \sum_{j\neq k}^{M}\wp_N(x_j - x_k). \tag{3}$$

This allows one in principle to find an ansatz for the eigenvectors and even try to describe them completely if the solutions to (3) are known. This has been done in the simplest nontrivial case $M = 3$ in [11], where I have used the result for three-particle elliptic Calogero-Moser problem [12].

At that time, the explicit form of the eigenfunctions of (3) at $M > 3$ has not been known. The situation has been changed after publishing the seminal paper [13] where these eigenfunctions have been obtained in the process of constructing solutions to the elliptic Knizhnik–Zamolodchikov–Bernard equations. It has been a main motivation for this paper in which I shall describe the complete set of the Bethe-ansatz-type equations for the eigenvectors of (1) at *arbitrary* $M \leq N/2$.

The Hamiltonian (1) commutes with the operator of total spin $\vec{\mathbf{S}} = \frac{1}{2}\sum_{j=1}^{N}\vec{\sigma}_j$. Then the eigenproblem for it is decomposed into the problems in the subspaces formed by the common eigenvectors of $\mathbf{S}_3$ and $\vec{\mathbf{S}}^2$ such that $S = S_3 = N/2 - M$, $0 \leq M \leq [N/2]$,

$$\mathcal{H}^{(s)}|\psi^{(M)}> = E_M|\psi^{(M)}> . \tag{4}$$

The eigenvectors $|\psi^{(M)}>$ are written in the usual form

$$|\psi^{(M)}> = \sum_{n_1..n_M}\psi_M(n_1..n_M)\prod_{\beta=1}^{M}s_{n_\beta}^-|0>, \tag{5}$$

where $|0> = |\uparrow\uparrow ... \uparrow>$ is the ferromagnetic ground state with all spins up and the summation is taken over all combinations of integers $\{n\} \leq N$ such that $\prod_{\mu<\nu}^{M}(n_\mu - n_\nu) \neq 0$. The substitution of (5) into (4) results in the *lattice* Schrödinger equation for completely symmetric wave function ψ_M

$$\sum_{s\neq n_1,..n_M}\sum_{\beta=1}^{M}\wp_N(n_\beta - s)\psi_M(n_1,..n_{\beta-1},s,n_{\beta+1},..n_M)$$
$$+ \left[\sum_{\beta\neq\gamma}^{M}\wp_N(n_\beta - n_\gamma) - \mathcal{E}_M\right]\psi_M(n_1,..n_M) = 0. \tag{6}$$

The eigenvalues $\{E_M\}$ are given by

$$E_M = J\left(\frac{\omega}{\pi}\sin\frac{\pi}{\omega}\right)^2\left\{\mathcal{E}_M + \frac{2}{\omega}\left[\frac{2M(2M-1)-N}{4}\zeta_N\left(\frac{\omega}{2}\right) - M\zeta_1\left(\frac{\omega}{2}\right)\right]\right\}, \tag{7}$$

where $\zeta_1(x)$ is the Weierstrass zeta function defined on the torus $T_1 = \mathbf{C}/\mathbf{Z} + \mathbf{Z}\omega$.

The solutions to (6) can be found with the use of the following ansatz for ψ_M:

$$\psi_M(n_1, ..n_M) = \sum_{P \in \pi_M} \varphi_M^{(p)}(n_{P1}, ..n_{PM}), \tag{8}$$

$$\varphi_M^{(p)}(n_1, ..n_M) = \exp\left(-i \sum_{\nu=1}^{M} \tilde{p}_\nu n_\nu\right) \chi_M^{(p)}(n_1, ..n_M), \tag{9}$$

where

$$\tilde{p}_\nu = p_\nu - 2\pi N^{-1} l_\nu, \qquad l_\nu \in \mathbf{Z}, \tag{10}$$

π_M is the group of all permutations $\{P\}$ of the numbers from 1 to M and $\chi_M^{(p)}$ is some special solution to the *continuum* quantum many-particle problem

$$\left[-\frac{1}{2}\sum_{\beta=1}^{M} \frac{\partial^2}{\partial x_\beta^2} + \sum_{\beta \neq \lambda}^{M} \wp_N(x_\beta - x_\lambda) - \mathsf{E}_M(p)\right] \chi_M^{(p)}(x_1, ..x_M) = 0. \tag{11}$$

