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BOUND $q\bar{q}$ SYSTEMS IN THE FRAMEWORK OF DIFFERENT VERSIONS OF $3D$ REDUCTIONS OF THE BETHE–SALPETER EQUATION

T.Kopaleishvili

HEPI, Tbilisi State University, University St. 9, 380086 Tbilisi, Georgia

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BOUND $q\bar{q}$ SYSTEMS IN THE FRAMEWORK OF DIFFERENT VERSIONS OF 3D REDUCTIONS OF THE BETHE–SALPETER EQUATION

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Five different versions of the three-dimensional (3D) reduction of the Bethe–Salpeter (BS) equation in the instantaneous approximation for kernel of BS equation for the two-fermion systems are formulated. The normalization conditions for the bound-state wave function in all versions are derived. Further, the 3D reduction of BS equation without instantaneous approximation for the kernel of BS equation is formulated in the quasi-potential approach. Except for the Salpeter version, other four versions have the correct one-body limit (Dirac equation) when mass of one of constituent fermions tends to infinity. Application of these versions for investigation of the different properties of the $q\bar{q}$ bound systems are considered.

Сформулированы пять различных версий трехмерной редукции уравнения Бете–Солпитера (БС) для двухфермионной системы в одновременном приближении для ядра уравнения БС. Получены условия нормировки волновой функции связанного состояния для всех версий редукции. Сформулирована также трехмерная редукция уравнения БС в квазипотенциальном подходе без применения одновременного приближения для ядра уравнения БС. Все версии редукции имеют правильный одночастичный предел (уравнение Дирака), в котором масса одного из составляющих фермионов устремляется в бесконечность, за исключением версии Солпитера. Рассмотрено также применение этих версий для исследования различных свойств связанной $q\bar{q}$-системы.

1. INTRODUCTION

After having firmly established the quark structure of mesons and baryons, there naturally arises the question: how to describe the properties of hadrons in terms of explicit quark and gluon degrees of freedom. The main feature of QCD at low energy — confinement of quarks and gluons into a colorless bound states — is still understood very little. For this reason, one has to resort to various kinds of QCD-inspired models. The simplest one is the so-called constituent quark model, where quarks have a given «constituent» mass, and the interactions between the «constituent» quarks within mesons $q\bar{q}$ and baryons qqq are described by «confining potentials», growing to infinity at the infinite quark separation. At the first stage, this intuitive picture has been implemented within the nonrelativistic approach. Despite the evident success of the nonrelativistic potential model, it has been understood long time ago, that one has to include the relativistic effects, at least when describing hadrons consisting of light u, d, s quarks. Field-theoretical Bethe–Salpeter (BS) equation provides a natural basis for a relativistic generalization of the potential model, where both light u, d, s and the heavy quark c, b bound states can be treated on the equal footing. A more sophisticated approach is based on a coupled set of Dyson–Schwinger (DS) and BS equations, that can be derived at QCD level [1–3]. In such an approach, one uses a model gluon propagator that through the solution of the DS equation leads to the quark propagator which is an entire function in a complex p^2 plane and therefore is believed to correspond to the confined quark. A full content of underlying QCD symmetries which are important at low energy, can be consistently embedded within this approach. In particular, the Goldstone bosons are properly described, and in the limit of the vanishing quark masses, the masses of Goldstone bosons obtained through the solution of the coupled DS and BS equations, also vanish (note that it is not the case in the simple potential-type models with quarks having the constant «constituent» mass).

In the following, we shall review the potential model based solely on the BS equation, which is the subject of intensive investigations during last twenty years.

2. BETHE–SALPETER EQUATION FOR THE TWO-FERMION BOUND STATE

To set up the notation, in this section we give a brief survey of the covariant BS approach to the two-fermion (fermion-antifermion) bound states. In order to derive the BS equation for the two-fermion bound state, we consider the full 4-fermion Green function G which in the momentum space is given by

$$G(p_1, p_2; p_1', p_2') = i^2 \int dx_1 dx_2 dx_1' dx_2' \, \mathrm{e}^{ip_1x_1 + ip_2x_2 - ip_1'x_1' - ip_2'x_2'} \times \\ \times \langle 0|T\psi_1(x_1)\psi_2(x_2)\bar{\psi}_1(x_1')\bar{\psi}_2(x_2')|0\rangle \,, \quad (2.1)$$

where, for simplicity, the fermions 1 and 2 are assumed to be distinguishable, and the spinor indices are suppressed.

The Green function satisfies the BS equation in the momentum space

$$G = G_0 + G_0KG = G_0 + GKG_0 \,, \quad (2.2)$$

where G_0 stands for the free 4-fermion Green function (the direct product of two fermion propagators), and K denotes the kernel of BS equation, given by the sum of all two-particle irreducible Feynman graphs.

In the momentum space, it is convenient to define the centre-of-mass (c.m.) and relative 4-momenta according to the following relations (with arbitrary α and β)*

$$P = p_1 + p_2, \quad p = \beta p_1 - \alpha p_2, \quad \alpha + \beta = 1,$$

$$\text{or} \qquad p_1 = \alpha P + p, \quad p_2 = \beta P - p\,. \tag{2.3}$$

For the basis vectors in the momentum space, the following notation is used

$$|p_1\rangle \otimes |p_2\rangle = |p_1 p_2\rangle = |Pp\rangle = |P\rangle \otimes |p\rangle\,. \tag{2.4}$$

These vectors satisfy the completeness and orthonormality conditions

$$\int |p_i\rangle \frac{d^4 p_i}{(2\pi)^4} \langle p_i| = \mathbf{1} \quad \text{for } i = 1,2\,, \quad \int |P\rangle \frac{d^4 P}{(2\pi)^4} \langle P| = \mathbf{1}\,, \qquad \int |p\rangle \frac{d^4 p}{(2\pi)^4} \langle p| = \mathbf{1}\,, \tag{2.5}$$

$$\langle p_i|p'_j\rangle = \delta_{ij}(2\pi)^4\delta^4(p_i - p'_j)\,, \qquad \langle P|P'\rangle = (2\pi)^4\delta^4(P-P')\,, \quad \langle p|p'\rangle = (2\pi)^4\delta^4(p-p')\,. \tag{2.6}$$

In these notations, we can write

$$\langle Pp|\mathbf{O}|P'p'\rangle = (2\pi)^4\delta^4(P-P')[\langle p|\mathbf{O}(P)|p'\rangle \equiv \mathbf{O}(P;p,p')], \qquad \mathbf{O} = G,\ G_0,\ K\,. \tag{2.7}$$

Further,

$$\langle p|G_0(P)|p'\rangle \equiv G_0(P;p,p') = (2\pi)^4\delta^4(p-p')\,G_0(P;p)\,, \tag{2.8}$$

$$G_0(P;p) = S_1(p_1) \otimes S_2(p_2) = -(\not p_1 + m_1) \otimes (\not p_2 + m_2)\ g_0(P;p)\,, \tag{2.9}$$

where $S_i(p_i) = i(\not p_i - m_i)^{-1}$ stands for the free fermion propagator with the mass m_i, and the quantity $g_0(P;p)$ is defined as follows

$$g_0(P;p) = \frac{1}{p_1^2 - m_1^2 + i0}\,\frac{1}{p_2^2 - m_2^2 + i0} = \frac{1}{p_{10}^2 - w_1^2 + i0}\,\frac{1}{p_{20}^2 - w_2^2 + i0}, \tag{2.10}$$

with $w_i = \sqrt{m_i^2 + \mathbf{p}_i^2}$.

*We choose the system of units, where $\hbar = c = 1$. Any 4-vector has the components $a = (a_0, \mathbf{a})$, and the metric is $g_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$.

The stable bound state with the mass M_B in quantum field theory is described by the 1-particle state vector in the Fock space

$$\langle \mathbf{P}_B|\mathbf{P}'_B\rangle = (2\pi)^3\, 2w_B\, \delta^3(\mathbf{P}-\mathbf{P}')\,, \qquad w_B = \sqrt{M_B^2+\mathbf{P}^2}\,. \tag{2.11}$$

However, there is no interpolating field in the Lagrangian corresponding to the bound state particle. The completeness condition of the Fock-state vectors in the presence of bound states reads

$$\mathbf{1} = \int\, |\mathbf{P}_B\rangle\, \frac{d^3\mathbf{P}_B}{(2\pi)^3}\, \langle \mathbf{P}_B| + \cdots\,, \tag{2.12}$$

where dots stand for the contributions of the states with elementary particles and from the multiparticle scattering states.

Using the completeness condition (2.12), it is straightforward to single out the bound-state contribution to the Green function (2.1) when $P^2 \to M_B^2$ (equivalently $P_0^2 \to w_B^2$). The quantity $\langle p|G(P)|p'\rangle$ exhibits the pole behavior at this point

$$\langle p|G(P)|p'\rangle = i\,\frac{\langle p|\Phi_{\mathbf{P}_B}\rangle\langle\bar{\Phi}_{\mathbf{P}_B}|p'\rangle}{P^2-M_B^2} + \langle p|R(P)|p'\rangle\,, \tag{2.13}$$

where $\langle p|R(P)|p'\rangle$ denotes the regular remainder of $\langle p|G(P)|p'\rangle$ at the bound-state pole that emerges from the contribution of other states in the sum over Fock-space vectors. Further, $\langle p|\Phi_{\mathbf{P}_B}\rangle$ stands for the BS wave function of the bound state

$$\begin{aligned}\langle p|\Phi_{\mathbf{P}_B}\rangle \equiv \Phi_{\mathbf{P}_B}(p) &= \textstyle\int dx\; \mathrm{e}^{ipx}\; \langle 0|T\psi_1(\beta x)\psi_2(-\alpha x)|\mathbf{P}_B\rangle,\\ \langle\bar{\Phi}_{\mathbf{P}_B}|p'\rangle \equiv \bar{\Phi}_{\mathbf{P}_B}(p') &= \textstyle\int dx\; \mathrm{e}^{-ip'x}\; \langle \mathbf{P}_B|T\bar{\psi}_1(\beta x)\bar{\psi}_2(-\alpha x)|0\rangle = \\ &= \Phi^+_{\mathbf{P}_B}(p')\,\gamma_0^{(1)}\otimes\gamma_0^{(2)}.\end{aligned} \tag{2.14}$$

The bound-state equation that can be derived for the state vector $|\Phi_{\mathbf{P}_B}\rangle$ by substituting Eq. (2.13) in the BS equation for the Green function (2.2), formally resembles the nonrelativistic Schrödinger equation for two fermions

$$\begin{aligned}G^{-1}(P_B)|\Phi_{\mathbf{P}_B}\rangle = 0\,, \qquad \langle\bar{\Psi}_{\mathbf{P}_B}|G^{-1}(P_B) = 0\,,\\ \text{with}\quad G_0^{-1}(P) - G^{-1}(P) = K(P)\,,\end{aligned} \tag{2.15}$$

or

$$|\Phi_{\mathbf{P}_B}\rangle = G_0(P_B)K(P_B)|\Phi_{\mathbf{P}_B}\rangle\,, \qquad \langle\bar{\Phi}_{\mathbf{P}_B}| = \langle\bar{\Phi}_{\mathbf{P}_B}|K(P_B)G_0(P_B)\,. \tag{2.16}$$

Here $P_B = (w_B, \mathbf{P}_B)$. Explicitly, in the momentum space, we arrive at the following equation for the bound-state wave function [4]

$$\begin{aligned}\Phi_{\mathbf{P}_B}(p) &= G_0(P;p)\int \frac{d^4p'}{(2\pi)^4}\, K(P;p,p')\Phi_{\mathbf{P}_B}(p')\,,\\ \bar\Phi_{\mathbf{P}_B}(p) &= \int \frac{d^4p'}{(2\pi)^4}\, \bar\Phi_{\mathbf{P}_B}(p')K(P;p',p)G_0(P;p)\,.\end{aligned} \tag{2.17}$$

This equation should be solved in order to obtain the mass M_B of the bound state. It is obvious that both equations: for $\Phi_{\mathbf{P}_B}(p)$, and for its conjugate $\bar\Phi_{\mathbf{P}_B}(p)$, lead to the same bound-state spectrum.

Next, we derive the normalization condition for the BS wave function. To this end, it is useful to start from the following identity

$$G(P)G^{-1}(P)G(P) = G(P) \Rightarrow G(P)(G_0^{-1}(P) - K(P))G(P) = G(P). \tag{2.18}$$

If P^2 is close to M_B^2, one can neglect the contribution from $R(P)$ in Eq. (2.13). We substitute the latter into Eq. (2.18), and perform the integration along the closed contour C that encircles only the bound-state pole at $P_0 = w_B$, in the complex P_0 plane.

$$\begin{aligned}& i\int_C |\Phi_{\mathbf{P}_B}\rangle\langle\bar\Phi_{\mathbf{P}_B}|\, \frac{(G_0^{-1}(P) - K(P))\,dP_0}{(P_0 + w_B - i0)^2(P_0 - w_B + i0)^2}\, |\Phi_{\mathbf{P}_B}\rangle\langle\bar\Phi_{\mathbf{P}_B}| = \\ &= \int_C |\Phi_{\mathbf{P}_B}\rangle\, \frac{dP_0}{(P_0 + w_B - i0)(P_0 - w_B + i0)}\, \langle\bar\Phi_{\mathbf{P}_B}|\,.\end{aligned} \tag{2.19}$$

From the Cauchy's theorem, one has

$$\int_C \frac{f(z)\,dz}{(z - z_S)^n} = \pm 2\pi i\, \frac{1}{(n-1)!}\, \frac{d^{n-1}}{dz^{n-1}}\, f(z)\Big|_{z=z_S}\,, \tag{2.20}$$

where the function $f(z)$ is analytic inside the contour C, and the choice of the $\pm$ sign depends on whether one integrates counterclockwise $(+)$ or clockwise $(-)$ along the contour. With the use of the above formula, from Eq. (2.19) one readily obtains the normalization condition for the BS wave function

$$\begin{aligned}i\int \frac{d^4p}{(2\pi)^4}\,\frac{d^4p'}{(2\pi)^4}\, \bar\Phi_{\mathbf{P}_B}(p)\, \left[\frac{\partial}{\partial P_0}(G_0^{-1}(P;p,p') - K(P;p,p'))\right]_{P_0=w_B} \times \\ \times\Phi_{\mathbf{P}_B}(p') = 2w_B.\end{aligned} \tag{2.21}$$

The equations (2.17), together with the normalization condition (2.21), completely determine the BS mass spectrum and the BS wave function.

At the end of this section, we shall consider in some detail the spin content of the BS wave function. In particular, we shall demonstrate that one can rewrite this equation in terms of «fermion-antifermion» rather than «two-fermion» wave function.

We work with the following representation of Dirac γ matrices

$$\gamma^0 = \gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \boldsymbol{\gamma} = \gamma^0 \left[\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \right] = -\boldsymbol{\alpha}\gamma^0. \quad (2.22)$$

The free two-fermion Green function given by Eqs. (2.9), (2.10), can be written as

$$G_0(P;p) = \begin{pmatrix} G_{aa}(p_1)\, G_{ab}(p_1) \\ G_{ba}(p_1)\, G_{bb}(p_1) \end{pmatrix} \otimes \begin{pmatrix} G_{aa}(p_2)\, G_{ab}(p_2) \\ G_{ba}(p_2)\, G_{bb}(p_2) \end{pmatrix}, \quad (2.23)$$

where $G_{uv}(p_i),\ u,v = a,b$ is the 2×2 matrix (operator) in the spin space of the i-th particle. Further, the BS wave function of the two-fermion system can be written as a column

$$\Phi_{\mathbf{P}_B}(p) = \begin{pmatrix} \Phi_{aa}(P;p) \\ \Phi_{ab}(P;p) \\ \Phi_{ba}(P;p) \\ \Phi_{bb}(P;p) \end{pmatrix}, \quad (2.24)$$

where, again, the components $\Phi_{uv}(P;p),\ u,v = a,b$ are the 2×2 matrices in the spin space of two fermions.

Now, it is straightforward to ensure that the BS equation (2.17) can be rewritten in terms of «fermion-antifermion» wave function $\Psi_{\mathbf{P}_B}(p)$

$$\Psi_{\mathbf{P}_B}(p) = S^{(1)}(p_1) \int \frac{d^4p'}{(2\pi)^4}\, K(P;p,p')\Psi_{\mathbf{P}_B}(p')\, S^{(2)}(-p_2)\,. \quad (2.25)$$

The wave functions $\Psi_{\mathbf{P}_B}(p)$ and $\Phi_{\mathbf{P}_B}(p)$ are related by (see [5])

$$\begin{aligned} \Psi_{\mathbf{P}_B}(p) &= \begin{pmatrix} \Phi_{aa}(P;p)\, \Phi_{ab}(P;p) \\ \Phi_{ba}(P;p)\, \Phi_{bb}(P;p) \end{pmatrix} C = \\ &= -i \begin{pmatrix} \Phi_{ab}(P;p)\sigma_2\ \Phi_{aa}(P;p)\sigma_2 \\ \Phi_{bb}(P;p)\sigma_2\ \Phi_{ba}(P;p)\sigma_2 \end{pmatrix}, \end{aligned} \quad (2.26)$$

where

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \quad (2.27)$$

denotes the charge conjugation matrix.

3. THREE-DIMENSIONAL REDUCTIONS OF THE BS EQUATION

One of the reasons why the three-dimensional ($3D$) reduction of the BS equation is necessary, is the absence of the usual quantum-mechanical probability interpretation for the wave function $\Phi_{\mathbf{P}_B}(p)$ due to the dependence of the latter on the 0-th component of the relative 4-momentum. Further, in the presence of the confining interactions, it is extremely difficult to construct a «reasonable» kernel K in four dimensions that describes these interactions — we are not aware of any, completely successful attempt. On the other hand, the concept of static ($3D$) confining kernels that corresponds to an intuitively clear picture of infinitely rising potentials in the coordinate space, has been extremely useful in many semiphenomenological applications to study, e.g., the characteristics of heavy quarkonia, etc.

For this reason, below we shall mainly consider the static BS kernels (i.e., the kernels which do not depend on the c.m. momentum P and on the 0-th components of the relative momenta $p_0,\ p'_0$)

$$K(P;p,p') \to K_{\rm st}(\mathbf{p},\mathbf{p}') \equiv -iV(\mathbf{p},\mathbf{p}')\,. \tag{3.1}$$

In this approximation, there are still different versions of the $3D$ equations for the bound-state wave function. Below, we shall consider these versions in detail.

3.1. The Salpeter Equation [6]. In the approximation (3.1), from Eq. (2.17) it is straightforwardly obtained

$$\begin{aligned}\Phi_{\mathbf{P}_B}(p) &= G_0(P;p)\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\,K_{\rm st}(\mathbf{p},\mathbf{p}')\,\tilde\Phi_{\mathbf{P}_B}(\mathbf{p}'),\\ \bar\Phi_{\mathbf{P}_B}(p) &= \int\frac{d^3\mathbf{p}'}{(2\pi)^3}\,\tilde{\bar\Phi}_{\mathbf{P}_B}(\mathbf{p}')\,K_{\rm st}(\mathbf{p}',\mathbf{p})\,G_0(P;p),\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\tilde\Phi_{\mathbf{P}_B}(\mathbf{p}) &= \tilde G_0(P;\mathbf{p})\int fracd^3\mathbf{p}'(2\pi)^3\,V(\mathbf{p},\mathbf{p}')\,\tilde\Phi_{\mathbf{P}_B}(\mathbf{p}')\,,\\ \tilde{\bar\Phi}_{\mathbf{P}_B}(\mathbf{p}) &= \int\frac{d^3\mathbf{p}'}{(2\pi)^3}\,\tilde{\bar\Phi}_{\mathbf{P}_B}(\mathbf{p}')\,V(\mathbf{p}',\mathbf{p})\,\tilde G_0(P;\mathbf{p})\,,\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\tilde\Phi_{\mathbf{P}_B}(\mathbf{p}) = \int\frac{dp_0}{2\pi}\,\Phi_{\mathbf{P}_B}(p)\,,\quad \tilde{\bar\Phi}_{\mathbf{P}_B}(\mathbf{p}) = \int\frac{dp_0}{2\pi}\,\bar\Phi_{\mathbf{P}_B}(p)\,,\\ \tilde G_0(P;\mathbf{p}) = \int\frac{dp_0}{2\pi i}\,G_0(P;p)\,.\end{aligned} \tag{3.4}$$

At the next step, we introduce the projection operators

$$\Lambda_i^{(\pm)}(\mathbf{p}_i) = \frac{w_i \pm h_i(\mathbf{p}_i)}{2w_i}\,,\qquad h_i(\mathbf{p}_i) = \boldsymbol{\alpha}^{(i)}\mathbf{p}_i + m_i\gamma_0^{(i)}\,,\qquad i=1,2 \tag{3.5}$$

with the properties

$$\sum_{\alpha_i=\pm}\Lambda_i^{(\alpha_i)}=1,\quad \Lambda_i^{(\alpha_i)}\Lambda_i^{(\alpha_i')}=\delta_{\alpha_i\alpha_i'}\Lambda_i^{(\alpha_i)},\quad h_i(\mathbf{p}_i)\Lambda_i^{(\pm)}=\pm w_i\Lambda_i^{(\pm)}. \tag{3.6}$$

With the use of the following identity

$$\not{p}_i+m_i=\Big\{(p_{i0}+w_i)\Lambda_i^{(+)}(\mathbf{p}_i)+(p_{i0}-w_i)\Lambda_i^{(-)}(\mathbf{p}_i)\Big\}\gamma_0^{(i)}\,, \tag{3.7}$$

it is straightforward to obtain

$$\tilde{G}_0(P;\mathbf{p})=\left\{\frac{\Lambda_{12}^{(++)}(\mathbf{p}_1,\mathbf{p}_2)}{P_0-w_1-w_2+i0}-\frac{\Lambda_{12}^{(--)}(\mathbf{p}_1,\mathbf{p}_2)}{P_0+w_1+w_2}\right\}\gamma_0^{(1)}\otimes\gamma_0^{(2)}=$$

$$=\big[P_0-h_1(\mathbf{p}_1)-h_2(\mathbf{p}_2)\big]^{-1}(\Lambda_{12}^{(++)}(\mathbf{p}_1,\mathbf{p}_2)-\Lambda_{12}^{(--)}(\mathbf{p}_1,\mathbf{p}_2))\,\gamma_0^{(1)}\otimes\gamma_0^{(2)}, \tag{3.8}$$

where $\Lambda_{12}^{(\alpha_1\alpha_2)}(\mathbf{p}_1,\mathbf{p}_2)=\Lambda_1^{(\alpha_1)}(\mathbf{p}_1)\otimes\Lambda_2^{(\alpha_2)}(\mathbf{p}_2)$.