It is specified up to a normalization factor by the particle pseudomomenta $(p_1, ..p_M)$. The standard argumentation of the Floquet–Bloch theory shows that due to periodicity of the potential term in (49) $\chi_M^{(p)}$ obeys the quasi-periodicity conditions

$$\chi_M^{(p)}(x_1, ..x_\beta + N, ..x_M) = \exp(ip_\beta N)\chi_M^{(p)}(x_1, ..x_M), \tag{12}$$

$$\chi_M^{(p)}(x_1, ..x_\beta + \omega, ..x_M) = \exp(2\pi i q_\beta(p) + ip_\beta\omega)\chi_M^{(p)}(x_1, ..x_M), \quad 0 \leq \Re e(q_\beta) < 1, \tag{13}$$

$$1 \leq \beta \leq M.$$

The eigenvalue $\mathsf{E}_M(p)$ is some symmetric function of $(p_1, ..p_M)$. As will be seen later, the set $\{q_\beta(p)\}$ is also completely determined by $\{p\}$.

It turns out [8] that the equation (6) with the use of (8),(9) can be recast in the form

$$\sum_{P \in \pi_M} \left[-\frac{1}{2}\sum_{\beta=1}^{M} \left(\frac{\partial}{\partial n_{P\beta}} - f_\beta(p)\right)^2 + \sum_{\beta \neq \gamma}^{M} \wp_N(n_{P\beta} - n_{P\gamma}) - \right.$$

$$\left. \mathcal{E}_M + \sum_{\beta=1}^{M} \varepsilon_\beta(p)\right] \varphi^{(p)}(n_{P1}, ..n_{PM}) = 0, \tag{14}$$

where

$$f_\beta(p) = 2\tilde{q}_\beta(p)\zeta_1(1/2) - \zeta_1(\tilde{q}_\beta(p)), \tag{15}$$

$$\varepsilon_\beta(p) = \frac{1}{2}\wp_1(\tilde{q}_\beta(p)), \tag{16}$$

$$\tilde{q}_\beta(p) = q_\beta(p) + \frac{l_\beta}{N}\omega. \tag{17}$$

where $\wp_1(x), \zeta_1(x)$ are the Weierstrass functions defined on the torus $T_1 = \mathbf{C}/\mathbf{Z} + \mathbf{Z}\omega$.

Turning to the definition (9) of $\varphi^{(p)}$, one observes that each term of the left-hand side of (14) has the same structure as the left-hand side of the many-particle Schrödinger equation (11) and vanishes if $\mathcal{E}_M$ and $f_\beta(p)$ are chosen as

$$f_\beta(p) = -i\tilde{p}_\beta, \qquad \beta = 1, ..M, \tag{18}$$

$$\mathcal{E}_M = \mathsf{E}_M(p) + \sum_{\beta=1}^{M}\varepsilon_\beta(p). \tag{19}$$

One can see from (15-19) that it remains now to find the explicit dependence of $\{q\}$ and E_M on $\{p\}$. It can be done by using the results given in [13] where the explicit form of $\chi_M^{(p)}(x)$ has been indicated. In suitable notations, it reads

$$\chi_M^{(p)}(x) \sim \exp(i\sum_{\beta=1}^{M}p_\beta x_\beta)\sum_{s\in\pi_m}l(s)\prod_{j=1}^{m}\tilde{\sigma}_{\sum_{k=1}^{j}(x_{c(s(k))}-x_{c(s(k))+1})}(t_{s(j)}-t_{s(j+1)}), \tag{20}$$

where $m = M(M-1)/2$, c is nondecreasing function $c : \{1,..,m\} \to \{1,..,M-1\}$ such that $|c^{-1}\{j\}| = M-j$, $l(s)$ is an integer which is defined for the permutation s by the relation $x_{c(s(1))+1}\partial/\partial x_{c(s(1))}...x_{c(s(m))+1}\partial/\partial x_{c(s(m))}x_1^M = l(s)(x_1...x_M)$, $\{t\}$ is a set of m complex parameters obeying m relations [13]

$$\sum_{l:|c(l)-c(j)|=1}\rho(t_j-t_l)-2\sum_{l:l\neq j,c(l)=c(j)}\rho(t_j-t_l)+M\delta_{cj,1}\rho(t_j) = i(p_{c(j)}-p_{c(j)+1}), \tag{21}$$

$$\rho(t) = \zeta_N(t) - \frac{2}{N}\zeta_N(N/2)t,$$

and

$$\tilde{\sigma}_w(t) = \exp((2/N)\zeta_N(N/2)wt)\frac{\sigma_N(w-t)}{\sigma_N(w)\sigma_N(t)},$$

σ_N being the Weierstrass sigma function on $\mathbf{T}_N$. The elementary building blocks of the χ function obey the useful quasi-periodicity relations

$$\tilde{\sigma}_{w+N}(t) = \tilde{\sigma}_w(t), \qquad \tilde{\sigma}_{w+\omega}(t) = e^{2\pi it/N}\tilde{\sigma}_w(t). \tag{22}$$

One can see that in this construction the color function $c(j)$ is of crucial role. It is useful to write it explicitly. Namely, define for every $k=1,..M-1$ the segment S_k

$$\frac{(k-1)(2M-k)}{2}+1 \leq j \leq \frac{k(2M-k-1)}{2}. \tag{23}$$

Then some calculation shows that

$$c(j) = k \qquad \text{if} \quad j \in S_k. \tag{24}$$

The main advantage of the explicit form of χ function is that it allows one to find the second set of relations between the Bloch factors $\{p\}, \{q\}$. It is easy to see from (21) that $\{p\}'s$ in the definitions (12) and (20) are the same. The problem consists in calculation of $\{q\}$. To do this, it is not necessary to analyze each term in the sum over permutations in (20) since all of them must have the same Bloch factors. It is convenient to choose the term which corresponds to the permutation

$$s_0: \quad s_0(j) = m+1-j, \quad j = 1,..m.$$

After some algebra, one finds that this permutation gives nontrivial contribution to the sum (20) with $l(s_0) = M!(M-1)!...2!$. Moreover, with the use of explicit form of the color function (23-24) one finds

$$c(s_0(l)) = M-q \quad \text{if} \quad q(q-1)/2+1 \leq l \leq q(q+1)/2.$$

Now the problem of calculation of the second Bloch factors reduces, due to second relation (21), to some long and tedious, but in fact simple calculations of the product of factors which various $\tilde{\sigma}$ functions acquire under changing arguments of χ function to the quasi-period ω. The final result is surprisingly simple,

$$q_\beta(p) = N^{-1}\left(\sum_{l:c(l)=\beta} t_l - \sum_{l:c(l)=\beta-1} t_l\right), \qquad 1 < \beta < M-1, \tag{25}$$

with the first and second term being omitted for $\beta = M$ and $\beta = 1$.

The equations (25), together with (18) and (21), form a closed set for finding Bloch factors $\{p\}, \{q\}$ at given integers $\{l_\beta\} \in \mathbf{Z}/M\mathbf{Z}$ and determining the eigenvalues of the spin Hamiltonian (1,2) completely. The corresponding eigenvalue of the continuum M-particle operator (11) is given by [13]

$$\mathsf{E}_M(p) = \frac{2M(M-1)}{N}\zeta\left(\frac{N}{2}\right) + \sum_{\beta=1}^{M} p_\beta^2/2$$

$$-\frac{1}{2}\left[\sum_{k<l}^{m}(2\delta_{c(k),c(l)}F(t_k-t_l) - \delta_{|c(k)-c(l)|,1}F(t_k-t_l)) - M\sum_{c(k)=1}F(t_k)\right], \tag{26}$$

where

$$F(t) = -\wp_N(t) + (\zeta_N(t) - 2/N\zeta_N(N/2))^2 + 4/N\zeta_N(N/2).$$

This allows one to find, via (7) and (19), the explicit form of the eigenvalues of spin Hamiltonian (1,2). It is worth noting that for their real calculation one has to solve the Bethe-type equations (18), (21), (25) at first.

In conclusion, it is demonstrated that the procedure of the exact diagonalization of the lattice Hamiltonian with the nonnearest-neighbor elliptic exchange can be reduced in each sector of the Hilbert space with given magnetization to the construction of the special double quasi-periodic eigenfunctions of the many-particle Calogero-Moser problem on a continuous line. The equations of the Bethe-ansatz form appear very naturally as a set of restrictions to the particle pseudomomenta. The proof of this correspodence between lattice and continuum integrable models is based only on analytic properties of the eigenfunctions. One can expect that the set of spin lattice states constructed by this way is complete. This is supported by exact analytic proof in the two-magnon case.

The analysis of explicit form of the equations (21) available for $M = 2, 3$ shows that the spectrum of the lattice Hamiltonian with the exchange (1) is *not* additive being given in terms of pseudomomenta $\{p\}$ or phases which parametrize the sets $\{p, q\}$ [11]. For arbitrary M, this can be seen directly from (26). The problem of finding appropriate set of parameters which gives the «separation» of the spectrum remains open. It would be also of interest to consider various limits $(N \to \infty, \alpha \to 0, \infty)$ so as to recover the results of the papers [2,3] and prove the validity of the approximate methods of asymptotic Bethe ansatz.

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