Now the Salpeter equation (3.3) in the c.m. frame ($\mathbf{P}_B$=$\mathbf{0}$) can be written as

$$\big[\,P_0-h_1(\mathbf{p}_1)-h_2(\mathbf{p}_2)\,\big]\,\tilde{\Phi}_{M_B}(\mathbf{p})=$$

$$=\Pi(\mathbf{p})\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\,\gamma_0^{(1)}\otimes\gamma_0^{(2)}\,V(\mathbf{p},\mathbf{p}')\,\tilde{\Phi}_{M_B}(\mathbf{p}')\,, \tag{3.9}$$

where

$$\Pi(\mathbf{p})=(\Lambda_{12}^{(++)}(\mathbf{p}_1,\mathbf{p}_2)-\Lambda_{12}^{(--)}(\mathbf{p}_1,\mathbf{p}_2))=\frac{h_1(\mathbf{p})}{2w_1}+\frac{h_2(-\mathbf{p})}{2w_2}\,. \tag{3.10}$$

Introducing the «frequency components» of the wave function according to

$$\tilde{\Phi}_{\mathbf{P}_B}(\mathbf{p})=\sum_{\alpha_1\alpha_2}\tilde{\Phi}_{\mathbf{P}_B}^{(\alpha_1\alpha_2)}(\mathbf{p})\,,\quad \tilde{\Phi}_{\mathbf{P}_B}^{(\alpha_1\alpha_2)}(\mathbf{p})=\Lambda_{12}^{(\alpha_1\alpha_2)}(\mathbf{p}_1,\mathbf{p}_2)\,\tilde{\Phi}_{\mathbf{P}_B}(\mathbf{p})\,, \tag{3.11}$$

the Eq. (3.9) can be reduced to the following system of equations

$$\big[M_B\mp(w_1+w_2)\big]\,\tilde{\Phi}_{M_B}^{(\pm\pm)}(\mathbf{p})=$$

$$=\pm\Lambda_{12}^{(\pm\pm)}(\mathbf{p},-\mathbf{p})\,\gamma_0^{(1)}\otimes\gamma_0^{(2)}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\,V(\mathbf{p},\mathbf{p}')\,\tilde{\Phi}_{M_B}(\mathbf{p}')\,, \tag{3.12}$$

with additional conditions

$$\tilde{\Phi}_{M_B}^{(\pm\mp)}(\mathbf{p})=0,\qquad \tilde{\Phi}_{M_B}(\mathbf{p})=\tilde{\Phi}_{M_B}^{(++)}(\mathbf{p})+\tilde{\Phi}_{M_B}^{(--)}(\mathbf{p})\,. \tag{3.13}$$

The normalization condition can be readily obtained from Eq. (2.21) by using the approximation (3.1) for the kernel, the relation between $4D$ and $3D$ wave functions (3.2), and the decomposition of the wave function (3.11), (3.13)

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \left\{ |\tilde{\Phi}_{M_B}^{(++)}(\mathbf{p})|^2 - |\tilde{\Phi}_{M_B}^{(--)}(\mathbf{p})|^2 \right\} = 2M_B \,. \tag{3.14}$$

Note that the wave function $\tilde{\Phi}_{M_B}(\mathbf{p})$ can be represented in a form analogous to (2.24)

$$\tilde{\Phi}_{M_B}(\mathbf{p}) = \begin{pmatrix} \tilde{\Phi}_{aa}(\mathbf{p}) \\ \tilde{\Phi}_{ab}(\mathbf{p}) \\ \tilde{\Phi}_{ba}(\mathbf{p}) \\ \tilde{\Phi}_{bb}(\mathbf{p}) \end{pmatrix} . \tag{3.15}$$

The constraints (3.13) can be considered as equations for the components $\tilde{\Phi}_{ab}(\mathbf{p})$ and $\tilde{\Phi}_{ba}(\mathbf{p})$. The solution of these equations gives

$$\begin{aligned} \tilde{\Phi}_{ab} &= (m_1 w_2 + m_2 w_1)^{-1} \left\{ w_1(\boldsymbol{\sigma}^{(2)}\mathbf{p}_2)\tilde{\Phi}_{aa} - w_2(\boldsymbol{\sigma}^{(1)}\mathbf{p}_1)\tilde{\Phi}_{bb} \right\} , \\ \tilde{\Phi}_{ba} &= (m_1 w_2 + m_2 w_1)^{-1} \left\{ w_2(\boldsymbol{\sigma}^{(1)}\mathbf{p}_1)\tilde{\Phi}_{aa} - w_1(\boldsymbol{\sigma}^{(2)}\mathbf{p}_2)\tilde{\Phi}_{bb}) \right\} . \end{aligned} \tag{3.16}$$

For the «frequency components» $\tilde{\Phi}_{xy}^{(\alpha_1\alpha_2)} = \Lambda_{12}^{(\alpha_1\alpha_2)}(\mathbf{p})\tilde{\Phi}_{xy}(\mathbf{p})$, $x, y = a, b$, we obtain the following relations

$$\begin{aligned} \tilde{\Phi}_{aa}^{(\pm\pm)} &= \pm(2(m_1 w_2 + m_2 w_1))^{-1} \times \\ &\times \left\{ (w_1 \pm m_1)(w_2 \pm m_2)\tilde{\Phi}_{aa} - (\boldsymbol{\sigma}^{(1)}\mathbf{p}_1)(\boldsymbol{\sigma}^{(2)}\mathbf{p}_2)\tilde{\Phi}_{bb} \right\} , \\ \tilde{\Phi}_{bb}^{(\pm\pm)} &= \pm(2(m_1 w_2 + m_2 w_1))^{-1} \times \\ &\times \left\{ (\boldsymbol{\sigma}^{(1)}\mathbf{p}_1)(\boldsymbol{\sigma}^{(2)}\mathbf{p}_2)\tilde{\Phi}_{aa} - (w_1 \mp m_1)(w_2 \mp m_2)\tilde{\Phi}_{bb} \right\} , \\ \tilde{\Phi}_{ab}^{(\pm\pm)} &= \pm(2(m_1 w_2 + m_2 w_1))^{-1} \times \\ &\times \left\{ (w_1 \pm m_1)(\boldsymbol{\sigma}^{(2)}\mathbf{p}_2)\tilde{\Phi}_{aa} - (w_2 \mp m_2)(\boldsymbol{\sigma}^{(1)}\mathbf{p}_1)\tilde{\Phi}_{bb} \right\} , \\ \tilde{\Phi}_{ba}^{(\pm\pm)} &= \pm(2(m_1 w_2 + m_2 w_1))^{-1} \times \\ &\times \left\{ (w_2 \pm m_2)(\boldsymbol{\sigma}^{(1)}\mathbf{p}_1)\tilde{\Phi}_{aa} - (w_1 \mp m_1)(\boldsymbol{\sigma}^{(2)}\mathbf{p}_2)\tilde{\Phi}_{bb} \right\} . \end{aligned} \tag{3.17}$$

The normalization condition (3.14) can be rewritten as

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \, \frac{2w_1 w_2}{m_1 w_2 + m_2 w_1} \, \left\{ |\tilde{\Phi}_{aa}(\mathbf{p})|^2 - |\tilde{\Phi}_{bb}(\mathbf{p})|^2 \right\} = 2M_B \,. \tag{3.18}$$

At the end of this subsection, we shall consider the existence of the one-body limit in the Salpeter equation. From the physical point of view, it is clear that if the mass of one of the particles in the two-particle bound state tends to infinity, the equation for the wave function should reduce to Dirac equation for the light particle with a given interaction potential. Let us check this property for the Salpeter equation assuming, e.g., that the mass of the first particle tends to infinity. In this limit:

$$m_1 \to \infty \quad \Rightarrow \quad w_1 \to m_1, \quad \gamma_0^{(1)} \to 1, \quad h_1 \to m_1 . \tag{3.19}$$

Then, the Salpeter equation for the bound-state vector $|\tilde{\Phi}_{M_B}\rangle$ is reduced to $(E_2 \equiv M_B - m_1)$

$$\big(E_2 - h_2 \big) \, |\tilde{\Phi}_{E_2}\rangle = \frac{1}{2} \left(1 + \frac{h_2}{w_2} \right) \gamma_0^{(2)} \, |\tilde{\Phi}_{E_2}\rangle . \tag{3.20}$$

Due to the presence of the prefactor $(1 + h_2/w_2)/2$, this equation differs from the Dirac equation for the particle 2 moving in the potential V — that is, the Salpeter equation does not possess the correct one-body limit.

Now there arises an important problem to solve. We are willing to obtain the $3D$ reduction of the BS equation in the static approximation, that correctly reproduces the dynamics of the system in the one-body limit — this property might be important for the description, e.g., the heavy-light $q\bar{q}$ bound states.

Below, we shall consider several versions of the $3D$ reduction procedure, which lead to the correct one-body limit.

3.2. The Gross Equation [7]. In the derivation of the Gross equation, first we assign $\alpha = 0$ and $\beta = 1$, in the definition of the c.m. and relative momentum variables (2.3). Physically, this means that the whole c.m. momentum is carried by the particle 2. The free Green function has the form

$$\begin{aligned} G_0(P;p) &= -(\not{p} + m_1) \otimes (\not{P} - \not{p} + m_2)\, g_0(P;p) , \\ g_0(P;p) &= \frac{1}{p^2 - m_1^2 + i0} \, \frac{1}{(P-p)^2 - m_2^2 + i0} . \end{aligned} \tag{3.21}$$

The first propagator can be rewritten as

$$\frac{1}{p^2 - m_1^2 + i0} = P \, \frac{1}{p^2 - m_1^2} - i\pi\delta(p^2 - m_1^2) , \tag{3.22}$$

where the symbol P stands for the principal-value prescription. The approximation that leads to the Gross equation, consists in the substitution

$$\frac{1}{p^2 - m_1^2 + i0} \ \Rightarrow \ -2\pi i \, \frac{\delta(p_0 - w_1)}{2w_1} . \tag{3.23}$$

This approximation is called the «spectator approximation». Note that in this approximation it is not only the principal-value term in the propagator of the first particle that is neglected, but also the term containing $\delta(p_0 + w_1)$ that emerges from $\delta(p^2 - m_1^2)$. Consequently, in this approximation the particle 1 always stays on its mass shell defined by the equation $p_0 = w_1$. In a result of this approximation, the free Green function in the c.m. frame ($P_\mu = (P_0, \mathbf{0})$) can be rewritten in the following form

$$G_0^{\rm GR}(P_0; p) = 2\pi i \delta(p_0 - w_1) \tilde{G}_0^{\rm GR}(P_0; \mathbf{p}) = 2\pi i \delta(p_0 - w_1) \times$$
$$\times \quad \left\{ \frac{\Lambda_1^{(+)}(\mathbf{p}) \otimes \Lambda_2^{(+)}(-\mathbf{p})}{P_0 - w_1 - w_2 + i0} + \frac{\Lambda_1^{(+)}(\mathbf{p}) \otimes \Lambda_2^{(-)}(-\mathbf{p})}{P_0 - w_1 + w_2 + i0} \right\} \gamma_0^{(1)} \otimes \gamma_0^{(2)}, \tag{3.24}$$

where the functions $G_0^{\rm GR}(P_0; p)$ and $\tilde{G}_0^{\rm GR}(P_0; \mathbf{p})$ are related by Eq. (3.4).

After substituting Eq. (3.24) into (2.16) and integrating over the variable p_0, in the c.m. frame (now $P^\mu = (M_B, \mathbf{0})$) we arrive at the Gross equation for the $3D$ bound-state wave function

$$\begin{aligned} \tilde{\Phi}_{M_B}(\mathbf{p}) &= \tilde{G}_0^{\rm GR}(M_B; \mathbf{p}) \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \, V(\mathbf{p}, \mathbf{p}') \, \tilde{\Phi}_{M_B}(\mathbf{p}'), \\ \tilde{\bar{\Phi}}_{M_B}(\mathbf{p}) &= \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \, \tilde{\bar{\Phi}}_{M_B}(\mathbf{p}') \, V(\mathbf{p}', \mathbf{p}) \, \tilde{G}_0^{\rm GR}(M_B; \mathbf{p}). \end{aligned} \tag{3.25}$$

Now, using again (3.24) together with (3.6), we arrive at

$$\left[M_B - h_1(\mathbf{p}) - h_2(-\mathbf{p}) \right] \tilde{\Phi}_{M_B}(\mathbf{p}) =$$
$$= \frac{1}{2} \left(1 + \frac{h_1}{w_1} \right) \gamma_0^{(1)} \otimes \gamma_0^{(2)} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \, V(\mathbf{p}, \mathbf{p}') \, \tilde{\Phi}_{M_B}(\mathbf{p}'), \tag{3.26}$$

which has the correct one-body limit when $m_1 \to \infty$.

The normalization condition for the $3D$ wave function that satisfies the Gross equation, cannot be obtained in a standard manner, by using Eq. (2.21). In order to demonstrate this, note that according to Eqs. (2.16), (3.24), and (3.25), $4D$ and $3D$ wave functions are related by

$$\Phi_{M_B}(p) = 2\pi\delta(p_0 - w_1) \, \tilde{\Phi}_{M_B}(\mathbf{p}), \;\; \bar{\Phi}_{M_B}(p) = 2\pi\delta(p_0 - w_1) \, \tilde{\bar{\Phi}}_{M_B}(\mathbf{p}). \tag{3.27}$$

Now if in the normalization condition (2.21) with the static kernel (3.1), the relation (3.27) between the $4D$ and $3D$ wave functions is substituted, one arrives at the ill-defined expression containing the product of δ functions with the same argument. For this reason, instead of the rigorous derivation, from the analogy with the Salpeter equation, one merely assumes that the solutions of the Gross equation satisfy the following normalization condition

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \left\{ |\tilde{\Phi}_{M_B}^{(++)}(\mathbf{p})|^2 + |\tilde{\Phi}_{M_B}^{(+-)}(\mathbf{p})|^2 \right\} = 2M_B. \tag{3.28}$$

3.3. The Mandelzweig–Wallace Equation [8]. In the derivation of the Mandelzweig–Wallace (MW) equation, the parameters α and β in the expression of the c.m. and relative momenta (2.3) are defined according to Wightmann and Garding

$$\alpha=\alpha(s)=\frac{s+m_1^2-m_2^2}{2s},\ \beta=\beta(s)=\frac{s-m_1^2+m_2^2}{2s},\ s=P^2=P_0^2-\mathbf{P}^2. \tag{3.29}$$

In the c.m. frame, from Eqs. (2.3) and (3.29) it follows

$$\begin{gathered} p_1=(E_1+p_0,\mathbf{p}),\quad p_2=(E_2-p_0,-\mathbf{p}),\\ E_1=\frac{M_B^2+m_1^2-m_2^2}{2M_B},\quad E_2=\frac{M_B^2-m_1^2+m_2^2}{2M_B},\\ E_1+E_2=M_B,\quad E_1-E_2=\frac{m_1^2-m_2^2}{M_B}. \end{gathered} \tag{3.30}$$

Further, we define in the c.m. frame

$$\tilde{G}_0^{\rm MW}(M_B;\mathbf{p})=\int\frac{dp_0}{2\pi i}\left[G_0(p_1,p_2)+G_0(p_1,p_2^{\rm cr})\right],\ p_2^{\rm cr}=(E_2+p_0,-\mathbf{p}), \tag{3.31}$$

where $G_0(p_1,p_2)$ is given by Eqs. (2.8), (2.9), and (2.10). After integrating over p_0, we obtain

$$\tilde{G}_0^{\rm MW}(M_B;\mathbf{p})=\left\{\frac{\Lambda_{12}^{++}(\mathbf{p},-\mathbf{p})}{E_1+E_2-w_1-w_2+i0}+\frac{\Lambda_{12}^{+-}(\mathbf{p},-\mathbf{p})}{-E_1+E_2+w_1+w_2}+\right.$$
$$\left.+\frac{\Lambda_{12}^{-+}(\mathbf{p},-\mathbf{p})}{E_1-E_2+w_1+w_2}-\frac{\Lambda_{12}^{--}(\mathbf{p},-\mathbf{p})}{E_1+E_2+w_1+w_2}\right\}\gamma_0^{(1)}\otimes\gamma_0^{(2)}. \tag{3.32}$$

The MW equation is obtained from the BS equation in the static approximation, by using the combination $G_0(p_1,p_2)+G_0(p_1,p_2^{\rm cr})$ instead of $G_0(p_1,p_2)$ alone. Unlike the Salpeter version, now all four possible projection operators $\Lambda_{12}^{(++)}$, $\Lambda_{12}^{(+-)}$, $\Lambda_{12}^{(-+)}$, $\Lambda_{12}^{(--)}$, enter the expression of $\tilde{G}_0^{\rm MW}(M_B;\mathbf{p})$, Eq. (3.32). For this reason, the inverse operator for the free Green function in the $3D$ space exists. Further, in analogy with Eq. (2.15), we can define the inverse of the full Green function in $3D$ space according to

$$\left[\tilde{G}_0^{\rm MW}\right]^{-1}(M_B;\mathbf{p},\mathbf{p}')-\left[\tilde{G}^{\rm MW}\right]^{-1}(M_B;\mathbf{p},\mathbf{p}')=V(\mathbf{p},\mathbf{p}'), \tag{3.33}$$

where

$$\tilde{G}_0^{\rm MW}(M_B;\mathbf{p},\mathbf{p}')=(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,\tilde{G}_0^{\rm MW}(M_B;\mathbf{p}). \tag{3.34}$$

The MW equation for the bound-state vector $|\tilde{\Phi}_{M_B}\rangle$ is given by

$$\big[\tilde{G}^{\rm MW}\big]^{-1}|\tilde{\Phi}_{M_B}\rangle = 0 \quad \Rightarrow \quad |\tilde{\Phi}_{M_B}\rangle = \tilde{G}_0^{\rm MW} V|\tilde{\Phi}_{M_B}\rangle\,. \qquad (3.35)$$

Note that we can rewrite the inverse of the free Green function in the MW equation as

$$\big[\tilde{G}_0^{\rm MW}\big]^{-1} = \gamma_0^{(1)}\otimes\gamma_0^{(2)}\left[(E_1-h_1)\otimes\frac{h_2}{w_2}+\frac{h_1}{w_1}\otimes(E_2-h_2)\right]. \qquad (3.36)$$

With the use of this identity, one can rewrite the MW equation as

$$\left[(E_1-h_1)\otimes\frac{h_2}{w_2}+\frac{h_1}{w_1}\otimes(E_2-h_2)\right]|\tilde{\Phi}_{M_B}\rangle = \gamma_0^{(1)}\otimes\gamma_0^{(2)}\,V\,|\tilde{\Phi}_{M_B}\rangle\,. \quad (3.37)$$

Let us now consider the limit of this equation when $m_1\to\infty$ (3.19). In this limit, according to Eq. (3.30), $E_1\to m_1$, $E_2\to M_B-m_1$, and the equation (3.37) simplifies to the Dirac equation

$$\big[E_2-h_2\big]\,|\tilde{\Phi}_{M_B}\rangle = \gamma_0^{(2)}V|\tilde{\Phi}_{M_B}\rangle\,. \qquad (3.38)$$

Consequently, the MW equation has the correct one-body limit.

3.4. The Cooper–Jennings Equation [9]. The parameters $\alpha(s)$ and $\beta(s)$ in the Cooper–Jennings (CJ) version are chosen as

$$\begin{aligned} \alpha(s) &= \frac{\alpha_1(s)}{\alpha_1(s)+\alpha_2(s)}, \qquad \beta(s) = \frac{\alpha_2(s)}{\alpha_1(s)+\alpha_2(s)},\\ \alpha_1(s) &= \frac{s+m_1^2-m_2^2}{2\sqrt{s}}, \qquad \alpha_2(s) = \frac{s-m_1^2+m_2^2}{2\sqrt{s}}. \end{aligned} \qquad (3.39)$$

The free Green function for the CJ equation is given by

$$G_0^{\rm CJ}(P;p) = -(\not{p}_1+m_1)\otimes(\not{p}_2+m_2)g_0^{\rm CJ}(P;p)\,, \qquad (3.40)$$

where $g_0^{\rm CJ}(P;p)$ is constrained by the elastic unitarity and can be written in the following form

$$\begin{aligned} g_0^{\rm CJ}(P;p) &= 2\pi i\int_{(m_1+m_2)^2}^{\infty}\frac{ds' f(s,s')}{s'-s-i0}\;\delta^+\big[(\alpha(s')P'+p)^2-m_1^2\big]\times\\ &\times\delta^+\big[(\beta(s')P'-p)^2-m_2^2\big]. \end{aligned} \qquad (3.41)$$

Here $\delta^+(x^2-a^2) = (2a)^{-1}\delta(x-a)$, $P' = \sqrt{s'/s}P$, and the function $f(s,s')$ satisfies the condition $f(s,s)=1$.

After integration, the expression (3.41) yields

$$g_0^{\mathrm{CJ}}(P;p) = -2\pi i \frac{\delta(2Pp)}{s-s_p}\ \sqrt{s s_p}\ \frac{f(s,s_p)}{\alpha_1(s_p)\,\alpha_2(s_p)}\,, \tag{3.42}$$

where $s_p = (\sqrt{m_1^2 - p^2} + \sqrt{m_2^2 - p^2})^2$.

Choosing the function $f(s,s_p)$ in the form

$$f(s,s_p) = \frac{4s\,\alpha_1(s_p)\,\alpha_2(s_p)}{s s_p - (m_1^2 - m_2^2)^2} \tag{3.43}$$

we arrive at the following expression for $g_0^{\mathrm{CJ}}(P;p)$

$$g_0^{\mathrm{CJ}}(P;p) = -2\pi i\, \frac{2s}{(s-(w_1+w_2)^2 + i0)(s-(w_1-w_2)^2)}\ \frac{2\sqrt{s}\,\delta(2Pp)}{w_1+w_2}. \tag{3.44}$$

Note, that in the c.m. frame, $2\sqrt{s}\,\delta(2Pp) = \delta(p_0)$. Because of the presence of the δ function, one can rewrite the free Green function from (3.40) in the following form (again, in the c.m. frame)

$$\begin{gathered} G_0^{\mathrm{CJ}}(M_B;p) = -(\tilde{\not{p}}_1 + m_1) \otimes (\tilde{\not{p}}_2 + m_2) g_0^{\mathrm{CJ}}(M_B;p)\,, \\ \tilde{p}_1 = (E_1, \mathbf{p}), \quad \tilde{p}_2 = (E_2, -\mathbf{p}), \end{gathered} \tag{3.45}$$

where E_1 and E_2 are given by Eq. (3.30). The free Green function for the CJ equation in $3D$ space is related to $4D$ Green function according to

$$G_0^{\mathrm{CJ}}(M_B;p) = 2\pi i\,\delta(p_0)\,\tilde{G}_0^{\mathrm{CJ}}(M_B;\mathbf{p})\,, \tag{3.46}$$

where $\tilde{G}_0^{\mathrm{CJ}}(M_B;\mathbf{p})$ is given by

$$\begin{aligned} \tilde{G}_0^{\mathrm{CJ}}(M_B;\mathbf{p}) &= \frac{1}{2(w_1+w_2)}\,\frac{(\tilde{\not{p}}_1 + m_1)\otimes(\tilde{\not{p}}_2 + m_2)}{E_1^2 - w_1^2} = \\ &= \frac{1}{2(w_1+w_2)}\,\frac{(\tilde{\not{p}}_1 + m_1)\otimes(\tilde{\not{p}}_2 + m_2)}{E_2^2 - w_2^2} = \\ &= \frac{1}{2(w_1+w_2)}\,\frac{\tilde{\not{p}}_1 + m_1}{\tilde{\not{p}}_2 - m_2} = \frac{1}{2(w_1+w_2)}\,\frac{\tilde{\not{p}}_2 + m_2}{\tilde{\not{p}}_1 - m_1}. \end{aligned} \tag{3.47}$$

In the limit, when one of the masses tends to infinity,

$$\frac{\tilde{\not{p}}_i + m_i}{2(w_1+w_2)} \to 1 \qquad \text{at} \quad m_i \to \infty, \quad i = 1,2. \tag{3.48}$$

Consequently, the CJ equation has the correct one-body limit.

Note that, using the properties of the projection operators, the free Green function in the $3D$ space can be rewritten in the following form

$$\tilde{G}_0^{\mathrm{CJ}}(M_B;\mathbf{p}) = \frac{1}{2(w_1+w_2)a}\left[(w_1+E_1)(w_2+E_2)\Lambda_{12}^{(++)}(\mathbf{p},-\mathbf{p}) - \right.$$
$$-(w_1+E_1)(w_2-E_2)\Lambda_{12}^{(+-)}(\mathbf{p},-\mathbf{p}) - (w_1-E_1)(w_2+E_2)\Lambda_{12}^{(-+)}(\mathbf{p},-\mathbf{p}) +$$
$$\left. +(w_1-E_1)(w_2-E_2)\Lambda_{12}^{(--)}(\mathbf{p},-\mathbf{p})\right]\gamma_0^{(1)}\otimes\gamma_0^{(2)}\,, \tag{3.49}$$

where

$$\begin{aligned} a &= E_1^2 - w_1^2 = E_2^2 - w_2^2 = \frac{1}{4}\left[M_B^2 + b_0^2 - 2(w_1^2+w_2^2)\right], \\ b_0 &= E_1 - E_2 = \frac{m_1^2 - m_2^2}{M_B}. \end{aligned} \tag{3.50}$$

3.5. The Maung–Norbury–Kahana Equation [10,11]. The free Green function for the Maung–Norbury–Kahana (MNK) equation is again given by Eq. (3.40), but with

$$g_0^{\mathrm{MNK}}(P;p) =$$
$$= -2\pi i\,\frac{\delta^+\left\{\left[(\alpha(s)P+p)^2 - m_1^2\right]\frac{1+y}{2} - \left[(\beta(s)P-p)^2 - m_2^2\right]\frac{1-y}{2}\right\}}{\left[(\alpha(s)P+p)^2 - m_1^2\right] + \left[(\beta(s)P-p)^2 - m_2^2\right] + i0}, \tag{3.51}$$

with $y = (m_1 - m_2)/(m_1 + m_2)$. This Green function, of course, satisfies the unitarity condition in the elastic channel. In addition, it has the property that the particles 1 and 2 in the intermediate states are now allowed to go off mass shell inverse proportionally to their masses — so that, if one of the particles becomes infinitely massive, it is automatically kept on its mass shell.

After some transformations, the Green function from Eq. (3.51) in the c.m. frame can be rewritten as

$$\begin{aligned} g_0^{\mathrm{MNK}}(M_B;p) &= -2\pi i\,\frac{\delta(p_0 - p_0^+)}{2R(p_0^+p_0^- + a)}\,, \\ p_0^+ &= \frac{R-b}{2y}, \quad p_0^- = p_0^+ + b_0, \\ R &= \sqrt{b^2 - 4y^2a}, \quad b = M_B + b_0 y, \end{aligned} \tag{3.52}$$

and b_0 is given by Eq. (3.50). By using the above expression, we obtain

$$\tilde{G}_0^{\rm MNK}(M_B,\mathbf{p}) = \frac{(\tilde{\not{p}}_1^+ + m_1)(\tilde{\not{p}}_2^+ + m_2)}{2R(p_0^+ p_0^- + a)},$$

$$\tilde{p}_1^+ = (E_1 + p_0^+, \mathbf{p}), \quad \tilde{p}_2^+ = (E_2 - p_0^+, -\mathbf{p}). \tag{3.53}$$

The relation between the $4D$ and $3D$ free Green functions in the MNK version is given by

$$G_0^{\rm MNK}(M_B;p) = 2\pi i\delta(p_0 - p_0^+)\,\tilde{G}_0^{\rm MNK}(M_B,\mathbf{p})\,. \tag{3.54}$$

Using the properties of the projection operators, the free Green function of the MNK equation can be recast in the following form

$$\begin{aligned}
&\tilde{G}_0^{\rm MNK}(M_B,\mathbf{p}) = \\
= \ & \frac{1}{2R(p_0^+ p_0^- + a)}\Big\{\big[(w_1+E_1)(w_2+E_2)\Lambda_{12}^{(++)}(\mathbf{p},-\mathbf{p}) - \\
- \ & (w_1+E_1)(w_2-E_2)\Lambda_{12}^{(+-)}(\mathbf{p},-\mathbf{p}) - (w_1-E_1)(w_2+E_2)\Lambda_{12}^{(-+)}(\mathbf{p},-\mathbf{p}) + \\
+ \ & (w_1-E_1)(w_2-E_2)\Lambda_{12}^{(--)}(\mathbf{p},-\mathbf{p})\big] - \\
- \ & \big[p_0^+ p_0^- + (w_1-w_2)p_0^+(\Lambda_{12}^{(++)}(\mathbf{p},-\mathbf{p}) - \Lambda_{12}^{(--)}(\mathbf{p},-\mathbf{p})) + \\
+ \ & (w_1+w_2)p_0^+(\Lambda_{12}^{(+-)}(\mathbf{p},-\mathbf{p}) - \Lambda_{12}^{(-+)}(\mathbf{p},-\mathbf{p}))\big]\Big\}\gamma_0^{(1)}\otimes\gamma_0^{(2)}.
\end{aligned} \tag{3.55}$$

In the limit when $m_1 \to \infty$, the function $g_0^{\rm MNK}(P;p)$ from Eq. (3.51) is reduced to

$$g_0^{\rm MNK}(P;p)\Big|_{m_1\to\infty} \to \frac{-2\pi i}{p_2^2 - m_2^2}\,\frac{\delta(p_0)}{2m_1}\,. \tag{3.56}$$

From Eq. (3.40) we can evaluate $G_0^{\rm MNK}(P;p)$ in this limit:

$$G_0^{\rm MNK}(P;p)\Big|_{m_1\to\infty} \to 2\pi i\ \delta(p_0)\ \frac{(\tilde{\not{p}}_1 + m_1)\otimes(\tilde{\not{p}}_2 + m_2)}{2m_1(\tilde{p}_2^2 - m_2^2)}\,, \tag{3.57}$$

where $\tilde{p}_1$ and $\tilde{p}_2$ are defined by Eq. (3.45). Integrating this relation over p_0, for the $3D$ free Green function in the c.m. frame we obtain

$$\tilde{G}_0^{\rm MNK}(M_B;\mathbf{p})\Big|_{m_1\to\infty} \to \frac{\tilde{\not{p}}_1 + m_1}{2m_1}\otimes\frac{\tilde{\not{p}}_2 + m_2}{\tilde{p}_2^2 - m_2^2}\,. \tag{3.58}$$

Since the factor $(\not{p}_1 + m_1)/(2m_1)$ tends to unity in the limit $m_1 \to \infty$, one concludes that the MNK equation has the correct one-body limit.

3.6. The Normalization Condition for the Wave Function in MW, CJ and MNK Versions. The $3D$ free Green function in either of MW, CJ, or MNK versions, in the c.m. frame can be rewritten in terms of the projection operators:

$$\tilde{G}_0(M_B, \mathbf{p}) = \sum_{\alpha_1,\alpha_2=\pm} \frac{D^{(\alpha_1\alpha_2)}(M_B;p)}{d(M_B;p)} \Lambda_{12}^{(\alpha_1\alpha_2)}(\mathbf{p},-\mathbf{p})\gamma_0^{(1)} \otimes \gamma_0^{(2)},$$
$$p = |\mathbf{p}|, \tag{3.59}$$

where

$$\text{MW} \quad : \quad D^{(\alpha_1\alpha_2)} = \frac{(-)^{(\alpha_1+\alpha_2)/2}}{(w_1+w_2)-(\alpha_1 E_1+\alpha_2 E_2)}\,, \qquad d = 1\,,$$
$$\text{CJ} \quad : \quad D^{(\alpha_1\alpha_2)} = (E_1+\alpha_1 w_1)(E_2+\alpha_2 w_2)\,, \qquad d = 2(w_1+w_2)a\,,$$
$$\text{MNK} \quad : \quad D^{(\alpha_1\alpha_2)} = (E_1+\alpha_1 w_1)(E_2+\alpha_2 w_2) - \tag{3.60}$$
$$-\frac{R-b}{2y}\left(\frac{R-b}{2y} + (E_1+\alpha_1 w_1) - (E_2+\alpha_2 w_2)\right),$$
$$d = 2RB\,, \quad B = \frac{R-b}{2y}\left(\frac{R-b}{2y} + b_0\right) + a\,,$$

with $E_1,\ E_2,\ a,\ b_0,\ R,\ b,\ y$ defined above.

The equation for the bound-state wave function frequency components (3.11) can be directly obtained from Eq. (3.3) by substituting the above expression for the free $3D$ Green function and using the properties of the projection operators

$$\left[M_B - (\alpha_1 w_1 + \alpha_2 w_2)\right]\tilde{\Phi}_{M_B}^{(\alpha_1\alpha_2)}(\mathbf{p}) = A^{(\alpha_1\alpha_2)}(M_B;p)\,\Lambda_{12}^{(\alpha_1\alpha_2)}(\mathbf{p},-\mathbf{p}) \times$$
$$\times\, \gamma_0^{(1)} \otimes \gamma_0^{(2)} \sum_{\alpha_1'\alpha_2'} \int \frac{d^3\mathbf{p}'}{(2\pi)^3}\, V(\mathbf{p},\mathbf{p}')\,\tilde{\Phi}_{M_B}^{(\alpha_1'\alpha_2')}(\mathbf{p}')\,, \tag{3.61}$$

where, for the different versions

$$\text{MW} \quad : \quad A^{(\pm\pm)} = 1\,, \quad A^{(\pm\mp)} = \frac{M_B}{w_1+w_2}\,,$$
$$\text{CJ} \quad : \quad A^{(\alpha_1\alpha_2)} = \frac{M_B + (\alpha_1 w_1 + \alpha_2 w_2)}{2(w_1+w_2)}\,,$$

$$\text{MNK} \quad : \quad A^{(\alpha_1\alpha_2)} = \frac{1}{2RB}\Big\{a\big[M_B + (\alpha_1 w_1 + \alpha_2 w_2)\big] -$$

$$-\big[M_B - (\alpha_1 w_1 + \alpha_2 w_2)\big]\frac{R-b}{2y}\times$$

$$\times\left(\frac{R-b}{2y} + (E_1 + \alpha_1 w_1) - (E_2 + \alpha_2 w_2)\right)\Big\}. \qquad (3.62)$$

Thus, the MW, CJ and MNK equations couple all four frequency components of the wave function: $\tilde{\Phi}^{(++)}_{M_B}$, $\tilde{\Phi}^{(+-)}_{M_B}$, $\tilde{\Phi}^{(-+)}_{M_B}$, and $\tilde{\Phi}^{(--)}_{M_B}$. One can formally extend these notations for the Salpeter (SAL) and Gross (GR) versions, defining

$$\text{SAL} \quad : \quad A^{(\pm\pm)} = \pm 1\,, \quad A^{(\pm\mp)} = 0\,,$$

$$\text{GR} \quad : \quad A^{(+\pm)} = +1\,, \quad A^{(-\mp)} = 0\,. \qquad (3.63)$$

It is immediately seen that Salpeter and Gross equations couple only two frequency components of the wave function, other two being equal to 0.

For the derivation of the wave function normalization condition in MW, CJ and MNK versions, let us consider the full $3D$ Green function that obeys the equation

$$\tilde{G}^i = \tilde{G}^i_0 + \tilde{G}^i_0 V \tilde{G}^i = \tilde{G}^i_0 + \tilde{G}^i V \tilde{G}^i_0\,, \qquad i = \text{MW, CJ, MNK}\,. \qquad (3.64)$$

In analogy with Eq. (2.13), this Green function develops a bound-state pole(s)

$$\langle\mathbf{p}|\tilde{G}(P)|\mathbf{p}'\rangle = \sum_B \frac{\langle\mathbf{p}|\tilde{\Phi}_{\mathbf{P}_B}\rangle\langle\bar{\tilde{\Phi}}_{\mathbf{P}_B}|\mathbf{p}'\rangle}{P^2 - M_B^2} + \langle\mathbf{p}|\tilde{R}(P)|\mathbf{p}'\rangle\,. \qquad (3.65)$$

This leads to the normalization condition in MW, CJ and MNK versions in analogy with Eq. (2.21)

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\,\bar{\tilde{\Phi}}^i_{M_B}(\mathbf{p})\left\{\frac{\partial}{\partial M_B}\Big((\tilde{G}^i_0(M_B;\mathbf{p},\mathbf{p}'))^{-1} - V(\mathbf{p},\mathbf{p}')\Big)\right\}\times$$
$$\times\tilde{\Phi}^i_{M_B}(\mathbf{p}') = 2M_B. \qquad (3.66)$$

Now using the relation

$$(\tilde{G}^i_0(M_B;\mathbf{p},\mathbf{p}'))^{-1} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,(\tilde{G}^i_0(M_B;\mathbf{p}))^{-1}\,, \qquad (3.67)$$

and the fact that $V(\mathbf{p},\mathbf{p}')$ does not depend on M_B, the normalization condition can be rewritten as

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\bar{\tilde{\Phi}}^i_{M_B}(\mathbf{p})\left\{\frac{\partial}{\partial M_B}\,(\tilde{G}^i_0(M_B;\mathbf{p}))^{-1}\right\}\tilde{\Phi}^i_{M_B}(\mathbf{p}) = 2M_B\,. \qquad (3.68)$$

If now one substitutes here the expression for the free Green function given by Eq. (3.59), one obtains (below, we drop the superscript «i» labeling various versions)

$$\sum_{\alpha_1\alpha_2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\tilde{\bar{\Phi}}^{(\alpha_1\alpha_2)}_{M_B}(\mathbf{p})\,f_{12}^{(\alpha_1\alpha_2)}(M_B;p)\,\tilde{\Phi}^{(\alpha_1\alpha_2)}_{M_B}(\mathbf{p})=2M_B\,, \tag{3.69}$$

where

$$f_{12}^{(\alpha_1\alpha_2)}(M_B;p)=\frac{\partial}{\partial M_B}\left(\frac{d(M_B;p)}{D^{\alpha_1\alpha_2}(M_B;p)}\right). \tag{3.70}$$

By using the explicit expressions for $D^{(\alpha_1\alpha_2)}$ and d given by Eq. (3.60), we obtain

$$\begin{aligned}
\text{MW} \quad &: \quad f_{12}^{(\alpha_1\alpha_2)}=\frac{\alpha_1E_1+\alpha_2E_2}{M_B}\,,\\
\text{CJ} \quad &: \quad f_{12}^{(\alpha_1\alpha_2)}=\frac{2(w_1+w_2)}{M_B}\,\frac{\alpha_1w_1E_1+\alpha_2w_2E_2}{(E_1+\alpha_1w_1)(E_2+\alpha_2w_2)}\,,\\
\text{MNK} \quad &: \quad f_{12}^{(\alpha_1\alpha_2)}=\frac{2}{D^{(\alpha_1\alpha_2)}}\left\{\left[\frac{M_BB}{R}(1-y^2)+\frac{M_B^2}{2}-2\left(\frac{R-M_B}{2y}\right)^2\right]-\right.\\
&-\frac{B}{D^{(\alpha_1\alpha_2)}}\left[\left(M_B+\frac{\alpha_1w_1+\alpha_2w_2}{2}\right)R-\frac{M_B^2}{2}+2\left(\frac{R-M_B}{2y}\right)^2+\right.\\
&\left.\left.+(\alpha_1w_1-\alpha_2w_2)\left(\frac{R-M_B}{2y}+\frac{M_By}{2}\right)\right]\right\}.
\end{aligned} \tag{3.71}$$

Let us note that the normalization condition (3.69) is valid for the Salpeter and Gross versions as well, provided we choose

$$\begin{aligned}
\text{SAL} \quad &: \quad f_{12}^{(\alpha_1\alpha_2)}=\frac{\alpha_1+\alpha_2}{2}\,,\\
\text{GR} \quad &: \quad f_{12}^{(+\pm)}=1\,,\quad f_{12}^{(-\pm)}=0\,.
\end{aligned} \tag{3.72}$$

Let us emphasize that the Salpeter, MW, CJ and MNK equations can be used for the bound systems with the equal masses of the constituents, whereas the Gross equation cannot — the particle «1» (spectator) should be heavier than the particle «2». This is due to the approximation (3.23) that was done in the free Green function of the Gross equation. Further, it is directly seen from Eqs. (3.62) and (3.63), that the Salpeter and Gross equations are linear eigenvalue equations for determining M_B (the functions $A^{(\alpha_1\alpha_2)}$ do not depend on M_B), whereas

MW, CJ and MNK equations are not, and M_B enters the right-hand side of these equations as well.

Let us now concentrate on the properties of the coefficient functions in detail. In the case of the equal-mass constituents $m_1 = m_2 = m$ and $w_1 = w_2 = w$, we obtain

$$\begin{aligned} \text{MW} &: \quad A^{(\pm\pm)} = 1\,, \quad A^{(\pm\mp)} = \frac{M_B}{2w}\,, \\ \text{CJ} &: \quad A^{(\pm\pm)} = \frac{M_B \pm 2w}{4w}\,, \quad A^{(\pm\mp)} = \frac{M_B}{4w}\,, \\ \text{MNK} &: \quad A^{(\pm\pm)} = \frac{M_B \pm 2w}{2M_B}\,, \quad A^{(\pm\mp)} = \frac{1}{2}\,. \end{aligned} \tag{3.73}$$

It is immediately seen that in the equal-mass case, the M_B drops out from the equations for mixed components $\tilde{\Phi}_{M_B}^{(\pm\mp)}$ in the MW and CJ versions — that is, these components are redundant and can be eliminated in this case.

The functions $f_{12}^{(\alpha_1\alpha_2)}$ in the equal-mass case are given by

$$\begin{aligned} \text{MW} &: \quad f_{12}^{(\pm\pm)} = \pm 1\,, \quad f_{12}^{(\pm\mp)} = 0\,, \\ \text{CJ} &: \quad f_{12}^{(\pm\pm)} = \pm\frac{16w^2}{(M_B \pm 2w)^2}\,, \quad f_{12}^{(\pm\mp)} = 0\,, \\ \text{MNK} &: \quad f_{12}^{(\pm\pm)} = \frac{2((M_B \pm 2w)^2 - 8w^2)}{(M_B \pm 2w)^2}\,, \quad f_{12}^{(\pm\mp)} = 2\,. \end{aligned} \tag{3.74}$$

From these expressions, we immediately see that in the CJ and MNK versions the function $f_{12}^{(--)}$ has the second-order pole at

$$M_B - 2w(p_s) = 0\,, \qquad p_s = \frac{1}{2}\sqrt{M_B^2 - 4m^2}\,. \tag{3.75}$$

It can be shown that in the nonequal mass case in the CJ and MNK versions the function $f_{12}^{(--)}$ has the second-order pole at

$$E_i^2 - w_i^2(p_s) = 0, \quad p_s = \frac{1}{2}\sqrt{M_B^2 + \left(\frac{m_1^2 - m_2^2}{M_B}\right)^2 - 2(m_1 + m_2)^2}, \tag{3.76}$$

while the other components $f_{12}^{(++)}$, $f_{12}^{(+-)}$ and $f_{12}^{(-+)}$ do not have any poles.

3.7. Logunov–Tavkhelidze Quasi-Potential Approach [12]. There exists the theoretical possibility to construct the $3D$ analogue of the BS equation without using the instantaneous approximation. To this end, one may use the Logunov–Tavkhelidze quasi-potential approach formulated in Ref. 12 for the case of two spinless particles, and generalized in Ref. 13 to the case of two fermions.

We introduce the following definition, for any operator $A(P)$ in the momentum space,

$$\langle \mathbf{p}|\tilde{A}(P)|\mathbf{p}'\rangle = \int \frac{dp_0}{2\pi}\,\frac{dp_0'}{2\pi}\,\langle p|A(P)|p'\rangle\,. \tag{3.77}$$

Then, from Eq. (2.2) one obtains

$$\tilde{G} = \tilde{G}_0 + \widetilde{G_0KG}, \tag{3.78}$$

where $\tilde{G}_0$ is given by Eq. (3.8).

Due to the fact that the operator Π defined by Eq. (3.10) cannot be inverted, the inverse operator of $\tilde{G}_0$ does not exist as well. As a result, one cannot define the interaction potential by the formula analogous to Eq. (3.33). In order to overcome this problem, it is convenient to introduce the Green function $\underline{\tilde{G}_0}$ defined by

$$\begin{aligned}\underline{\tilde{G}_0}(P;\mathbf{p}) &= i\big[P_0 - h_1(\mathbf{p}_1) - h_2(\mathbf{p}_2)\big]^{-1}\gamma_0^{(1)}\otimes\gamma_0^{(2)}\,,\\ \tilde{G}_0(P;\mathbf{p},\mathbf{p}') &= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,\underline{\tilde{G}_0}(P;\mathbf{p})\,\gamma_0^{(1)}\otimes\gamma_0^{(2)}\,\hat{\Pi}\,,\end{aligned} \tag{3.79}$$

where $\hat{\Pi} = \Pi\gamma_0^{(1)}\otimes\gamma_0^{(2)}$.

Now, the inverse of the operator $\underline{\tilde{G}_0}(P;\mathbf{p},\mathbf{p}') = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,\underline{\tilde{G}_0}(P;\mathbf{p})$ exists, and one may define

$$\underline{\tilde{G}} = \underline{\tilde{G}_0} + \widetilde{G_0KG}\,, \tag{3.80}$$

from which follows that

$$\underline{\tilde{G}} = \tilde{G} + \underline{\tilde{G}_0}\,(1 - \gamma_0^{(1)}\otimes\gamma_0^{(2)}\hat{\Pi})\,. \tag{3.81}$$

It is clear, that near the bound-state pole the Green functions $\tilde{G}$ and $\underline{\tilde{G}}$ differ only by the regular term, since $\underline{\tilde{G}_0}$ is regular in the vicinity of the pole. Consequently, in order to derive the bound-state equation, one may use $\underline{\tilde{G}}$ instead of $\tilde{G}$, and define the interaction potential according to

$$\big[\,\underline{\tilde{G}_0}\,\big]^{-1} - \big[\,\underline{\tilde{G}}\,\big]^{-1} = \underline{\tilde{V}}\,, \tag{3.82}$$

from which it follows that

$$\underline{\tilde{G}} = \underline{\tilde{G}_0} + \underline{\tilde{G}_0}\,\underline{\tilde{V}}\,\underline{\tilde{G}}\,, \tag{3.83}$$

and

$$\underline{\tilde{V}} = \left[\,\underline{\tilde{G}_0}\,\right]^{-1} \widetilde{G_0 K G} \left[\,\underline{\tilde{G}}\,\right]^{-1}. \tag{3.84}$$

For any given kernel $K(P; p, p')$ the interaction potential can be constructed by using Eq. (3.84). The equation for the bound-state wave function in the c.m. frame can be obtained directly from Eq. (3.82)

$$\underline{\tilde{G}}^{-1}(M_B)\,|\tilde{\Phi}_{M_B}\rangle = 0\,, \qquad |\tilde{\Phi}_{M_B}\rangle = \underline{\tilde{G}}\,\underline{\tilde{V}}\,|\tilde{\Phi}_{M_B}\rangle\,. \tag{3.85}$$

Defining the quasi-potential as

$$V_q(M_B; \mathbf{p}, \mathbf{p}') = i\gamma_0^{(1)} \otimes \gamma_0^{(2)}\, \underline{\tilde{V}}(M_B; \mathbf{p}, \mathbf{p}')\,, \tag{3.86}$$

we obtain

$$\left[M_B - h_1(\mathbf{p}) - h_2(-\mathbf{p})\right] \tilde{\Phi}_{M_B}(\mathbf{p}) = \int \frac{d^3\mathbf{p}'}{(2\pi)^3}\; V_q(M_B; \mathbf{p}, \mathbf{p}')\; \tilde{\Phi}_{M_B}(\mathbf{p}'). \tag{3.87}$$

Note that in the instantaneous approximation (3.1), the interaction potential reduces to

$$\underline{\tilde{V}}(M_B; \mathbf{p}, \mathbf{p}') = \gamma_0^{(1)} \otimes \gamma_0^{(2)}\, \Pi(\mathbf{p})\, \gamma_0^{(1)} \otimes \gamma_0^{(2)}\, K_{\rm st}(\mathbf{p}, \mathbf{p}')\,. \tag{3.88}$$

As a result, the quasi-potential equation reduces to the Salpeter equation.

The first-order quasi-potential is defined by Eqs. (3.82) and (3.86), if in the former the full Green function G is substituted by the free Green function G_0

$$V_q^{(1)}(M_B; \mathbf{p}, \mathbf{p}') = i\gamma_0^{(1)} \otimes \gamma_0^{(2)}\; \langle \mathbf{p}| \left[\,\underline{\tilde{G}_0}\,\right]^{-1} \widetilde{G_0 K G_0} \left[\,\underline{\tilde{G}_0}\,\right]^{-1} |\mathbf{p}'\rangle \tag{3.89}$$

from which, in the static approximation one obtains

$$V_q^{(1)}(M_B; \mathbf{p}, \mathbf{p}') = \Pi(\mathbf{p})\, \gamma_0^{(1)} \otimes \gamma_0^{(2)}\; iK_{\rm st}(\mathbf{p}, \mathbf{p}')\Pi(\mathbf{p})\,. \tag{3.90}$$

It is seen that, unlike the full quasi-potential equation, the first-order equation does not reduce to the Salpeter equation in the static limit. Only when one may neglect the negative-frequency component of the bound-state wave function, the first-order equation again reduces to the Salpeter equation in the static limit. Here we note, that the first-order quasi-potential equation was used in Ref. 14 in order to evaluate the dynamical retardation effect in the $q\bar{q}$ bound system mass spectrum (i.e., the effect that stems from the deviation of the BS kernel from the static one).

In the rest of this subsection, we consider the normalization condition for the quasi-potential bound-state wave function. Near the bound-state pole, the $3D$ Green function $\tilde{G}(P)$ develops a pole (3.65). Using the fact that in the vicinity of the bound-state pole the Green functions $\tilde{G}(P)$ and $\underline{\tilde{G}(P)}$ coincide up to the regular term, it is straightforward to obtain the normalization condition

$$i \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} \bar{\tilde{\Phi}}_{M_B}(\mathbf{p}) \left[\frac{\partial}{\partial M_B} \left((\underline{\tilde{G}_0}(M_B;\mathbf{p},\mathbf{p}'))^{-1} - \underline{\tilde{V}}(M_B;\mathbf{p},\mathbf{p}') \right) \right] \times \\ \times \tilde{\Phi}_{M_B}(\mathbf{p}') = 2M_B. \quad (3.91)$$

From this equation, using the definition of the conjugate wave function (2.14) and Eqs. (3.79), (3.86), we obtain

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} \tilde{\Phi}^{+}_{M_B}(\mathbf{p}) \left[\mathbf{1} - \frac{\partial}{\partial M_B} \left(V_q(M_B;\mathbf{p},\mathbf{p}') \right) \right] \tilde{\Phi}_{M_B}(\mathbf{p}') = 2M_B. \quad (3.92)$$

As is seen from Eq. (3.92), in the static limit the above normalization condition reduces to the normalization condition for the Salpeter wave function only if one neglects the contribution from the negative-energy component of the wave function.

4. MESON SPECTROSCOPY

4.1. Partial-Wave Decomposition. The properties of the $q\bar{q}$ bound systems in the $3D$ formalism obtained from the BS equation in the static approximation, were studied in Refs. 11,15–35, without making any additional assumptions. Note that these $3D$ equations can be written either, as in Eq. (3.9), for the 2-fermion bound-state wave function [11, 15–19, 21–23, 26, 27, 33, 35], or for the fermion-antifermion bound-state wave function [20,24,25,28–32,34] (the latter is obtained from Eq. (2.25) in the static approximation). Further, one may write down these equations in terms of either the frequency components of the $3D$ wave functions $\tilde{\Phi}^{(\pm\pm)}_{M_B}(\mathbf{p})$ and $\tilde{\Psi}^{(\pm\pm)}_{M_B}(\mathbf{p})$ (the latter denotes the frequency components of the fermion-antifermion wave function), or in terms of their linear combinations $\tilde{\Phi}_{aa}(\mathbf{p})$, $\tilde{\Phi}_{bb}(\mathbf{p})$, etc., $\tilde{\Psi}_{aa}(\mathbf{p})$, $\tilde{\Psi}_{bb}(\mathbf{p})$, etc., see Eqs. (2.24), (2.26). Below, we shall use the form of the $3D$ equations given by (3.61)–(3.63), with the normalization condition given by (3.69)–(3.72) [11, 26, 35].

In order to rewrite the equations explicitly in either of the forms above, one has to specify the explicit spin structure of the interaction potential. This potential consists of several parts. First, there is the one-gluon (OG) exchange piece dominating at short distances. In the Feynman gauge, the spin structure of this piece is given by $\gamma^{(1)}_{\mu} \otimes \gamma^{(2)\mu} = \gamma^{(1)}_0 \otimes \gamma^{(2)}_0 - \boldsymbol{\gamma}^{(1)} \otimes \boldsymbol{\gamma}^{(2)}$. In accordance with the static approximation, however, we neglect the second term in this expression [5]. In addition, there is the confinement (C) piece in the potential that dominates at large distances and leads to the formation of the $q\bar{q}$ bound states. The spin structure of this piece is not known *a priori*. We choose it to be the mixture of a scalar and the zeroth component of a vector. Further, sometimes an additional «instanton-induced» piece, corresponding to the t'Hooft interaction, is included in the potential [25]. The spin structure of this term is given by the equal mixture of scalar and pseudoscalar parts. The rationale for including the latter piece is the following. In the absence of the proper treatment of the Goldstone nature of light pseudoscalar bosons that is due to the spontaneous breaking of chiral symmetry in QCD, the t'Hooft interaction mimics this effect, leading to the large mass splitting between the pseudoscalar and vector mesons. Note that the chiral symmetry can be consistently incorporated in the $3D$ framework (see, e.g., [20]), albeit at a cost of the more involved formalism. For example, in this case the Hamiltonian of the free quark is replaced by

$$h_i(\mathbf{p}_i) = \boldsymbol{\alpha}^{(i)}\mathbf{p}_i + m_i\gamma_0^{(i)} \to B_i(\mathbf{p}_i)\boldsymbol{\alpha}^{(i)}\mathbf{p}_i + A_i(\mathbf{p}_i)\gamma_0^{(i)}\,, \tag{4.1}$$

where $A_i(\mathbf{p}_i)$ and $B_i(\mathbf{p}_i)$ are determined by solving the gap equation for the quark propagator with the static potential. Below, however, we do not consider this approach.

Thus, the spin structure of the static potential we shall be using, is given by

$$\begin{aligned} V &= \gamma_0^{(1)}\otimes\gamma_0^{(2)}\,V_{\rm OG}(r) + (x\gamma_0^{(1)}\otimes\gamma_0^{(2)} + (1-x)I^{(1)}\otimes I^{(2)})\,V_{\rm C}(r) + \\ &+ (I^{(1)}\otimes I^{(2)} + \gamma_5^{(1)}\otimes\gamma_5^{(2)})\,V_{\rm T}(r)\,, \end{aligned} \tag{4.2}$$

where the last term corresponds to the t'Hooft interaction, all potentials are assumed to be local, and $0 \le x \le 1$.

Let us now turn to the wave function. It is possible to «solve» the constraints imposed on the frequency components, defining

$$\begin{aligned} \tilde{\Phi}_{M_B}^{(\alpha_1\alpha_2)}(\mathbf{p}) &= \mathcal{N}_{12}^{(\alpha_1\alpha_2)}(p)\begin{pmatrix} 1 \\ \dfrac{\alpha_1(\boldsymbol{\sigma}^{(1)}\mathbf{p})}{w_1+\alpha_1 m_1}\end{pmatrix}\otimes\begin{pmatrix} 1 \\ \dfrac{-\alpha_2(\boldsymbol{\sigma}^{(2)}\mathbf{p})}{w_2+\alpha_2 m_2}\end{pmatrix}, \\ \chi_{M_B}^{(\alpha_1\alpha_2)}(\mathbf{p}) &= \begin{pmatrix} \tilde{\Phi}_{aa}^{(\alpha_1\alpha_2)}(\mathbf{p}) \\ \tilde{\Phi}_{ab}^{(\alpha_1\alpha_2)}(\mathbf{p}) \\ \tilde{\Phi}_{ba}^{(\alpha_1\alpha_2)}(\mathbf{p}) \\ \tilde{\Phi}_{bb}^{(\alpha_1\alpha_2)}(\mathbf{p}) \end{pmatrix}, \end{aligned} \tag{4.3}$$

where

$$\mathcal{N}_{12}^{(\alpha_1\alpha_2)}(p) = \sqrt{\frac{w_1+\alpha_1 m_1}{2w_1}}\,\sqrt{\frac{w_2+\alpha_2 m_2}{2w_2}} = \mathcal{N}_1^{(\alpha_1)}(p)\,\mathcal{N}_2^{(\alpha_2)}(p)\,, \tag{4.4}$$

and $\chi_{M_B}^{(\alpha_1\alpha_2)}(\mathbf{p})$ is the unconstrained Pauli 2×2 spinor. For this spinor, the following system of equations is obtained

$$\big[M_B-(\alpha_1 w_1+\alpha_2 w_2)\big]\chi_{M_B}^{(\alpha_1\alpha_2)}(\mathbf{p}) = \\ = A^{(\alpha_1\alpha_2)}(M_B;p)\sum_{\alpha_1'\alpha_2'}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\,V_{\text{eff}}^{(\alpha_1\alpha_2\alpha_1'\alpha_2')}(\mathbf{p},\mathbf{p}')\chi_{M_B}^{(\alpha_1'\alpha_2')}(\mathbf{p}'), \quad (4.5)$$

where

$$V_{\text{eff}}^{(\alpha_1\alpha_2\alpha_1'\alpha_2')}(\mathbf{p},\mathbf{p}') = \mathcal{N}_{12}^{(\alpha_1\alpha_2)}(p)\big(V_1(\mathbf{p}-\mathbf{p}')B_1^{(\alpha_1\alpha_2\alpha_1'\alpha_2')}(\mathbf{p},\mathbf{p}') + \\ +V_2(x;\mathbf{p}-\mathbf{p}')B_2^{(\alpha_1\alpha_2\alpha_1'\alpha_2')}(\mathbf{p},\mathbf{p}')\big) + \\ +V_{\text{T}}(\mathbf{p}-\mathbf{p}')\left(B_1^{(\alpha_1\alpha_2\alpha_1'\alpha_2')}(\mathbf{p},\mathbf{p}') - B_2^{(\alpha_1\alpha_2\alpha_1'\alpha_2')}(\mathbf{p},\mathbf{p}') - \\ -B_3^{(\alpha_1\alpha_2\alpha_1'\alpha_2')}(\mathbf{p},\mathbf{p}')\big)\,\mathcal{N}_{12}^{(\alpha_1'\alpha_2')}(p') \quad (4.6)$$

and

$$B_1^{(\alpha_1\alpha_2\alpha_1'\alpha_2')}(\mathbf{p},\mathbf{p}')=1+\frac{\alpha_1\alpha_2\alpha_1'\alpha_2'\,(\boldsymbol{\sigma}^{(1)}\mathbf{p})(\boldsymbol{\sigma}^{(2)}\mathbf{p})(\boldsymbol{\sigma}^{(1)}\mathbf{p}')(\boldsymbol{\sigma}^{(2)}\mathbf{p}')}{(w_1+\alpha_1 m_1)(w_2+\alpha_2 m_2)(w_1'+\alpha_1' m_1)(w_2'+\alpha_2' m_2)},$$

$$B_2^{(\alpha_1\alpha_2\alpha_1'\alpha_2')}(\mathbf{p},\mathbf{p}') = \frac{\alpha_1\alpha_1'\,(\boldsymbol{\sigma}^{(1)}\mathbf{p})(\boldsymbol{\sigma}^{(1)}\mathbf{p}')}{(w_1+\alpha_1 m_1)(w_1'+\alpha_1' m_1)} + \frac{\alpha_2\alpha_2'\,(\boldsymbol{\sigma}^{(2)}\mathbf{p})(\boldsymbol{\sigma}^{(2)}\mathbf{p}')}{(w_2+\alpha_2 m_2)(w_2'+\alpha_2' m_2)},$$

$$B_3^{(\alpha_1\alpha_2\alpha_1'\alpha_2')}(\mathbf{p},\mathbf{p}') = \frac{\alpha_1\alpha_2\,(\boldsymbol{\sigma}^{(1)}\mathbf{p})(\boldsymbol{\sigma}^{(2)}\mathbf{p})}{(w_1+\alpha_1 m_1)(w_2+\alpha_2 m_2)} + \frac{\alpha_1'\alpha_2'\,(\boldsymbol{\sigma}^{(1)}\mathbf{p}')(\boldsymbol{\sigma}^{(2)}\mathbf{p}')}{(w_1'+\alpha_1' m_1)(w_2'+\alpha_2' m_2)} - \\ -\frac{\alpha_1\alpha_2'\,(\boldsymbol{\sigma}^{(1)}\mathbf{p})(\boldsymbol{\sigma}^{(2)}\mathbf{p}')}{(w_1+\alpha_1 m_1)(w_2'+\alpha_2' m_2)} - \frac{\alpha_1'\alpha_2\,(\boldsymbol{\sigma}^{(1)}\mathbf{p}')(\boldsymbol{\sigma}^{(2)}\mathbf{p})}{(w_1'+\alpha_1' m_1)(w_2+\alpha_2 m_2)}, \quad (4.7)$$

$$V_1 = V_{\text{OG}} + V_{\text{C}}\,, \qquad V_2(x) = V_{\text{OG}} + (2x-1)V_{\text{C}}\,. \quad (4.8)$$

The functions $V_{\text{OG}}(\mathbf{p}-\mathbf{p}')$, $V_{\text{C}}(\mathbf{p}-\mathbf{p}')$ and $V_{\text{T}}(\mathbf{p}-\mathbf{p}')$ are the Fourier-transform of the local potentials $V_{\text{OG}}(r)$, $V_{\text{C}}(r)$ and $V_{\text{T}}(r)$, respectively.

The normalization condition for the Pauli spinors $\chi_{M_B}^{(\alpha_1\alpha_2)}(\mathbf{p})$ follows from (3.69)

$$\sum_{\alpha_1\alpha_2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\chi_{M_B}^{+(\alpha_1\alpha_2)}(\mathbf{p})\,f_{12}^{(\alpha_1\alpha_2)}(p)\,\chi_{M_B}^{(\alpha_1\alpha_2)}(\mathbf{p}) = 2M_B\,. \quad (4.9)$$

The partial-wave expansion of the Pauli spinor $\chi_{M_B}^{(\alpha_1\alpha_2)}(\mathbf{p})$ is given by

$$\chi_{M_B}^{(\alpha_1\alpha_2)}(\mathbf{p})=\sum_{LSJM_J}\chi_{LSJM_J}^{(\alpha_1\alpha_2)}(\mathbf{p})=\sum_{LSJM_J}\langle\mathbf{n}|LSJM_J\rangle R_{LSJ}^{(\alpha_1\alpha_2)}(p),\ \mathbf{n}=\frac{\mathbf{p}}{p},\tag{4.10}$$

where $R_{LSJ}^{(\alpha_1\alpha_2)}(p)$ denote the radial wave functions, and L, S, J, M_J stand for the total orbital angular momentum, total spin, total angular momentum, and the projection of the total angular momentum of the $q\bar{q}$ system, respectively.

The partial-wave expansion of the potentials reads as

$$\begin{gathered}V(\mathbf{p}-\mathbf{p}')=(2\pi)^3\sum_{\bar{L}\bar{S}\bar{J}\bar{M}_{\bar{J}}}\langle\mathbf{n}|\bar{L}\bar{S}\bar{J}\bar{M}_{\bar{J}}\rangle V_{\mathrm{I}}^{\bar{L}}(p,p')\langle\bar{L}\bar{S}\bar{J}\bar{M}_{\bar{J}}|\mathbf{n}'\rangle,\\ \mathrm{I}=\mathrm{OG},\mathrm{C},\mathrm{T},\end{gathered}\tag{4.11}$$

where

$$V_{\mathrm{I}}^{\bar{L}}(p,p')=\frac{2}{\pi}\int_0^\infty r^2dr\, j_{\bar{L}}(pr)\,V_{\mathrm{I}}(r)\,j_{\bar{L}}(p'r)\,,\tag{4.12}$$

$j_{\bar{L}}$ being the spherical Bessel function.

Using the fact that for the spherical potentials $V_{\mathrm{I}}(\mathbf{p}-\mathbf{p}')=V_{\mathrm{I}}(|\mathbf{p}-\mathbf{p}'|)$, one may write

$$V_{\mathrm{I}}^{\bar{L}}(p,p')=\frac{1}{4\pi^2}\int_{-1}^{1}dz\,P_{\bar{L}}(z)\,V_{\mathrm{I}}\Big(\sqrt{p^2+p'^2-2pp'z}\Big)\,,\tag{4.13}$$

where $P_{\bar{L}}(z)$ denotes the Legendre polynomial. The above form is convenient when the function $V_{\mathrm{I}}(p,p';z)$ can be written in the analytic form.

In order to carry out the partial-wave expansion in the bound-state equation, it is convenient to introduce the operators $\mathbf{S}=\frac{1}{2}(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})$, $\boldsymbol{\sigma}=\frac{1}{2}(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)})$, instead of the individual spin operators $\boldsymbol{\sigma}^{(i)}$, $i=1,2$. At the next step, one uses the known values of matrix elements of the operators $\mathbf{Sn}$, $\boldsymbol{\sigma}\mathbf{n}$, and the tensor operators

$$\begin{gathered}S_{12}=3(\boldsymbol{\sigma}^{(1)}\mathbf{n})(\boldsymbol{\sigma}^{(2)}\mathbf{n})-(\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)})=6(\mathbf{Sn})^2-2\mathbf{S}^2,\\ (\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)})=2\mathbf{S}^2-3\end{gathered}\tag{4.14}$$

between the different spin-angular momentum states

$$\begin{aligned}&\langle LSJM_J|\begin{pmatrix}\mathbf{Sn}\\ \boldsymbol{\sigma}\mathbf{n}\end{pmatrix}|L'S'J'M_{J'}\rangle=\\ &=\delta_{JJ'}\delta_{M_JM_{J'}}\langle L||\mathbf{n}||L'\rangle\langle S||\begin{pmatrix}\mathbf{S}\\ \boldsymbol{\sigma}\end{pmatrix}||S'\rangle\times\,W(LL'SS';1J)(-1)^{S'+L-J}\,,\end{aligned}$$

$$\begin{aligned}\langle L||\mathbf{n}||L'\rangle &= \sqrt{2L'+1}\;\langle L'100|L0\rangle, \\ \langle S||\mathbf{S}||S'\rangle &= \delta_{SS'}\sqrt{(2S+1)S(S+1)}\,, \\ \langle S||\boldsymbol{\sigma}||S'\rangle &= (-1)^{S+1}\sqrt{3}\delta_{S|S'-1|},\end{aligned} \tag{4.15}$$

$$\langle LSJM_J|\hat{S}_{12}|L'S'J'M_{J'}\rangle = \delta_{JJ'}\delta_{M_JM'_J}\,\frac{\langle LSJ||\hat{S}_{12}||L'S'J'\rangle}{\sqrt{2J+1}} =$$

$$= (-1)^{L-L'}\delta_{S1}\delta_{SS'}\sqrt{120(2L'+1)}\langle L'200|L0\rangle W(LJ21';1L')\,,$$

where W stand for the conventional Racah coefficients. The bound-state equations for the radial wave functions $R^{(\alpha_1\alpha_2)}_{LSJ}(p)$ are then obtained straightforwardly

$$\begin{aligned}
&\left[\,M_B-(\alpha_1 w_1+\alpha_2 w_2)\,\right] R^{(\alpha_1\alpha_2)}_{J\binom{0}{1}J}(p) = A^{(\alpha_1\alpha_2)}(M_B;p)\sum_{\alpha_1\alpha_2}\int_0^\infty p'^2dp'\times\\
&\times\Bigg\{\Bigg[\Bigg((\mathcal{N}^{(\alpha_1\alpha_2)}_{12}(p)\mathcal{N}^{(\alpha'_1\alpha'_2)}_{12}(p')+\alpha_1\alpha_2\alpha'_1\alpha'_2\mathcal{N}^{(-\alpha_1-\alpha_2)}_{12}(p)\times\\
&\times\mathcal{N}^{(-\alpha'_1-\alpha'_2)}_{12}(p'))V^J_1(p,p')+\\
&+(\alpha_1\alpha'_1\mathcal{N}^{(-\alpha_1\alpha_2)}_{12}(p)\mathcal{N}^{(-\alpha'_1\alpha'_2)}_{12}(p')+\alpha_2\alpha'_2\mathcal{N}^{(\alpha_1-\alpha_2)}_{12}(p)\mathcal{N}^{(\alpha'_1-\alpha'_2)}_{12}(p'))\times\\
&\times V^{\binom{0}{1}}_{2\oplus J}(x;p,p')\Bigg) R^{(\alpha'_1\alpha'_2)}_{J\binom{0}{1}J}(p')-\\
&-(\alpha_1\alpha'_1\mathcal{N}^{(-\alpha_1\alpha_2)}_{12}(p)\mathcal{N}^{(-\alpha'_1\alpha'_2)}_{12}(p')-\alpha_2\alpha'_2\mathcal{N}^{(\alpha_1-\alpha_2)}_{12}(p)\times\\
&\times\mathcal{N}^{(\alpha'_1-\alpha'_2)}_{12}(p'))V_{2\ominus J}(x;p,p')\,R^{(\alpha'_1\alpha'_2)}_{J\binom{1}{0}J}(p')\Bigg]+\\
&+\Bigg[\Bigg((\mathcal{N}^{(\alpha_1\alpha_2)}_{12}(p)\mathcal{N}^{(\alpha'_1\alpha'_2)}_{12}(p')+\alpha_1\alpha_2\alpha'_1\alpha'_2\mathcal{N}^{(-\alpha_1-\alpha_2)}_{12}(p)\mathcal{N}^{(-\alpha'_1-\alpha'_2)}_{12}(p'))\pm\\
&\pm(\alpha_1\alpha_2\mathcal{N}^{(-\alpha_1-\alpha_2)}_{12}(p)+\alpha'_1\alpha'_2\mathcal{N}^{(-\alpha'_1-\alpha'_2)}_{12}(p'))\,V^J_{\mathrm{T}}(p,p')-\\
&-(\alpha_1\alpha'_1\mathcal{N}^{(-\alpha_1\alpha_2)}_{12}(p)\mathcal{N}^{(-\alpha'_1\alpha'_2)}_{12}(p')+\alpha_2\alpha'_2\mathcal{N}^{(\alpha_1-\alpha_2)}_{12}(p)\mathcal{N}^{(\alpha'_1-\alpha'_2)}_{12}(p')\pm\\
&\pm(\alpha_1\alpha_2\mathcal{N}^{(-\alpha_1-\alpha_2)}_{12}(p)+\alpha'_1\alpha'_2\mathcal{N}^{(-\alpha'_1-\alpha'_2)}_{12}(p')))V^{\binom{0}{1}}_{\mathrm{T}\oplus J}(p,p')\Bigg) R^{(\alpha'_1\alpha'_2)}_{J\binom{0}{1}J}(p')+\\
&+\Bigg((\alpha_1\alpha'_1\mathcal{N}^{(-\alpha_1\alpha_2)}_{12}(p)\mathcal{N}^{(-\alpha'_1\alpha'_2)}_{12}(p')-\alpha_2\alpha'_2\mathcal{N}^{(\alpha_1-\alpha_2)}_{12}(p)\mathcal{N}^{(\alpha'_1-\alpha'_2)}_{12}(p')\mp\\
&\mp(\alpha_1\alpha_2\mathcal{N}^{(-\alpha_1-\alpha_2)}_{12}(p)-\alpha'_1\alpha'_2\mathcal{N}^{(-\alpha'_1-\alpha'_2)}_{12}(p')))V_{\mathrm{T}\ominus J}(p,p')\Bigg)R^{(\alpha'_1\alpha'_2)}_{J\binom{1}{0}J}(p')\Bigg]\Bigg\},
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
&\left[\, M_B-(\alpha_1 w_1+\alpha_2 w_2)\,\right] R^{(\alpha_1\alpha_2)}_{J\pm 11J}(p)=A^{(\alpha_1\alpha_2)}(M_B;p)\sum_{\alpha_1\alpha_2}\int_0^\infty p'^2dp'\times\\
&\times\Bigg\{\Bigg[\Bigg(\mathcal{N}^{(\alpha_1\alpha_2)}_{12}(p)\mathcal{N}^{(\alpha'_1\alpha'_2)}_{12}(p')V_1^{J\mp1}(p,p')+\alpha_1\alpha_2\alpha'_1\alpha'_2\mathcal{N}^{(-\alpha_1-\alpha_2)}_{12}(p)\times\\
&\times\mathcal{N}^{(-\alpha'_1-\alpha'_2)}_{12}(p')V_{1(J\pm1)J}(p,p')+(\alpha_1\alpha'_1\mathcal{N}^{(-\alpha_1\alpha_2)}_{12}(p)\mathcal{N}^{(-\alpha'_1\alpha'_2)}_{12}(p')+\\
&+\alpha_2\alpha'_2\mathcal{N}^{(\alpha_1-\alpha_2)}_{12}(p)\mathcal{N}^{(\alpha'_1-\alpha'_2)}_{12}(p'))V_2^J(x;p,p')\Bigg)R^{(\alpha'_1\alpha'_2)}_{J\pm11J}(p')+\\
&+\Bigg(\alpha_1\alpha_2\alpha'_1\alpha'_2\mathcal{N}^{(-\alpha_1-\alpha_2)}_{12}(p)\mathcal{N}^{(-\alpha'_1-\alpha'_2)}_{12}(p')\frac{2}{2J{+}1}V_{1\ominus J}(p,p')\Bigg)R^{(\alpha'_1\alpha'_2)}_{J\mp11J}(p')\Bigg]+\\
&+\Bigg[\Bigg((\mathcal{N}^{(\alpha_1\alpha_2)}_{12}(p)\mathcal{N}^{(\alpha'_1\alpha'_2)}_{12}(p')\pm(\alpha_1\alpha_2\mathcal{N}^{(-\alpha_1-\alpha_2)}_{12}(p)+\alpha'_1\alpha'_2\mathcal{N}^{(-\alpha'_1-\alpha'_2)}_{12}(p'))\times\\
&\times\frac{1}{2J+1})V_{\mathrm{T}}^{J\pm1}(p,p')+\alpha_1\alpha_2\alpha'_1\alpha'_2\mathcal{N}^{(-\alpha_1-\alpha_2)}_{12}(p)\mathcal{N}^{(-\alpha'_1-\alpha'_2)}_{12}(p')\times\\
&\times V_{\mathrm{T}(J\pm1)J}(p,p')-(\alpha_1\alpha'_1\mathcal{N}^{(-\alpha_1\alpha_2)}_{12}(p)\mathcal{N}^{(-\alpha'_1\alpha'_2)}_{12}(p')+\\
&+\alpha_2\alpha'_2\mathcal{N}^{(\alpha_1-\alpha_2)}_{12}(p)\mathcal{N}^{(\alpha'_1-\alpha'_2)}_{12}(p')\pm(\alpha_1\alpha'_2\mathcal{N}^{(-\alpha_1\alpha_2)}_{12}(p)\mathcal{N}^{(\alpha'_1-\alpha'_2)}_{12}(p')+\\
&+\alpha_2\alpha'_1\mathcal{N}^{(\alpha_1-\alpha_2)}_{12}(p)\mathcal{N}^{(-\alpha'_1\alpha'_2)}_{12}(p'))\frac{1}{2J+1})V_{\mathrm{T}}^J(p,p')\Bigg)R^{(\alpha'_1\alpha'_2)}_{J\pm11J}(p')+\\
&+\Bigg(\alpha_1\alpha_2\alpha'_1\alpha'_2\mathcal{N}^{(-\alpha_1-\alpha_2)}_{12}(p)\mathcal{N}^{(-\alpha'_1-\alpha'_2)}_{12}(p')\frac{2}{2J+1}V_{\mathrm{T}\ominus J}(p,p')-\\
&-(\alpha_1\alpha'_1\mathcal{N}^{(-\alpha_1\alpha_2)}_{12}(p)\mathcal{N}^{(-\alpha'_1\alpha'_2)}_{12}(p')+\alpha_2\alpha'_2\mathcal{N}^{(\alpha_1-\alpha_2)}_{12}(p)\times\\
&\times\mathcal{N}^{(\alpha'_1-\alpha'_2)}_{12}(p'))\frac{2\sqrt{J(J{+}1)}}{2J{+}1}V_{\mathrm{T}}^{J\mp1}(p,p')\mp(\alpha_1\alpha'_2\mathcal{N}^{(-\alpha_1\alpha_2)}_{12}(p)\mathcal{N}^{(\alpha'_1-\alpha'_2)}_{12}(p')+\\
&+\alpha_2\alpha'_1\mathcal{N}^{(\alpha_1-\alpha_2)}_{12}(p)\mathcal{N}^{(-\alpha'_1\alpha'_2)}_{12}(p'))\frac{2\sqrt{J(J+1)}}{2J+1}V_{\mathrm{T}}^J(p,p')\Bigg)R^{(\alpha'_1\alpha'_2)}_{J\mp11J}(p')\Bigg]\Bigg\},
\end{aligned}\tag{4.17}$$

where

$$\begin{aligned}
&V_{\mathrm{A}\oplus J}^{\binom{0}{1}}(p,p')=\frac{1}{2J{+}1}\left[\begin{pmatrix}J\\J+1\end{pmatrix}V_{\mathrm{A}}^{J-1}(p,p')+\begin{pmatrix}J+1\\J\end{pmatrix}V_{\mathrm{A}}^{J+1}(p,p')\right],\\
&V_{\mathrm{A}\ominus J}(p,p')=\frac{\sqrt{J(J+1)}}{2J+1}\left[V_{\mathrm{A}}^{J-1}(p,p')-V_{\mathrm{A}}^{J+1}(p,p')\right],\\
&V_{\mathrm{A}(J\pm1)J}(p,p')=\frac{1}{(2J+1)^2}\left[V_{\mathrm{A}}^{J\pm1}(p,p')+4J(J+1)V_{\mathrm{A}}^{J\mp1}(p,p')\right],
\end{aligned}$$

$$\mathrm{A}=1,2,\mathrm{T}. \tag{4.18}$$

The normalization condition in terms of radial wave functions has a particularly simple form

$$\sum_{LS}\sum_{\alpha_1\alpha_2}\int_0^\infty \frac{p^2dp}{(2\pi)^3}\, f_{12}^{(\alpha_1\alpha_2)}(M_B;p)\,\left[R_{LSJ}^{(\alpha_1\alpha_2)}(p)\right]^2 = 2M_B\,, \tag{4.19}$$

where $LS = J0,\ J1$ or $J \pm 11$ corresponding to the system of equations (4.16) and (4.17).

4.2. Dynamical Input. For solving the bound-state equation, one needs further to specify the interquark potentials $V_{\rm OG},\ V_{\rm C},\ V_{\rm T}$, introduced above. Let us start from the confining part of the potential. It is believed that the explicit form of this potential (i.e., its dependence on the interquark distance) is in principle, derivable from QCD. At present, however, the only tangible theoretical constraint on the form of this potential is the linear growth at large distances obtained within the quenched lattice QCD [36]. Less compelling arguments based on the background field technique, were provided to justify the harmonic oscillator-type ($\sim r^2$) behavior of the confining potential at small distances. With no rigorous solution of the problem in sight, one may use the potential that interpolates between the «known» behavior of the potential in different limiting situations [38, 39] (for a slightly modified version, see [26])

$$V_{\rm C}(r) = \frac{4}{3}\,\alpha_{\rm S}(m_{12}^2)\left(\frac{\mu_{12}\omega_0^2 r^2}{2\sqrt{1+A_0m_1m_2r^2}} - V_0\right), \tag{4.20}$$

$$\alpha_{\rm S} = \frac{12\pi}{33-2n_f}\left(\ln\frac{Q^2}{\Lambda_{\rm QCD}^2}\right)^{-1},\ m_{12}=m_1+m_2,\ \mu_{12}=\frac{m_1m_2}{m_{12}}, \tag{4.21}$$

where Q^2 is the momentum transfer squared, and the factor $\frac{4}{3}$ comes from the color-dependent part of the $q\bar{q}$ interaction. n_f is the number of flavors ($n_f = 3$ for u, d, s quarks; $n_f = 4$, for u, d, s, c quarks; $n_f = 5$, for u, d, s, c, b quarks). $\omega_0,\ V_0,\ A_0,\ \Lambda_{\rm QCD}$ are considered to be the free parameters of the model. The potential given by Eq. (4.20), effectively reduces to the harmonic oscillator potential for the light quarks u, d, s, and to the linear potential for the heavy b, c quarks, that meets our expectations. In these limiting cases, the potential takes the form

$$\text{LINEAR}\quad:\quad V_{\rm C}(r) = \frac{4}{3}\alpha_{\rm S}(m_{12}^2)\left(\frac{\omega_0^2}{2m_{12}}\sqrt{\frac{m_1m_2}{A_0}}\,r - V_0\right) \equiv a_1 + b_1 r,$$

$$\text{HARMONIC}\quad:\quad V_{\rm C}(r) = \frac{4}{3}\alpha_{\rm S}(m_{12}^2)\left(\frac{\mu_{12}\omega_0^2}{2}\,r^2 - V_0\right) \equiv a_2 + b_2 r^2. \tag{4.22}$$

The one-gluon exchange potential is given by the standard expression [26]

$$V_{\rm OG}(r) = -\frac{4}{3}\frac{\alpha_{\rm S}(m_{12}^2)}{r} \equiv b_{-1}r^{-1}\,. \qquad (4.23)$$

Noting that

$$r^n = \lim_{\eta\to 0}(-)^n\frac{\partial^n}{\partial\eta^n}\left(\frac{{\rm e}^{-\eta r}}{r}\right), \qquad (4.24)$$

one can rewrite the potentials in the momentum space

$$\text{LINEAR} \;:\; V_{\rm C}(\mathbf{p}-\mathbf{p}') = a_1(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}') + +b_1\lim_{\eta\to 0}\frac{\partial^2}{\partial\eta^2}\left(\frac{4\pi}{|\mathbf{p}-\mathbf{p}'|^2+\eta^2}\right), \qquad (4.25)$$

$$\text{HARMONIC} \;:\; V_{\rm C}(\mathbf{p}-\mathbf{p}') = \left(a_2 - b_2\triangle_{\mathbf{p}'}\right)(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}'), \qquad (4.26)$$

$$\text{ONE-GLUON} \;:\; V_{\rm C}(\mathbf{p}-\mathbf{p}') = b_{-1}\,\frac{4\pi}{|\mathbf{p}-\mathbf{p}'|^2}\,. \qquad (4.27)$$

In order to investigate the properties of the $q\bar{q}$ bound systems, the linear potential was used both in the configuration space [15, 20, 23, 31], and in the momentum space [21, 22, 33, 40]. In the latter case, a special numerical algorithm based on the regularization (4.25), was utilized [22, 40]. In Refs. 25, 28–30, 32, the matrix elements of $V_{\rm C}(r)$ were calculated in the configuration-space basis, in order to encompass the difficulties related to the singular character of the linear potential in the momentum space.

The investigation of the $q\bar{q}$ systems in the framework of Salpeter equation was carried out [17–20, 26], using the harmonic confining potential. MW, CJ and MNK equations with the harmonic confinement were considered in Refs. 11, 35.

Some mathematical problems arise if the one-gluon exchange potential with the *fixed* coupling constant b_{-1} is used for the calculation of the characteristics of $q\bar{q}$ bound systems. Namely, as it was shown in Ref. 28, 20, in this case the Salpeter wave function is divergent at $r \to 0$. For the running coupling constant this divergence is less pronounced but still present — now, the problem occurs in the decay observables which depend on the value of the wave function at $r \to 0$. In order to cure this divergence, in Refs. 28, 30 the following regularization was proposed

$$V_{\rm OG}(r) = -\frac{4}{3}\frac{\alpha_{\rm S}(r)}{r}\,, \qquad \text{for } r > r_0\,,$$

$$V_{\rm OG}(r) = a_g r^2 + b_g\,, \qquad \text{for } r < r_0\,, \qquad (4.28)$$

where

$$\alpha_{\rm S}(r) = \frac{A}{2\ln({\rm e}^{-(\gamma+\mu a)}/a + {\rm e}^{A/(2\alpha_{\rm sat})})}\left[1 - B\ \frac{\ln(2\ln({\rm e}^{-\tilde{\mu}a}/a + {\rm e}^{1/2}))}{2\ln({\rm e}^{-\tilde{\mu}a}/a + {\rm e}^{B/2})}\right], \quad (4.29)$$

where $a = \Lambda_{\rm QCD} r$, $\gamma = 0.577215\ldots$ is the Euler–Mascheroni constant, and $\alpha_{\rm sat} = 0.4$, $\mu = 4$, $\tilde{\mu} = 20$. Further,

$$A = \frac{12\pi}{33 - 2n_f}\,, \qquad B = \frac{6(153 - 19n_f)}{(33 - 2n_f)^2}\,. \quad (4.30)$$

Note, that in Ref. 23, the choice $B = 0$ is adopted. The constants a_g and b_g from Eq. (4.28) are determined from matching of the potential and its first derivative at $r = r_0$. It turns out that the dependence of the $q\bar{q}$ system mass spectrum on the regularization parameter r_0 is very weak provided the latter is chosen to be sufficiently small.

4.3. t'Hooft Interaction. The t'Hooft interaction is used in the form suggested in Ref. 25. The point-like potential in the configuration space would lead to the divergences. For this reason, the following regularization of the potential is considered

$$V_{\rm T}(r) \to 4\hat{g} V_{\rm T,reg}(r;\Lambda)\,, \quad V_{\rm T,reg}(r;\Lambda) = \frac{1}{(\Lambda\sqrt{\pi})^3}\ \exp\left(-\frac{r^2}{\Lambda^2}\right). \quad (4.31)$$

In the momentum space, we have

$$V_{\rm T,reg}(\mathbf{p} - \mathbf{p}';\Lambda) = \exp\left(-\frac{\Lambda^2(\mathbf{p} - \mathbf{p}')^2}{4}\right). \quad (4.32)$$

Now, using the following representation of the Legendre polynomials

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\,(z^2 - 1)^n\,, \quad (4.33)$$

with the use of the identity

$$\int {\rm e}^{az} z^n dz = \frac{\partial^n}{\partial a^n}\int {\rm e}^{az} dz\,, \quad (4.34)$$

after the partial-wave expansion of the t'Hooft potential we obtain

$$\lim_{\Lambda\to 0} V^{\bar{L}}_{\rm T,reg}(p,p';\Lambda) = \delta_{\bar{L}0} V^{0}_{\rm T,reg}(p,p';0)\,, \quad (4.35)$$

that reflects the point-like character of the t'Hooft interaction. Here,

$$V^{0}_{\rm T,reg}(p,p';\Lambda) = \frac{\exp(-\Lambda^2(p^2 + {p'}^2)/4)}{4\pi^2}\ \frac{4}{\Lambda^2 pp'}\ \sinh\frac{\Lambda^2 pp'}{2}\,. \quad (4.36)$$

In accordance with the Eq. (4.35), all partial waves except $\bar{L}=0$ in the partial-wave expansion of the t'Hooft potential are neglected even at nonzero Λ.

As was mentioned above, the t'Hooft interaction was introduced in order to provide the mass splitting between the pseudoscalar and vector octets within the framework of the constituent quark model, which in QCD is due to the spontaneous breaking of chiral symmetry. The quantity $\hat{g}$ that appears in Eq. (4.31), is the matrix in the flavor space. The matrix elements of this matrix between various meson states

$$\pi,\ \rho,\ K,\ \omega=\eta_n=\frac{1}{\sqrt{2}}(u\bar{u}+d\bar{d})\,,\ \Phi=\eta_s=s\bar{s} \tag{4.37}$$

are [25]

$$\begin{gathered}\langle\pi|\hat{g}|\pi\rangle=-g=\langle\rho|\hat{g}|\rho\rangle\,,\quad\langle K|\hat{g}|K\rangle=-g',\\ \langle\eta_s|\hat{g}|\eta_s\rangle=0\,,\quad\langle\eta_n|\hat{g}|\eta_n\rangle=g\,,\quad\langle\eta_n|\hat{g}|\eta_s\rangle=\sqrt{2}g'\langle\eta_s|\hat{g}|\eta_n\rangle,\end{gathered} \tag{4.38}$$

where g and g' are two independent coupling constants that are considered to be the free parameters of the model.

The η and η' mesons are the superpositions of η_n and η_s. In order to take the mixing into account, we introduce the matrix notations

$$\begin{aligned}\tilde{\Phi}_{M_B}(\mathbf{p})&=\begin{pmatrix}\tilde{\Phi}_{n,M_B}(\mathbf{p})\\ \tilde{\Phi}_{s,M_B}(\mathbf{p})\end{pmatrix},\\ \tilde{G}_0(M_B;\mathbf{p})&=\begin{pmatrix}\tilde{G}_{n,0}(M_B;\mathbf{p}) & 0\\ 0 & \tilde{G}_{s,0}(M_B;\mathbf{p})\end{pmatrix},\\ V_{\mathrm{A}}(\mathbf{p},\mathbf{p}')&=\begin{pmatrix}V_{n,\mathrm{A}}(\mathbf{p},\mathbf{p}') & 0\\ 0 & V_{n,\mathrm{A}}(\mathbf{p},\mathbf{p}')\end{pmatrix},\quad \mathrm{A}=\mathrm{C},\mathrm{OG}\,,\\ V_{\mathrm{T}}(\mathbf{p},\mathbf{p}')&=\begin{pmatrix}V_{nn,\mathrm{T}}(\mathbf{p},\mathbf{p}') & V_{ns,\mathrm{T}}(\mathbf{p},\mathbf{p}')\\ V_{sn,\mathrm{T}}(\mathbf{p},\mathbf{p}') & V_{ss,\mathrm{T}}(\mathbf{p},\mathbf{p}')\end{pmatrix}.\end{aligned} \tag{4.39}$$

The radial wave functions $R^{(\alpha_1\alpha_2)}_{f,000}(p),\ f=n,s$ describing the η and η' mesons, obey the following system of equations

$$\begin{gathered}\left[M_B-(\alpha_1w_1+\alpha_2w_2)\right]R^{(\alpha_1\alpha_2)}_{f,000}(p)=A^{(\alpha_1\alpha_2)}_f(M_B;p)\sum_{\alpha_1'\alpha_2'}\int_0^\infty p'^2dp'\times\\ \times\Bigg\{\Bigg[(\mathcal{N}^{(\alpha_1\alpha_2)}_{f,12}(p)\mathcal{N}^{(\alpha_1'\alpha_2')}_{f,12}(p')+\alpha_1\alpha_2\alpha_1'\alpha_2'\mathcal{N}^{(-\alpha_1-\alpha_2)}_{f,12}(p)\mathcal{N}^{(-\alpha_1'-\alpha_2')}_{f,12}(p'))\times\\ \times V^0_1(p,p')+(\alpha_1\alpha_1'\mathcal{N}^{(-\alpha_1\alpha_2)}_{f,12}(p)\mathcal{N}^{(-\alpha_1'\alpha_2')}_{f,12}(p')+\\ +\alpha_2\alpha_2'\mathcal{N}^{(\alpha_1-\alpha_2)}_{f,12}(p)\mathcal{N}^{(\alpha_1'-\alpha_2')}_{f,12}(p'))V^1_2(p,p')\Bigg]\times\end{gathered}$$

$$\times R_{f,000}^{(\alpha_1'\alpha_2')}(p') + \sum_{f'} \Bigg[(\mathcal{N}_{f,12}^{(\alpha_1\alpha_2)}(p)\mathcal{N}_{f',12}^{(\alpha_1'\alpha_2')}(p') +$$

$$+ \alpha_1\alpha_2\alpha_1'\alpha_2'\mathcal{N}_{f,12}^{(-\alpha_1-\alpha_2)}(p)\mathcal{N}_{f',12}^{(-\alpha_1'-\alpha_2')}(p') +$$

$$+ \alpha_1\alpha_2\mathcal{N}_{f,12}^{(-\alpha_1-\alpha_2)}(p) + \alpha_1'\alpha_2'\mathcal{N}_{f',12}^{(-\alpha_1'-\alpha_2')}(p')) \times$$

$$\times V_{\mathrm{T,reg}}^0(p,p';\Lambda)\langle\eta_f|4\hat{g}|\eta_{f'}\rangle \Bigg] R_{f',000}^{(\alpha_1'\alpha_2')}(p') \Bigg\} . \tag{4.40}$$

The functions $R_{f,000}^{(\alpha_1\alpha_2)}(p),\ f = n, s$ satisfy the normalization condition

$$\int_0^\infty \frac{p^2 dp}{(2\pi)^3} \sum_{\alpha_1\alpha_2} \Bigg[f_{n,12}^{(\alpha_1\alpha_2)}(M_B, p) \Bigg(R_{n,000}^{(\alpha_1\alpha_2)}(p) \Bigg)^2 +$$

$$+ f_{s,12}^{(\alpha_1\alpha_2)}(M_B, p) \Bigg(R_{s,000}^{(\alpha_1\alpha_2)}(p) \Bigg)^2 \Bigg] = 2M_B \, , \tag{4.41}$$

where M_B is either M_η or $M_{\eta'}$.

The equations for other mesonic states can be obtained, replacing $\langle\eta_f|4\hat{g}|\eta_{f'}\rangle$, $f, f' = n, s$ by the corresponding matrix elements from Eq. (4.38).

Note that the mixing in $\Phi - \omega$ and $\eta - \eta'$ systems has been recently also investigated in Refs. [41] within the Nambu–Jona-Lasinio (NJL) model, with an account of the relativistic confinement potential (Lorentz vector structure only) and the t'Hooft interaction.

4.4. Solution of the Equations. One has to specify the numerical procedure for the solution of the system of radial equations (4.16)–(4.17). A possible algorithm looks as follows. One chooses the known basis functions denoted by $R_{nL}(p)$. The radial wave functions are expanded in the linear combinations of the basis functions

$$R_{LSJ}^{(\alpha_1\alpha_2)}(p) = \sqrt{2M_B(2\pi)^3}\, \bar{R}_{LSJ}^{(\alpha_1\alpha_2)}(p) =$$

$$= \sqrt{2M_B(2\pi)^3} \sum_{n=0}^{\infty} c_{nLSJ}^{(\alpha_1\alpha_2)} R_{nL}(p) \, , \tag{4.42}$$

where $c_{nLSJ}^{(\alpha_1\alpha_2)}$ are the coefficients of the expansion. The integral equation for the radial wave functions is then transformed into the system of linear equations for these coefficients. If the truncation is carried out, the finite system of equations is obtained that can be solved by using conventional numerical methods. The convergence of the whole procedure, with more terms taken into account in the expansion (4.42), depends on the successful choice of the basis. In

Refs. 24, 25, 28–32, 34, where the linear confining potential is assumed, the basis functions are chosen in the following manner

$$R_{nL}(y) = N_{nL} y^L L_n^{2L+2}(y) \mathrm{e}^{-y/2}\,, \qquad y = \beta p\,, \tag{4.43}$$

where $L_n^{2L+2}(y)$ are the Laguerre polynomials, and β is the free parameter. In Refs.15, 27, 33, the nonrelativistic oscillator wave functions (again containing the free parameter), were used in spite of the fact that the linear confining potential was assumed. In Refs. 11, 17–19, 21, 35, the same basis functions were used, but without the free parameter, due to the fact that the confining potential was taken in the harmonic form, with the parameters already fixed. Finally, in Ref. 26, the harmonic oscillator basis was used, whereas the confining potential had the general form given by Eq. (4.20).

To clarify the choice of the basis functions, let us consider the nonrelativistic limit of the equations (3.61). In this limit, one can replace $\gamma_0 \to 1$, $\gamma_5 \to 0$, $\boldsymbol{\gamma} \to 0$. Consequently,

$$V \to V_{\mathrm{OG}} + V_{\mathrm{C}} + V_{\mathrm{T}}\,. \tag{4.44}$$

Further, to derive the nonrelativistic limit of the equations, we expand the kinetic term $\alpha_1 w_1 + \alpha_2 w_2$ in Eq. (3.61), retaining terms up to (including) $O(\mathbf{p}^2/m_i^2)$. In the right-hand side of this equation, the function $A^{(\alpha_1\alpha_2)}(M_B;p)$ can be replaced by its value at $p = 0$. In the result, we obtain

$$\tilde{\Phi}^{(\pm\mp)}_{M_B;NR}(\mathbf{p}) = 0\,, \qquad \tilde{\Phi}^{(--)}_{M_B;NR}(\mathbf{p}) = 0\,, \tag{4.45}$$

$$\begin{aligned}\left[\varepsilon_B - \frac{\mathbf{p}^2}{2\mu_{12}}\right]\tilde{\Phi}^{(++)}_{\varepsilon_B;NR}(\mathbf{p}) \;&=\; A^{(++)}(M_B;0)\int \frac{d^3\mathbf{p}'}{(2\pi)^3}\big[V_{\mathrm{OG}}(\mathbf{p}-\mathbf{p}') + \\ +V_{\mathrm{C}}(\mathbf{p}-\mathbf{p}') \;&+\; V_{\mathrm{T}}(\mathbf{p}-\mathbf{p}')\big]\,\tilde{\Phi}^{(++)}_{\varepsilon_B;NR}(\mathbf{p}')\,,\end{aligned} \tag{4.46}$$

where $\varepsilon_B = M_B - m_{12}$, and

$$\begin{aligned}\text{SAL, GR, MW} \;&:\; A^{(++)}(M_B;0) = 1\,, \\ \text{CJ} \;&:\; A^{(++)}(M_B;0) = \frac{1}{2}\left(1 + \frac{M_B}{m_{12}}\right).\end{aligned} \tag{4.47}$$

The nonrelativistic limit in the MNK version is more tricky. For a general M_B, there emerges an arbitrary function of the ratio M_B/m_{12}. However, if one uses the nonrelativistic approximation also for the bound-state mass $M_B = m_{12}$, then $A^{(++)}(M_B;0) = 1$ for both the CJ and MNK versions. Below, we shall use this approximation.

Since in the nonrelativistic limit (see Eq. (4.3))

$$\tilde{\Phi}^{(++)}_{\varepsilon_B;NR}(\mathbf{p}) = \chi^{(++)}_{\varepsilon_B;NR}(\mathbf{p})\,, \tag{4.48}$$

the nonrelativistic limit of Eq. (3.61) with the harmonic confinement potential (4.22) only, is given by

$$\left[\varepsilon_B - \frac{\mathbf{p}^2}{2\mu_{12}} + \frac{4}{3}\,\alpha_{\rm S}(m_{12}^2)\Big(\frac{\mu_{12}\omega_0^2}{2}\,\triangle_{\mathbf{p}} + V_0\Big)\right]\chi^{(++)}_{\varepsilon_B;NR}(\mathbf{p}) = 0\,. \tag{4.49}$$

Performing the partial-wave expansion of Eq. (4.49), we obtain the equation for the radial wave functions

$$\left[\frac{d^2}{dz^2} + \frac{2}{z}\,\frac{d}{dz} - \frac{L(L+1)}{z^2} - z^2 + \frac{2}{\omega_0}\sqrt{\frac{3}{4\alpha_{\rm S}(m_{12}^2)}}\,\Big(\varepsilon_B^{(n)} + \frac{4}{3}\,\alpha_{\rm S}(m_{12}^2)V_0\Big)\right]\times$$

$$\times R_L(z) = 0, \tag{4.50}$$

where $z = p/\bar{p}$, and $\bar{p} = \sqrt{\mu_{12}\omega_0\sqrt{\frac{4}{3}\alpha_{\rm S}(m_{12}^2)}}$. The solutions of this equation with the energy spectrum

$$\varepsilon_B^{(n)} = -\frac{4}{3}\,\alpha_{\rm S}(m_{12}^2)V_0 + \sqrt{\frac{4}{3}\,\alpha_{\rm S}(m_{12}^2)}\,\omega_0\Big(2n + L + \frac{3}{2}\Big)\,, \tag{4.51}$$

are the well-known harmonic oscillator wave functions

$$\begin{aligned} R_{nL}(p) &= \bar{p}^{-3/2}R_{nL}(z)\,,\\ R_{nL}(z) &= c_{nL}z^L\exp\Big(-\frac{z^2}{2}\Big)\ {}_1F_1\Big(-n, L+\frac{3}{2}, z^2\Big)\,,\\ c_{nL} &= \sqrt{\frac{2\Gamma(n+L+3/2)}{\Gamma(n+1)}}\,\frac{1}{\Gamma(L+3/2)}\,, \end{aligned} \tag{4.52}$$

where ${}_1F_1$ denotes the confluent hypergeometric function.

The functions $R_{nL}(p)$ can be used as a basis for the expansion in the general case (4.42). The system of equations for the coefficients is given by

$$M_B c^{(\alpha_1\alpha_2)}_{nLSJ} = \sum_{\alpha_1'\alpha_2'}\sum_{L'S'}\sum_{n'} H^{(\alpha_1\alpha_2;\alpha_1'\alpha_2')}_{LSJn;L'S'J'n'}(M_B)c^{(\alpha_1'\alpha_2')}_{n'L'S'J'}\,, \tag{4.53}$$

where the matrix $H(M_B)$ is given by the convolution of the potential and various kinematic factors that appear in Eqs. (4.16)–(4.17), with the wave functions of the basis. From Eq. (4.53) it is immediately seen that, in general, the eigenvalue equation for M_B is not a linear one, and should be solved, e.g., by iterations.

In order to actually solve the system of equations (4.53), one has to truncate it at some fixed $n = N_{\max}$. Then, $c^{(\alpha_1\alpha_2)}_{LSJn}$ are determined from the system of $4(N_{\max}+1)$ $(2(N_{\max}+1)$ in Salpeter and Gross versions) linear equations. This procedure determines the eigenvalue M_B as well, either directly, when the matrix $H(M_B)$ does not depend on M_B, or by using the iterative procedure. Having solved the eigenvalue problem at a fixed value of $N_{\max}$, one has then to check the stability with respect to the change of $N_{\max}$ — if the calculated eigenvalues do not converge with the increase of $N_{\max}$, the original system of integral equations is declared to have no solutions.

Note that the system of equations (4.53) is homogeneous in $c^{(\alpha_1\alpha_2)}_{LSJn}$. This means that the solution of the eigenvalue problem determines these coefficients up to an overall factor that can be fixed from the normalization condition

$$\sum_{LS}\sum_{\alpha_1\alpha_2}\sum_{nn'} c^{(\alpha_1\alpha_2)}_{LSJn} c^{(\alpha_1\alpha_2)}_{LSJn'} \int_0^\infty p^2 dp\, f^{(\alpha_1\alpha_2)}_{12}(M_B;p)\, R_{nL}(p) R_{n'L}(p) = 1. \quad (4.54)$$

In the CJ and MNK versions, the function $f^{(--)}_{12}(M_B;p)$ has the second-order pole, so in the normalization condition one encounters singular integrals of the following type

$$I(x_0) = \int_0^\infty \frac{f(x)dx}{(x-x_0)^2}\,, \quad (4.55)$$

where $f(x)$ is the regular function that obeys the conditions $f(0) = f(\infty) = 0$. The integral in (4.55) can be regularized according to

$$\int_0^\infty \frac{f(x)dx}{(x-x_0)^2} = \int_0^{2x_0} \frac{(f'(x)-f'(x_0))dx}{x-x_0} + \int_{2x_0}^\infty \frac{f'(x)dx}{x-x_0}\,. \quad (4.56)$$

The first question, which one may be willing to investigate, is the manifestation of the Lorentz structure of the confining interaction in the bound-state mass spectrum, especially in the case of light quarks. This question was addressed, e.g., in Ref. 18, where the scalar, timelike vector, and their equal-weight mixture were studied on the basis of Salpeter equation (this corresponds to the choice $x = 0; 1; 0.5$ in Eq. (4.2), respectively). It was demonstrated that the stable solutions of the Salpeter equation in the light quark sector do not exist for the scalar confining potential $x = 0$, and do exist for $x = 0.5$ and $x = 1$. Further, in Ref. 18, the structure $\gamma^{(1)}_\mu \otimes \gamma^{\mu(2)}$ was considered as well — it was demonstrated that in the case the stable solutions do not exist. In Ref. 21, more general conclusion was obtained — it was demonstrated that the stable solutions in the light quark sector exist for any x from the interval $0.5 \le x \le 1$. This result was confirmed in Refs. 27, 31. Further, in Ref. 19, it was shown that in the heavy quark sector nothing really depends on the mixing parameter x — the solutions exist everywhere and practically do not change when x varies in the whole interval $0 \le x \le 1$. This result is easy to understand. Indeed, the projection operator $(\Lambda_{12}^{(++)} - \Lambda_{12}^{(--)})\gamma_0^{(1)} \otimes \gamma_0^{(2)}$, that is present in the Salpeter equation, in the heavy quark limit is equal to $\frac{1}{2}(\gamma_0^{(1)} + \gamma_0^{(2)})\gamma_0^{(1)} \otimes \gamma_0^{(2)}$, so that the confining interaction in this limit equals

$$(\Lambda_{12}^{(++)} - \Lambda_{12}^{(--)})\gamma_0^{(1)} \otimes \gamma_0^{(2)} \left[x\gamma_0^{(1)} \otimes \gamma_0^{(2)} + (1-x)I^{(1)} \otimes I^{(2)}\right] V_{\rm C}(\mathbf{p} - \mathbf{p}') \to$$
$$\to \frac{1}{2}(\gamma_0^{(1)} + \gamma_0^{(2)})V_{\rm C}(\mathbf{p} - \mathbf{p}') \quad (4.57)$$

at $m_1, m_2 \to \infty$, and does not depend on x at all. Note that in the literature we encounter the different choice of the parameter x: $x = 1$ [15, 20, 27], $x = 0.5$ [28, 30, 32], $x = 0$ [20, 25, 28]. Note also, that, as it was shown in Ref. 26, the nonexistence of the stable solutions at small x in the light-quark sector is related to the presence of the «negative-energy» component in the Salpeter wave function.

The same question can be studied in other — GR, MW, CJ and MNK — versions that, unlike the Salpeter equation, have the correct one-body limit. For the MW and CJ versions the investigations were carried out in Ref. 23. Here, the problem was studied in the configuration space, and for the confining potential the following Lorentz structure was assumed: $V_{\rm C}(\mathbf{p}, \mathbf{p}') = \left[x\gamma_\mu^{(1)} \otimes \gamma^{\mu(2)} + (1 - x)I^{(1)} \otimes I^{(2)}\right] V_{\rm C}(\mathbf{p} - \mathbf{p}')$, where for $V_{\rm C}(\mathbf{p} - \mathbf{p}')$ a linear form was chosen. It was demonstrated that this potential should be «more scalar than vector» in order to provide the existence of the stable solutions. More detailed study of MW, CJ and MNK versions in the momentum space was carried out in Refs. 11, 35, where the harmonic confining potential was used, with the Lorentz structure given by Eq. (4.2). The following states $d\bar{s}:\ {}^1S_0, {}^3S_1, {}^1P_1, {}^3P_0, {}^3P_1, {}^3P_2, {}^1P_2, {}^3D_1, {}^3D_3$, $c\bar{u}$ and $c\bar{s}:\ {}^1S_1, {}^1P_1, {}^3P_2$, were considered. It was demonstrated, that in all versions the solutions always exist at $x = 0$, whereas for $x = 1$ for the majority of the states there is no solution. This is just the opposite to the Salpeter equation case (see above) — there, at $x = 1$, there are the solutions, whereas at $x = 0$, the solutions for majority of states cease to exist. Put differently, the existence/nonexistence of the solutions depends critically on the value of x, and the criteria vary from version to version. In addition, the criteria depend on the details of the potential — in particular, on the strength parameter ω_0 introduced in Eq. (4.22). Note that the instability mentioned, is now caused by the admixture of the mixed $(+-), (-+)$ frequency components in the bound-state wave function. One may look for the admissible window in the parameter space, where the solutions of all versions simultaneously exist and approximately coincide. In this way, one may judge on the Lorentz structure — assuming that the whole physical picture of the $q\bar{q}$ bound states based on the $3D$ reduction of the BS equation, is viable. From this study, one has to reject the MW version that poorly agrees either with other versions or with data. Further, on the basis of SAL, CJ and MNK versions, one can determine the acceptable interval for the mixing parameter x: $0.3 \leq x \leq 0.6$.

Both the fine structure (P-wave splitting), and the hyperfine structure ($^3S_1 - {}^3D_1$ splitting) of the $q\bar{q}$ states depends on the value of the mixing parameter x. As was shown in Refs. 25, 33 on the basis of Salpeter equation, the spin-orbit splitting in the light quarkonia can only be described by the mixture of scalar and timelike vector confinement. However, as was shown in Ref. 33, the fine structure and the hyperfine structure cannot be simultaneously described by simply varying the value of the mixing parameter. Finally, in Ref. 35, more general — and pessimistic — conclusion was drawn: neither of the versions — SAL, MW, CJ or MNK — with the dynamical input specified above, does not describe even qualitative features of the whole mass spectrum of $q\bar{q}$ bound states with x inside the interval $0.3 \leq x \leq 0.6$. Clearly, the problem calls for the further investigation. Note that some aspects of the dependence on x the existence of stable solutions of the different three-dimensional relativistic equations is studied in Refs. 44, 45.

5. DECAYS OF THE MESONS IN THE C.M. FRAME

Further information about the bound $q\bar{q}$ systems may be gained, investigating their decays. Below, we consider exclusively the decays that proceed into the c.m. frame of the bound state*. These are: the weak decays of the pseudoscalar mesons $P \to \mu\bar{\nu}$, the leptonic decays of the neutral vector mesons $V \to e^+e^-$, and the two-photon decays $M \to \gamma\gamma$. The corresponding characteristics are: the weak decay constant f_P, the leptonic decay width $\Gamma(V \to e^+e^-)$ (or the leptonic constant f_V), and the two-photon decay width $\Gamma(M \to \gamma\gamma)$.

The expressions for the quantities f_P and $\Gamma(V \to e^+e^-)$ were obtained in Refs. 18,25,28,29 in the framework of Salpeter equation, directly in terms of $\tilde{\Phi}^{(\pm)}(\mathbf{p}) = \tilde{\Phi}_{aa}(\mathbf{p}) \pm \tilde{\Phi}_{bb}(\mathbf{p})$, or $\tilde{\Psi}_{aa}(\mathbf{p}) = \tilde{\Phi}^{(++)}(\mathbf{p})$, $\tilde{\Psi}_{bb}(\mathbf{p}) = \tilde{\Phi}^{(--)}(\mathbf{p})$ (see above). In Ref. 35, these quantities were evaluated in the framework of SAL, CJ and MNK versions written in the form (4.5)–(4.1), that corresponds to the

*The treatment of the decays which cannot be confined to the c.m. frame, implies the specification of the Lorentz-transformation rules for the instantaneous potentials and $3D$ wave functions. Due to the Lorentz covariance, the dependence on the 0-th component of the relative momentum emerges into the transformed wave functions, that renders the problem extremely complicated, and the further assumptions are necessary. We do not consider such processes here.

representation of the wave function in the form (4.3)–(4.4). The main conclusion that comes from this investigation, is that the results do not depend much on the choice of the different $3D$ reduction scheme. The quantity $\Gamma(M \to \gamma\gamma)$ was evaluated in Refs. 28, 29, 32, 34 for the systems (π, η, η'). Below, we shall follow the derivation presented in Ref. 35.

For the calculation of the quantities listed above, we need the wave function $\tilde{\Phi}^{(\alpha_1\alpha_2)}_{LSJM_J}(\mathbf{p})$ which, according to Eq. (4.3), is expressed via $\tilde{\chi}^{(\alpha_1\alpha_2)}_{LSJM_J}(\mathbf{p})$. The partial-wave expansion for the components of the wave function reads

$$\begin{aligned}
\big[\tilde{\Phi}^{(\alpha_1\alpha_2)}_{LSJM_J}(\mathbf{p})\big]_{aa} &= \langle \mathbf{n}|LSJM_J\rangle\, \mathcal{N}^{(\alpha_1\alpha_2)}_{12}(p)\, R^{(\alpha_1\alpha_2)}_{LSJ}(p)\,, \\
\big[\tilde{\Phi}^{(\alpha_1\alpha_2)}_{LSJM_J}(\mathbf{p})\big]_{ab} &= -(\mathbf{sn}-\boldsymbol{\sigma}\mathbf{n})\,\langle \mathbf{n}|LSJM_J\rangle\, \frac{\alpha_2 p \mathcal{N}^{(\alpha_1\alpha_2)}_{12}(p)}{w_2+\alpha_2 m_2}\, R^{(\alpha_1\alpha_2)}_{LSJ}(p)\,, \\
\big[\tilde{\Phi}^{(\alpha_1\alpha_2)}_{LSJM_J}(\mathbf{p})\big]_{ba} &= (\mathbf{sn}+\boldsymbol{\sigma}\mathbf{n})\,\langle \mathbf{n}|LSJM_J\rangle\, \frac{\alpha_1 p \mathcal{N}^{(\alpha_1\alpha_2)}_{12}(p)}{w_1+\alpha_1 m_1}\, R^{(\alpha_1\alpha_2)}_{LSJ}(p)\,, \\
\big[\tilde{\Phi}^{(\alpha_1\alpha_2)}_{LSJM_J}(\mathbf{p})\big]_{bb} &= -\frac{\mathbf{S}_{12}+\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}}{3}\,\langle \mathbf{n}|LSJM_J\rangle\,\times \\
&\quad \times \frac{\alpha_1\alpha_2 p^2 \mathcal{N}^{(\alpha_1\alpha_2)}_{12}(p)}{(w_1+\alpha_1 m_1)(w_2+\alpha_2 m_2)}\, R^{(\alpha_1\alpha_2)}_{LSJ}(p)\,, \qquad (5.1)
\end{aligned}$$

where $\mathbf{S} = \frac{1}{2}(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})$, $\boldsymbol{\sigma} = \frac{1}{2}(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)})$, and the operators $\mathbf{S}_{12}$ and $\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}$ are given by Eq. (4.14).

Using now the identity which is valid for any operator $\hat{\mathbf{O}}$

$$\begin{aligned}
&\hat{\mathbf{O}}(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, \mathbf{n})\,\langle \mathbf{n}|LSJM_J\rangle = \\
&\qquad = \sum_{L'S'J'M_{J'}} \langle \mathbf{n}|L'S'J'M_{J'}\rangle\,\langle L'S'J'M_{J'}|\hat{\mathbf{O}}|LSJM_J\rangle\,, \qquad (5.2)
\end{aligned}$$

and the expressions for the matrix elements of the operators $\mathbf{Sn}$, $\boldsymbol{\sigma}\mathbf{n}$, $\mathbf{S}_{12}$ (4.15), from Eq. (5.1) it is straightforward to obtain

$$\big[\tilde{\Phi}^{(\alpha_1\alpha_2)}_{LSJM_J}(\mathbf{p})\big]_{aa} = \langle \mathbf{n}|LSJM_J\rangle\, \mathcal{N}^{(\alpha_1\alpha_2)}_{12}(p)\, R^{(\alpha_1\alpha_2)}_{LSJ}(p),$$

$$\Big[\tilde{\Phi}^{(\alpha_1\alpha_2)}_{J\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)JM_J}(\mathbf{p})\Big]_{ab} =$$

$$= \left[\mp\sqrt{\frac{\begin{pmatrix} J+1 \\ J\end{pmatrix}}{2J+1}}\,\langle \mathbf{n}|J+11JM_J\rangle + \sqrt{\frac{\begin{pmatrix} J \\ J+1\end{pmatrix}}{2J+1}}\,\langle \mathbf{n}|J-11JM_J\rangle\right] \times$$

$$\times\,\alpha_2\mathcal{N}_{12}^{(\alpha_1-\alpha_2)}(p)\,R^{(\alpha_1\alpha_2)}_{J\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)J}(p)\,,$$

$$\big[\tilde{\Phi}^{(\alpha_1\alpha_2)}_{J\pm 11JM_J}(\mathbf{p})\big]_{ab}=$$
$$=\left[\sqrt{\frac{\begin{pmatrix}J\\J+1\end{pmatrix}}{2J+1}}\,\langle\mathbf{n}|J1JM_J\rangle\mp\sqrt{\frac{\begin{pmatrix}J+1\\J\end{pmatrix}}{2J+1}}\,\langle\mathbf{n}|J0JM_J\rangle\right]\times$$
$$\times\,\alpha_2\mathcal{N}_{12}^{(\alpha_1-\alpha_2)}(p)\,R^{(\alpha_1\alpha_2)}_{J\pm 11J}(p)\,,$$

$$\big[\tilde{\Phi}^{(\alpha_1\alpha_2)}_{J\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)JM_J}(\mathbf{p})\big]_{ba}=$$
$$=\left[-\sqrt{\frac{\begin{pmatrix}J+1\\J\end{pmatrix}}{2J+1}}\,\langle\mathbf{n}|J+11JM_J\rangle\pm\sqrt{\frac{\begin{pmatrix}J\\J+1\end{pmatrix}}{2J+1}}\,\langle\mathbf{n}|J-11JM_J\rangle\right]\times$$
$$\times\,\alpha_1\mathcal{N}_{12}^{(-\alpha_1\alpha_2)}(p)\,R^{(\alpha_1\alpha_2)}_{J\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)J}(p)\,,$$

$$\big[\tilde{\Phi}^{(\alpha_1\alpha_2)}_{J\pm 11JM_J}(\mathbf{p})\big]_{ba}=$$
$$=\left[-\sqrt{\frac{\begin{pmatrix}J\\J+1\end{pmatrix}}{2J+1}}\,\langle\mathbf{n}|J1JM_J\rangle\mp\sqrt{\frac{\begin{pmatrix}J+1\\J\end{pmatrix}}{2J+1}}\,\langle\mathbf{n}|J0JM_J\rangle\right]\times$$
$$\times\,\alpha_1\mathcal{N}_{12}^{(-\alpha_1\alpha_2)}(p)\,R^{(\alpha_1\alpha_2)}_{J\pm 11J}(p)\,,$$

$$\big[\tilde{\Phi}^{(\alpha_1\alpha_2)}_{J\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)JM_J}(\mathbf{p})\big]_{bb}=\pm\langle\mathbf{n}|J\begin{pmatrix}0\\1\end{pmatrix}JM_J\rangle\,\alpha_1\alpha_2\mathcal{N}_{12}^{(-\alpha_1-\alpha_2)}(p)\,R^{(\alpha_1\alpha_2)}_{J\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)J}(p)\,,$$

$$\big[\tilde{\Phi}^{(\alpha_1\alpha_2)}_{J\pm 11JM_J}(\mathbf{p})\big]_{bb}=$$
$$=\left[\pm\frac{1}{2J+1}\,\langle\mathbf{n}|J\pm 11JM_J\rangle-\frac{2\sqrt{J(J+1)}}{2J+1}\,\langle\mathbf{n}|J\mp 11JM_J\rangle\right]\times$$
$$\times\alpha_1\alpha_2\mathcal{N}_{12}^{(-\alpha_1-\alpha_2)}(p)\,R^{(\alpha_1\alpha_2)}_{J\pm 11J}(p)\,. \tag{5.3}$$

With the use of these expressions, we can explicitly calculate the quantity

$$\big[\hat{\tilde{\Phi}}^{(\alpha_1\alpha_2)}_{LSJM_J}(\mathbf{p})\big]_{ij}=\big[\tilde{\Phi}^{(\alpha_1\alpha_2)}_{LSJM_J}(\mathbf{p})\big]_{ij}(-i\sigma_y)\,,\qquad i,j=a,b\,, \tag{5.4}$$

as a 2×2 matrix in the fermion spin space. To this end, we explicitly introduce the fermion spin coordinates σ_1 and σ_2 ($\sigma_i=\pm 1/2,\ i=1,2$). Then, we have

$$\langle\mathbf{n}|LSJM_J\rangle\equiv\langle\mathbf{n}\sigma_1\sigma_2|LSJM_J\rangle=$$
$$=\sum_{m_Lm_S}\langle LSm_Lm_S|JM_J\rangle\,\langle\frac{1}{2}\frac{1}{2}\sigma_1\sigma_2|Sm_S\rangle\,\langle\mathbf{n}|Lm_L\rangle\,. \tag{5.5}$$

Further, with the account of the following relations

$$\begin{aligned} \langle\frac{1}{2}\frac{1}{2}\sigma_1\sigma_2|00\rangle(-i\sigma_y) &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\, I \equiv \hat{\varphi}_0\,, \\ \langle\frac{1}{2}\frac{1}{2}\sigma_1\sigma_2|10\rangle(-i\sigma_y) &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\,\sigma_z \equiv \hat{\varphi}_{10}\,, \\ \langle\frac{1}{2}\frac{1}{2}\sigma_1\sigma_2|1\pm 1\rangle(-i\sigma_y) &= \begin{pmatrix} 0 & \binom{-1}{0} \\ \binom{0}{1} & 0 \end{pmatrix} = \\ &= \frac{1}{2}\,(\mp\sigma_x - i\sigma_y) = \frac{1}{\sqrt{2}}\,\sigma_{\pm} \equiv \hat{\varphi}_{1\pm 1}\,, \end{aligned} \tag{5.6}$$

it follows that

$$\langle \mathbf{n}|LSJM_J\rangle = \sum_{m_L m_S} \langle LSm_L m_S|JM_J\rangle\, \langle \mathbf{n}|Lm_L\rangle\, \hat{\varphi}_{Sm_S} \equiv$$
$$\equiv (\langle \mathbf{n}|L\rangle \otimes \hat{\varphi}_S)^{JM_J}\,. \tag{5.7}$$

For the quantity $\hat{\bar{\Phi}}_{LSJM_J}(\mathbf{p}) = \sum_{\alpha_1\alpha_2} \hat{\bar{\Phi}}^{(\alpha_1\alpha_2)}_{LSJM_J}(\mathbf{p})$ we obtain

$$\big[\hat{\bar{\Phi}}_{LSJM_J}(\mathbf{p})\big]_{aa} = (\langle \mathbf{n}|L\rangle \otimes \hat{\varphi}_S)^{JM_J} \textstyle\sum_{\alpha_1\alpha_2} \mathcal{N}_{12}^{(\alpha_1\alpha_2)}(p) R_{LSJ}^{(\alpha_1\alpha_2)}(p),$$

$$\big[\hat{\bar{\Phi}}_{J\binom{0}{1}JM_J}(\mathbf{p})\big]_{ab} = \Bigg[\mp\sqrt{\frac{\binom{J}{J+1}}{2J+1}}\,(\langle \mathbf{n}|J+1\rangle \otimes \hat{\varphi}_1)^{JM_J} +$$
$$+\sqrt{\frac{\binom{J+1}{J}}{2J+1}}\,(\langle \mathbf{n}|J-1\rangle \otimes \hat{\varphi}_1)^{JM_J}\Bigg] \textstyle\sum_{\alpha_1\alpha_2} \alpha_2 \mathcal{N}_{12}^{(\alpha_1-\alpha_2)}(p)\, R^{(\alpha_1\alpha_2)}_{J\binom{0}{1}J}(p)\,,$$

$$\big[\hat{\bar{\Phi}}_{J\pm 11JM_J}(\mathbf{p})\big]_{ab} = \Bigg[\sqrt{\frac{\binom{J}{J+1}}{2J+1}}\,(\langle \mathbf{n}|J\rangle \otimes \hat{\varphi}_1)^{JM_J} \mp$$
$$\mp\sqrt{\frac{\binom{J+1}{J}}{2J+1}}\,(\langle \mathbf{n}|J\rangle \otimes \hat{\varphi}_0)^{JM_J}\Bigg] \textstyle\sum_{\alpha_1\alpha_2} \alpha_2 \mathcal{N}_{12}^{(\alpha_1-\alpha_2)}(p)\, R^{(\alpha_1\alpha_2)}_{J\pm 11J}(p)\,,$$

$$\big[\hat{\bar{\Phi}}_{J\binom{0}{1}JM_J}(\mathbf{p})\big]_{ba} = \Bigg[-\sqrt{\frac{\binom{J+1}{J}}{2J+1}}\,(\langle \mathbf{n}|J+1\rangle \otimes \hat{\varphi}_1)^{JM_J} \pm$$
$$\pm\sqrt{\frac{\binom{J+1}{J}}{2J+1}}\,(\langle \mathbf{n}|J-1\rangle \otimes \hat{\varphi}_1)^{JM_J}\Bigg] \textstyle\sum_{\alpha_1\alpha_2} \alpha_1 \mathcal{N}_{12}^{(-\alpha_1\alpha_2)}(p)\, R^{(\alpha_1\alpha_2)}_{J\binom{0}{1}J}(p)\,,$$

$$[\hat{\tilde{\Phi}}_{J\pm 11JM_J}(\mathbf{p})]_{ba} = \left[-\sqrt{\frac{\begin{pmatrix} J \\ J+1 \end{pmatrix}}{2J+1}}\,(\langle\mathbf{n}|J\rangle\otimes\hat{\varphi}_1)^{JM_J}\mp\right.$$

$$\left.\mp\sqrt{\frac{\begin{pmatrix} J+1 \\ J \end{pmatrix}}{2J+1}}\,(\langle\mathbf{n}|J\rangle\otimes\hat{\varphi}_0)^{JM_J}\right]\sum_{\alpha_1\alpha_2}\alpha_1\mathcal{N}_{12}^{(-\alpha_1\alpha_2)}(p)\,R_{J\pm 11J}^{(\alpha_1\alpha_2)}(p),$$

$$[\hat{\tilde{\Phi}}_{J\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)JM_J}(\mathbf{p})]_{bb} = \left[\pm\left(\langle\mathbf{n}|J\rangle\otimes\begin{pmatrix}\hat{\varphi}_0\\ \hat{\varphi}_1\end{pmatrix}\right)^{JM_J}\right]\times$$

$$\times\sum_{\alpha_1\alpha_2}\alpha_1\alpha_2\mathcal{N}_{12}^{(-\alpha_1-\alpha_2)}(p)R^{(\alpha_1\alpha_2)}_{J\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)J}(p), \tag{5.8}$$

$$[\hat{\tilde{\Phi}}_{J\pm 11JM_J}(\mathbf{p})]_{bb} = \left[\pm\frac{1}{2J+1}\,(\langle\mathbf{n}|J\pm 1\rangle\otimes\hat{\varphi}_1)^{JM_J}-\right.$$

$$\left.-\frac{2\sqrt{J(J+1)}}{2J+1}\,(\langle\mathbf{n}|J\mp 1\rangle\otimes\hat{\varphi}_1)^{JM_J}\right]\sum_{\alpha_1\alpha_2}\alpha_1\alpha_2\mathcal{N}_{12}^{(-\alpha_1-\alpha_2)}(p)R^{(\alpha_1\alpha_2)}_{J\pm 11J}(p).$$

In order to evaluate the constants f_P and f_V, we need the bound-state wave function of the $q\bar{q}$ state at $\mathbf{r}=0$

$$\tilde{\Psi}_{LSJM_J}(\mathbf{r}=0)\equiv\tilde{\Psi}_{LSJM_J}(\mathbf{r}=0,\sigma_1,\sigma_2)=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\tilde{\Psi}_{LSJM_J}(\mathbf{p})\equiv$$
$$\equiv\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\tilde{\Psi}_{LSJM_J}(\mathbf{p},\sigma_1\sigma_2)\,, \tag{5.9}$$

where, according to Eqs. (2.26), (5.7) and (5.8)

$$\tilde{\Psi}_{LSJM_J}(\mathbf{p})=\begin{pmatrix}(\hat{\tilde{\Phi}}_{LSJM_J}(\mathbf{p}))_{ab} & (\hat{\tilde{\Phi}}_{LSJM_J}(\mathbf{p}))_{aa}\\ (\hat{\tilde{\Phi}}_{LSJM_J}(\mathbf{p}))_{bb} & (\hat{\tilde{\Phi}}_{LSJM_J}(\mathbf{p}))_{ba}\end{pmatrix}. \tag{5.10}$$

The decay constants f_P and f_V for the pseudoscalar ($L=S=J=0$) and vector ($L=0,\ S=J=1$) mesons, respectively, are given by [46]

$$\begin{aligned}\delta_{\mu 0}M_B f_P &= \sqrt{3}\,\mathrm{tr}\left[\tilde{\Psi}_{0000}(\mathbf{r}=0)\gamma^\mu(1-\gamma^5)\right],\\ f_V(\lambda) &= \sqrt{3}\,\mathrm{tr}\left[\tilde{\Psi}_{011\lambda}(\mathbf{r}=0)\gamma^\mu\right]\varepsilon_\mu(\lambda)\,,\qquad \lambda=\pm 1,0,\end{aligned} \tag{5.11}$$

where the factor $\sqrt{3}$ stems from the color part of the wave function, and $\varepsilon_\mu(\lambda)$ is the polarization vector of the vector meson [47]

$$\varepsilon_\mu(\lambda=\pm 1)=\mp\frac{1}{\sqrt{2}}\,(0,1,\pm i,0),\ \varepsilon_\mu(\lambda=0)=(0,0,0,1),\ \text{in c.m. frame.} \tag{5.12}$$

Now, using the equations (5.9)–(5.11), we obtain

$$f_P = \frac{\sqrt{24\pi}}{M_B} \int_0^\infty \frac{p^2 dp}{(2\pi)^3} \sum_{\alpha_1\alpha_2} \left[\mathcal{N}_{12}^{(\alpha_1\alpha_2)}(p) - \alpha_1\alpha_2 \mathcal{N}_{12}^{(-\alpha_1\alpha-2)}(p)\right] R_{000}^{(\alpha_1\alpha_2)}(p),$$

$$f_V(\lambda) = -\delta_{\lambda 0}\sqrt{24\pi} \int_0^\infty \frac{p^2 dp}{(2\pi)^3} \sum_{\alpha_1\alpha_2} \left[\mathcal{N}_{12}^{(\alpha_1\alpha_2)}(p) + \frac{\alpha_1\alpha_2}{3} \mathcal{N}_{12}^{(-\alpha_1\alpha-2)}(p)\right] \times \times R_{011}^{(\alpha_1\alpha_2)}(p) \equiv \delta_{\lambda 0} f_V. \tag{5.13}$$

The leptonic decay width of the vector mesons (ρ^0, ω, Φ) is given by

$$\Gamma(V \to e^+e^-) = 4\pi \frac{\alpha_{\text{eff}}^2}{M_B^3} \frac{1}{3} \sum_{\lambda=\pm 1,0} |f_V(\lambda)|^2 = \frac{4\pi\alpha_{\text{eff}}^2 |f_V|^2}{3M_B^3}, \tag{5.14}$$

where

$$\alpha_{\text{eff}}^2 = \alpha^2 \bar{e}_q^2, \qquad \bar{e}_q = e_q/e, \qquad \bar{e}_q^2 = \left(\frac{1}{2}, \frac{1}{18}, \frac{1}{9}\right) \tag{5.15}$$

for ρ^0, ω, Φ mesons, respectively. Here, $\bar{e}_q$ denotes the expectation value of the quark charge in the units of the elementary charge e.

In order to explain the reason, why the quantity $\bar{e}_q$ appears in the expression (5.14), let us note that the leptonic decay of the vector meson in the lowest order in e is described by the diagram depicted in Fig. 1. Taking into account the flavor structure of the wave functions

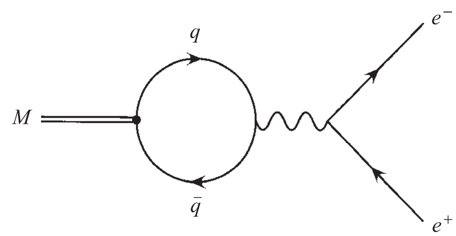

Fig. 1. Decay of the meson into electron-positron pair

$$\rho^0 \sim \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}), \qquad \omega \sim \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}), \qquad \Phi \sim s\bar{s}, \tag{5.16}$$

we obtain, that the transition amplitudes of the vector mesons into the photon are proportional to

$$(\rho^0 \to \gamma) \sim \frac{e}{\sqrt{2}}, \qquad (\omega \to \gamma) \sim \frac{e}{3\sqrt{2}}, \qquad (\Phi \to \gamma) \sim -\frac{e}{3}, \tag{5.17}$$

from which the Eq. (5.15) follows directly.

Further, taking into account Eqs. (4.42), (4.52) and (4.54), we can express the quantity f_P and the leptonic decay width in terms of the dimensionless wave functions $\bar{R}^{(\alpha_1\alpha_2)}_{LSJ}(z)$

$$f_P = \frac{\sqrt{6}\bar{p}^{3/2}}{\pi\sqrt{M_B}}\left|\sum_{\alpha_1\alpha_2}\int\limits_0^\infty z^2dz\left[\mathcal{N}^{(\alpha_1\alpha_2)}_{12}(\bar{p},z) - \alpha_1\alpha_2\mathcal{N}^{(-\alpha_1-\alpha_2)}_{12}(\bar{p},z)\right]\bar{R}^{(\alpha_1\alpha_2)}_{000}(z)\right|,$$

$$\Gamma(V\to e^+e^-) = \frac{8\alpha^2_{\rm eff}\bar{p}^3}{\pi M_B^2}\left|\sum_{\alpha_1\alpha_2}\int\limits_0^\infty z^2dz\times\right.$$
$$\left.\times\left[\mathcal{N}^{(\alpha_1\alpha_2)}_{12}(\bar{p},z) + \frac{\alpha_1\alpha_2}{3}\mathcal{N}^{(-\alpha_1-\alpha_2)}_{12}(\bar{p},z)\right]\bar{R}^{(\alpha_1\alpha_2)}_{011}(z)\right|^2, \tag{5.18}$$

where the functions

$$\bar{R}^{(\alpha_1\alpha_2)}_{LSJ}(z) = \sum_{n=0}^\infty C^{(\alpha_1\alpha_2)}_{LSJn}\,\bar{R}_{nL}(z) \tag{5.19}$$

satisfy the normalization condition

$$\sum_{LS}\sum_{\alpha_1\alpha_2}\int_0^\infty z^2dz f^{(\alpha_1\alpha_2)}_{12}(M_B;\bar{p},z)\left[\bar{R}^{(\alpha_1\alpha_2)}_{LSJ}(z)\right]^2 = 1\,. \tag{5.20}$$

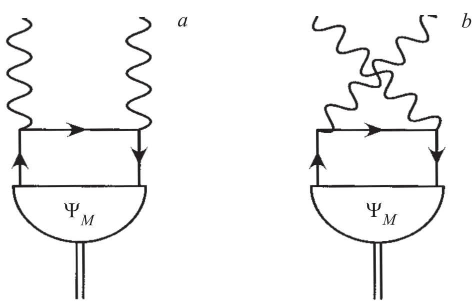

Fig. 2. Two-photon decay of the meson

Next, we consider the two-photon decays of the neutral mesons. The amplitude of the two-photon decay of the $q\bar{q}$ bound state with equal-mass quarks in the lowest order in the coupling constant e is given by the diagrams depicted in Fig. 2. In the c.m. frame, where $P = (M_B, \mathbf{0})$, this amplitude is equal to [24]

$$T(\lambda_1\lambda_2) = i\sqrt{3}e_q^2\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left\{\Psi_{M_B}(p)\left[\not{\epsilon}_1 S\left(\frac{P}{2}+p-k_1\right)\not{\epsilon}_2 +\right.\right.$$
$$\left.\left. + \not{\epsilon}_2 S\left(\frac{P}{2}+p-k_2\right)\not{\epsilon}_1\right]\right\}, \tag{5.21}$$

where $\Psi_{M_B}(p)$ is the BS amplitude of the $q\bar{q}$ bound state which satisfies Eq. (2.25) and is written in the form (2.26). Further, $k_1 = (M_B/2, \mathbf{k})$ and

$k_2 = (M_B/2, -\mathbf{k})$, where $\mathbf{k}$ is the relative 3-momentum of photons in the c.m. frame directed along the z axis, and $\epsilon_i \equiv \epsilon(\lambda_i)$ are the polarization vectors for the emitted photons. Due to the fact that the emitted physical photons are transversely polarized, one needs to consider only the values $\lambda_i = \pm 1$ for which

$$\epsilon(\lambda_1 = \pm 1) = \mp \frac{1}{\sqrt{2}}\,(0, 1, \pm i, 0)\,, \quad \epsilon(\lambda_2 = \pm 1) = \pm \frac{1}{\sqrt{2}}\,(0, 1, \mp i, 0)\,, \tag{5.22}$$

and

$$\not\epsilon(\lambda_1 = \pm 1) = \frac{1}{\sqrt{2}}\,(\pm\gamma_x + i\gamma_y)\,, \quad \not\epsilon(\lambda_2 = \pm 1) = \frac{1}{\sqrt{2}}\,(\mp\gamma_x + i\gamma_y)\,. \tag{5.23}$$

Further, one may rewrite the expression, entering the integrand in Eq. (5.21) in the following manner (below, we follow the derivation given in Ref. 48)

$$\begin{aligned} f_M(p, \mathbf{k}) &= -i \not\epsilon_1 S\left(\frac{P}{2} + p - k_1\right) \not\epsilon_2 - i \not\epsilon_2 S\left(\frac{P}{2} + p - k_2\right) \not\epsilon_1 = \\ &= \frac{a_{12}^{(+)}(\mathbf{p} - \mathbf{k})}{p_0 - w(\mathbf{p} - \mathbf{k}) + i0} + \frac{a_{12}^{(-)}(\mathbf{p} - \mathbf{k})}{p_0 + w(\mathbf{p} - \mathbf{k}) - i0} + \\ &+ \frac{a_{21}^{(+)}(\mathbf{p} + \mathbf{k})}{p_0 - w(\mathbf{p} + \mathbf{k}) + i0} + \frac{a_{21}^{(-)}(\mathbf{p} + \mathbf{k})}{p_0 + w(\mathbf{p} + \mathbf{k}) - i0}\,, \end{aligned} \tag{5.24}$$

where $w(\mathbf{p} \pm \mathbf{k}) = \sqrt{m^2 + (\mathbf{p} \pm \mathbf{k})^2}$, and

$$a_{12}^{(\alpha)}(\mathbf{p} - \mathbf{k}) = \not\epsilon_1 \Lambda^{(\alpha)}(\mathbf{p} - \mathbf{k})\gamma_0 \not\epsilon_2, \quad a_{21}^{(\alpha)}(\mathbf{p} + \mathbf{k}) = \not\epsilon_2 \Lambda^{(\alpha)}(\mathbf{p} + \mathbf{k})\gamma_0 \not\epsilon_1. \tag{5.25}$$

Note that, of course, the relation of the BS amplitude $\Psi_{M_B}(p)$ and the $3D$ amplitude $\tilde{\Psi}_{M_B}(\mathbf{p})$ is different in different versions of the $3D$ reduction. In particular, in the Salpeter version,

$$\Psi_{M_B}(p) = S\left(\frac{P}{2} + p\right)\Gamma(\mathbf{p})S\left(-\frac{P}{2} + p\right), \tag{5.26}$$

where, taking into account Eq. (2.26), we have

$$\begin{aligned} \Gamma(\mathbf{p}) &= -i \int \frac{d^3\mathbf{p}'}{(2\pi)^3}\, V(\mathbf{p}, \mathbf{p}')\, \tilde{\Psi}_{M_B}(\mathbf{p}')\,, \\ \tilde{\Psi}_{M_B}(\mathbf{p}') &= -i \begin{pmatrix} \tilde{\phi}_{ab}(\mathbf{p}')\sigma_y & \tilde{\phi}_{aa}(\mathbf{p}')\sigma_y \\ \tilde{\phi}_{bb}(\mathbf{p}')\sigma_y & \tilde{\phi}_{ba}(\mathbf{p}')\sigma_y \end{pmatrix}. \end{aligned} \tag{5.27}$$

On the other hand, from Eq. (5.26) one may obtain

$$\Psi_{M_B}(p) = -\frac{\Gamma^{(+-)}(\mathbf{p})}{M_B - 2w + i0}\left(\frac{1}{p_0 - \frac{M_B}{2} + w - i0} - \frac{1}{p_0 + \frac{M_B}{2} - w + i0}\right) +$$
$$+ \frac{\Gamma^{(-+)}(\mathbf{p})}{M_B + 2w}\left(\frac{1}{p_0 - \frac{M_B}{2} + w - i0} - \frac{1}{p_0 + \frac{M_B}{2} - w + i0}\right) +$$
$$+ \frac{\Gamma^{(++)}(\mathbf{p})}{M_B}\left(\frac{1}{p_0 - \frac{M_B}{2} - w + i0} - \frac{1}{p_0 + \frac{M_B}{2} - w + i0}\right) +$$
$$+ \frac{\Gamma^{(--)}(\mathbf{p})}{M_B}\left(\frac{1}{p_0 - \frac{M_B}{2} + w - i0} - \frac{1}{p_0 + \frac{M_B}{2} + w + i0}\right), \quad (5.28)$$

where

$$\Gamma^{(\alpha\beta)}(\mathbf{p}) = \Lambda^{(\alpha)}(\mathbf{p})\gamma_0\Gamma(\mathbf{p})\gamma_0\Lambda^{(\beta)}(-\mathbf{p})\,. \quad (5.29)$$

After integrating Eq. (5.28) over p_0, we obtain the Salpeter equation for the equal-time amplitude

$$\tilde{\Psi}_{M_B}(\mathbf{p}) = -\frac{i\Gamma^{(+-)}(\mathbf{p})}{M_B - 2w + i0} + \frac{i\Gamma^{(-+)}(\mathbf{p})}{M_B + 2w}\,. \quad (5.30)$$

Now, substituting (5.28) into the expression of the two-photon decay amplitude (5.21) and integrating over p_0, we obtain: $T(\pm\mp) = 0$ and

$$T(\pm\pm) = i\sqrt{3}e_q^2 \int \frac{d^3\mathbf{p}}{(2\pi)^3}\left\{\frac{1}{\frac{1}{4}M_B^2 - (w + w(\mathbf{p}-\mathbf{k}))^2} \times\right.$$
$$\times \mathrm{tr}\left[\frac{1}{2}\,(\Gamma^{(++)}(\mathbf{p}) - \Gamma^{(--)}(\mathbf{p}))(\gamma_0 \mp \gamma_5\gamma_z) +\right.$$
$$+\left(\frac{1}{2}\,(\Gamma^{(++)}(\mathbf{p}) - \Gamma^{(--)}(\mathbf{p})) + \left(\frac{M_B}{M_B + 2w}\,\Gamma^{(-+)}(\mathbf{p}) +\right.\right.$$
$$\left.\left. + i(\frac{M_B}{2} + w + w(\mathbf{p}-\mathbf{k}))\right)\tilde{\Psi}_{M_B}(\mathbf{p})\right)\times$$

$$\times \frac{1}{w(\mathbf{p}-\mathbf{k})}\left((1 \pm \gamma_5\gamma_0\gamma_z)m + (\gamma_z \mp \gamma_5\gamma_0)(p_z - k)\right)\Big] +$$

$$+ \frac{1}{\frac{1}{4}M_B^2 - (w + w(\mathbf{p}+\mathbf{k}))^2}\,\mathrm{tr}\left[\frac{1}{2}\left(\Gamma^{(++)}(\mathbf{p}) - \Gamma^{(--)}(\mathbf{p})\right)(\gamma_0 \mp \gamma_5\gamma_z) +\right.$$

$$+ \Bigg(\frac{1}{2}\left(\Gamma^{(++)}(\mathbf{p}) + \Gamma^{(--)}(\mathbf{p})\right) + \Bigg(\frac{M_B}{M_B + 2w}\,\Gamma^{(-+)}(\mathbf{p}) +$$

$$+ i\left(\frac{M_B}{2} + w + w(\mathbf{p}+\mathbf{k})\right)\Bigg)\tilde{\Psi}_{M_B}(\mathbf{p})\Bigg) \times$$

$$\times \frac{1}{w(\mathbf{p}+\mathbf{k})}\left((1 \mp \gamma_5\gamma_0\gamma_z)m + (\gamma_z \mp \gamma_5\gamma_0)(p_z + k)\right)\Bigg]\Bigg\}. \tag{5.31}$$

For the further transformation of this expression, one may use the fact that the static potential $V(\mathbf{p},\mathbf{p}')$ has the Lorentz structure given by Eq. (4.2). Then,

$$V(\mathbf{p},\mathbf{p}')\tilde{\Psi}_{M_B}(\mathbf{p}') = V_{\mathrm{OG}}(\mathbf{p}-\mathbf{p}')\gamma_0\tilde{\Psi}_{M_B}(\mathbf{p}')\gamma_0 + V_{\mathrm{C}}(\mathbf{p}-\mathbf{p}') \times$$

$$\times\left(x\gamma_0\tilde{\Psi}_{M_B}(\mathbf{p}')\gamma_0 + (1-x)\tilde{\Psi}_{M_B}(\mathbf{p}')\right) +$$

$$+ V_{\mathrm{T}}(\mathbf{p}-\mathbf{p}')4\hat{g}\left(\mathrm{tr}\,(\tilde{\Psi}_{M_B}(\mathbf{p}')) + \gamma_5\mathrm{tr}\,(\tilde{\Psi}_{M_B}(\mathbf{p}')\gamma_5)\right). \tag{5.32}$$

From this, one can directly obtain

$$\Gamma^{(\alpha\beta)}(\mathbf{p}) = -i\Lambda^{(\alpha)}(\mathbf{p})\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\Bigg[V_{\mathrm{OG}}(\mathbf{p}-\mathbf{p}')\begin{pmatrix} \hat{\tilde{\phi}}_{ab}(\mathbf{p}') & \hat{\tilde{\phi}}_{aa}(\mathbf{p}') \\ \hat{\tilde{\phi}}_{bb}(\mathbf{p}') & \hat{\tilde{\phi}}_{ba}(\mathbf{p}') \end{pmatrix} +$$

$$+ V_{\mathrm{C}}(\mathbf{p}-\mathbf{p}')\begin{pmatrix} \hat{\tilde{\phi}}_{ab}(\mathbf{p}') & (2x-1)\hat{\tilde{\phi}}_{aa}(\mathbf{p}') \\ (2x-1)\hat{\tilde{\phi}}_{bb}(\mathbf{p}') & \hat{\tilde{\phi}}_{ba}(\mathbf{p}') \end{pmatrix} +$$

$$+ V_{\mathrm{T}}(\mathbf{p}-\mathbf{p}')4\hat{g}\begin{pmatrix} \mathrm{tr}\,(\hat{\tilde{\phi}}_{ab}(\mathbf{p}') + \hat{\tilde{\phi}}_{ba}(\mathbf{p}')) & -\,\mathrm{tr}\,(\hat{\tilde{\phi}}_{aa}(\mathbf{p}') + \hat{\tilde{\phi}}_{bb}(\mathbf{p}')) \\ -\mathrm{tr}\,(\hat{\tilde{\phi}}_{aa}(\mathbf{p}') + \hat{\tilde{\phi}}_{bb}(\mathbf{p}')) & \mathrm{tr}\,(\hat{\tilde{\phi}}_{ab}(\mathbf{p}') + \hat{\tilde{\phi}}_{ba}(\mathbf{p}')) \end{pmatrix}\Bigg] \times$$

$$\times\,\Lambda^{(\beta)}(-\mathbf{p})\,, \tag{5.33}$$

where

$$\hat{\tilde{\phi}}_{\alpha\beta}(\mathbf{p}) = -i\tilde{\phi}_{\alpha\beta}(\mathbf{p})\sigma_y \tag{5.34}$$

are the components of the meson amplitude in the spin space. After performing the partial-wave decomposition of these amplitudes, the expression for the quantity

$T(\lambda_1\lambda_2)$ takes the form (note that we have replaced t'Hooft interaction by its regularized version).

$^1S_0$ **state:**

$$T(\pm\pm)=\pm\frac{\sqrt{3}e_q^2}{(\sqrt{2}\pi)^5}\int_0^\infty\frac{p^2dp}{w^2}\int p'^2dp'\Bigg\{\Bigg[\Bigg(\frac{2mw}{M_B(M_B+2w)}\,J_0(M_B;p)+$$

$$+\frac{2mw}{M_Bp}\,I_1(M_B;p)+\frac{2mw^2}{M_B^2p}\,I_2(M_B;p)\Bigg)V_{\rm C}^1(p,p')R_{000(ab+ba)}(p')-(2x-1)\times$$

$$\times\Bigg(\Big(\frac{p}{M_B}+\frac{m^2}{p(M_B+2w)}\Big)J_0(M_B;p)+\frac{2w}{M_B}\,I_1(M_B;p)+\frac{2w^2}{M_B^2}\,I_2(M_B;p)\Bigg)\times$$

$$\times V_{\rm C}^0(p,p')R_{000(aa-bb)}(p')+$$

$$+(2x-1)\frac{mw}{p(M_B+2w)}J_0(M_B;p)V_{\rm C}^0(p,p')R_{000(aa+bb)}(p')\Bigg]+[x=1,V_{\rm C}\to V_{\rm OG}]+$$

$$+4\hat{g}\Bigg[\frac{2mw}{p(M_B+2w)}J_0(M_B;p)V^0_{\rm T,reg}(p,p';\Lambda)R_{000(aa-bb)}(p')\Bigg]+$$

$$+\frac{4w^2}{M_B}\Bigg[-\frac{m}{p}I_3(M_B;p)R_{000(ab+ba)}(p)+\Big(\frac{M_B}{2p}\,I_3^0(M_B;p))+$$

$$+I_3(M_B;p)\Big)R_{000(aa-bb)}(p)\Bigg]\Bigg\}, \tag{5.35}$$

$^3P_0$ **state:**

$$T(\pm\pm)=\pm\frac{\sqrt{3}e_q^2}{(\sqrt{2}\pi)^5}\int_0^\infty\frac{p^2dp}{w^2}\int p'^2dp'\Bigg\{\Bigg[\Bigg(\frac{2m}{M_B}\Big(\frac{m^2}{pM_B}+\frac{p}{M_B+2w}\Big)\,J_0(M_B;p)+$$

$$+\frac{2mw}{M_Bp}\,I_1^0(M_B;p)+\frac{2mw}{M_B(M_B+2w)}\,I_2(M_B;p)+\frac{mp}{M_B^2}\,I_2^2(M_B;p)\Bigg)\times$$

$$\times V_{\rm C}^0(p,p')R_{110(ab+ba)}(p')-(2x-1)\Bigg(\frac{4m^2w}{M_B^2(M_B+2w)}\,J_0(M_B;p)+$$

$$+\frac{2w}{M_B}\,I_1^0(M_B;p)+\Bigg(\frac{p}{M_B}+\frac{m^2}{p(M_B+2w)}\Bigg)\,I_2(M_B;p)+$$

$$+2\Big(\frac{p^2}{M_B^2}+\frac{m^2}{M_B(M_B+2w)}\Big)I_2^2(M_B;p)\bigg)V_{\rm C}^1(p,p')R_{110(aa-bb)}(p')-(2x-1)\times$$

$$\times\bigg(\frac{M_B}{M_B+2w}\,\frac{2mw}{M_B^2}\,(J_0(M_B;p)-I_2^2(M_B;p))-\frac{mw}{M_Bp}\,I_2(M_B;p)\bigg)\times$$

$$\times V_{\rm C}^1(p,p')R_{110(aa+bb)}(p')\bigg]+[x=1,V_{\rm C}\to V_{\rm OG}]+$$

$$+8\hat{g}\bigg[4\bigg(\frac{mw}{M_Bp}\,I_1^0(M_B;p)+\frac{m^3}{M_B^2p}\,J_0(M_B;p)+\frac{m}{2M_B}\,I_2(M_B;p)+$$

$$+\frac{mp}{M_B^2}\,I_2^2(M_B;p)\bigg)+\frac{4M_B}{M_B+2w}\bigg(\frac{mp}{M_B^2}\,J_0(M_B;p)-$$

$$-\frac{m}{M_B}\,I_2(M_B;p)-\frac{mp}{M_B^2}\,I_2^2(M_B;p)\bigg)\bigg]\times$$

$$\times V_{\rm T,reg}^0(p,p';\Lambda)R_{110(ab+ba)}(p')+\frac{4w^2}{M_B}\bigg[-\frac{m}{p}\,I_3^0(M_B;p)R_{110(ab+ba)}(p)+$$

$$+\left(\frac{M_B}{2p}I_3(M_B;p)-I_3^2(M_B;p)\right)R_{110(aa-bb)}(p)\bigg]\bigg\}\,, \qquad (5.36)$$

where

$$R_{LSJ(aa+bb)}(p)=R_{LSJ}^{(++)}(p)+R_{LSJ}^{(--)}(p), \qquad (5.37)$$

$$R_{LSJ(aa-bb)}(p)=\frac{m}{w}(R_{LSJ}^{(++)}(p)-R_{LSJ}^{(--)}(p))+\frac{p}{w}(R_{LSJ}^{(+-)}(p)+R_{LSJ}^{(-+)}(p)),$$

$$R_{LSJ(ab+ba)}(p)=\frac{p}{w}(R_{LSJ}^{(++)}(p)-R_{LSJ}^{(--)}(p))+\frac{m}{w}(R_{LSJ}^{(+-)}(p)+R_{LSJ}^{(-+)}(p)),$$

and

$$
\begin{aligned}
J_0 &= \ln\frac{2(w+w_+)-M_B}{2(w+w_+)+M_B} - \ln\frac{2(w+w_-)-M_B}{2(w+w_-)+M_B}\,,\\
J &= \ln\frac{2w(w+w_+)+M_Bp}{2w(w+w_-)-M_Bp}\,,\\
I_1^0 &= \frac{1}{2}\,J - \frac{w}{M_B}\,J_0\,,\\
I_1 &= 1 - \frac{4w}{w_+ + w_-} + \frac{w^2}{M_Bp}\,J - \frac{w}{2p}\,J_0\,,\\
I_2 &= \frac{2M_B}{w_+ + w_-} - \frac{w}{p}\,J\,,\\
I_2^2 &= -\frac{2w}{p} + \frac{2}{3}\,\frac{5w^2 - \frac{1}{4}\,M_B^2 + w_+w_-}{p(w_+ + w_-)} + \frac{w^2}{p^2}\,J_0\,,\\
I_3^0 &= \ln\frac{2(w+w_+)-M_B}{2(w+w_-)-M_B}\,,\\
I_3 &= 1 + \frac{M_B - 2w}{w_+ + w_-} - \frac{w}{p}\,I_3^0\,,\\
I_3^2 &= \frac{w}{p} + \frac{(M_B-2w)(w^2 + \frac{1}{4}\,M_B^2 - w_+w_- + 3M_Bw)}{3M_Bp(w_+ + w_-)} + \frac{w^2}{p^2}\,I_3^0\,,
\end{aligned}
\tag{5.38}
$$

with $w_\pm = \sqrt{w^2 + \frac{1}{4}M_B^2 \pm M_Bp}$.

Now, we consider other versions of the $3D$ equations. Since we consider the equal-mass case, the Gross equation cannot be used. For this reason, we shall restrict ourselves to the study of two-photon decay processes in CJ and MNK versions. In these versions, there exists a relation between $4D$ and $3D$ free Green functions given by Eqs. (3.46) and (3.54). This relation can be immediately translated into the relation between the $4D$ and $3D$ wave functions

$$
\Psi_{M_B}(p) = 2\pi i\delta(p_0)\,\tilde{\Psi}_{M_B}(\mathbf{p})\,, \tag{5.39}
$$

where for the MNK version the equality $p_0^+ = 0$ holds for the equal-mass case. As to the MW version, here the relation between $\Psi_{M_B}(p)$ and $\tilde{\Psi}_{M_B}(\mathbf{p})$ does not exist due to the definition of $\tilde{G}_0^{\rm MW}(M_B, \mathbf{p})$ (3.31). For the above reasons, below we restrict ourselves to the CJ and MNK versions only. Substituting the expression (5.39) into Eq. (5.21), with an account of (5.24) one obtains

$$T(\lambda_1\lambda_2) = i\sqrt{3}e_q^2 \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\mathrm{tr}\left\{\tilde{\Psi}_{M_B}(\mathbf{p})\left[\frac{a_{12}^{(+)}(\mathbf{p}-\mathbf{k})}{w(\mathbf{p}-\mathbf{k})} - \frac{a_{12}^{(-)}(\mathbf{p}-\mathbf{k})}{w(\mathbf{p}-\mathbf{k})} + \right.\right.$$

$$\left.\left. + \frac{a_{21}^{(+)}(\mathbf{p}+\mathbf{k})}{w(\mathbf{p}+\mathbf{k})} - \frac{a_{21}^{(-)}(\mathbf{p}+\mathbf{k})}{w(\mathbf{p}+\mathbf{k})}\right]\right\}. \tag{5.40}$$

From this equation one readily obtains

$$T(\pm,\pm) = \pm i\sqrt{3}e_q^2 \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\mathrm{tr}\left\{\tilde{\Psi}_{M_B}(\mathbf{p}) \times \right.$$

$$\times \left[\frac{(1 \pm \gamma_5\gamma_0\gamma_z)m + (\gamma_z \mp \gamma_5\gamma_0)(p_z - k)}{w^2(\mathbf{p}-\mathbf{k})} + \right.$$

$$\left.\left. + \frac{(1 \mp \gamma_5\gamma_0\gamma_z)m + (\gamma_z \pm \gamma_5\gamma_0)(p_z + k)}{w^2(\mathbf{p}+\mathbf{k})}\right]\right\}. \tag{5.41}$$

Substituting $\tilde{\Psi}_{M_B}(\mathbf{p})$ in the matrix form given by Eq. (5.10), we finally obtain for the CJ and MNK versions

$^1S_0$ **state:**

$$T(\pm\pm) = \pm i e_q^2 \frac{2}{(2\pi)^{5/2}} \int\limits_0^\infty p dp \left[\left(-2 + \frac{w^2 + \frac{1}{4}M_B^2}{M_B p}\,\tilde{J}(M_B;p)\right)\times\right.$$

$$\left. \times \frac{m}{M_B} R_{000(ab+ba)}(p) + \left(\frac{2p}{M_B} - \frac{w^2 - \frac{1}{4}M_B^2}{2M_B^2}\,\tilde{J}(M_B;p)\right) R_{000(aa-bb)}(p)\right], \tag{5.42}$$

$^3P_0$ **state:**

$$T(\pm\pm) = -i\sqrt{3}e_q^2 \frac{2}{(2\pi)^{5/2}} \int\limits_0^\infty p dp \left[\tilde{J}(M_B;p)\frac{m}{M_B}\, R_{110(ab+ba)}(p) + \right.$$

$$\left. +2\left(\frac{w^2 - \frac{1}{4}M_B^2}{M_B^2} - \frac{w^4 - \frac{1}{16}M_B^4}{2M_B^3 p}\,\tilde{J}(M_B;p)\right) R_{110(aa-bb)}(p)\right], \tag{5.43}$$

where

$$\tilde{J}(M_B;p) = \ln \frac{w^2 + \frac{1}{4}M_B^2 + M_B p}{w^2 + \frac{1}{4}M_B^2 - M_B p}\,. \tag{5.44}$$

It is important to note that in the Salpeter version the two-photon decay amplitude depends on the potential both directly and indirectly, through the radial wave functions, whereas in CJ and MNK versions this dependence enters only through the radial wave functions.

For a given meson, the two-photon decay amplitude can be rewritten as

$$T(\lambda_1\lambda_2) = e^2\tilde{e}^2_{q,\mathrm{eff}}\sqrt{3}\tilde{T}(\lambda_1\lambda_2; LSJM_J)\,. \tag{5.45}$$

The decay width is given by

$$\Gamma(\text{meson}\to\gamma\gamma) = 3\pi\frac{\alpha^2}{M_B}\,\frac{1}{2(2J+1)}\sum_{\lambda_1\lambda_2 M_J}\left|\tilde{e}^2_{q,\mathrm{eff}}\tilde{T}(\lambda_1\lambda_2; LSJM_J)\right|^2, \tag{5.46}$$

where $\tilde{e}^2_{q,\mathrm{eff}}$ depends on the choice of the meson flavor wave function. If this function has a simple form $q\bar{q}$, then $\tilde{e}^2_{q,\mathrm{eff}} = \tilde{e}^2_q$. However, if the meson wave function is made up of different flavor states $\alpha q_1\bar{q}_1 + \beta q_2\bar{q}_2$, the expression for $\tilde{e}^2_{q,\mathrm{eff}}$ is more complicated. Consider as an example calculation of this factor for π^0 and η_n states. The flavor structure of the wave functions is given by

$$\pi^0 \sim \frac{1}{\sqrt{2}}\,(u\bar{u} - d\bar{d})\,, \qquad \eta_n \sim \frac{1}{\sqrt{2}}\,(u\bar{u} + d\bar{d})\,. \tag{5.47}$$

It follows then straightforwardly that $\tilde{e}^2_{q,\mathrm{eff}} = \dfrac{1}{3\sqrt{2}}$ and $\tilde{e}^2_{q,\mathrm{eff}} = \dfrac{5}{9\sqrt{2}}$ for π^0 and η_n states, respectively. Further, the decay amplitudes for the physical η and η' mesons are the linear superposition of the ones corresponding to η_n and $\eta_s \sim s\bar{s}$ states.

Note that the two-photon decays of $\pi^0,\ \eta,\ \eta'$ mesons were also studied in the NJL model, taking into account the relativistic confinement and the t'Hooft interaction [42].

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