QUANTUM ZENO EFFECT FOR *N*-LEVEL FRIEDRICHS MODEL

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We study the short-time behaviour of the survival probability in the framework of the N-level Friedrichs model. We show that depending on initial conditions the decay can be considerably slowed down or even stopped. By choosing proper parameters of the system, the Zeno time can also be considerably extended.

INTRODUCTION

Since the very beginning of the quantum mechanics, the measurement process has been a most fundamental issue. The main characteristic feature of the quantum measurement is that the measurement changes the dynamical evolution. This is the main difference of the quantum measurement compared to its classical analogue. On this framework, Misra and Sudarshan pointed out [1] that repeated measurements can prevent an unstable system from decaying (the quantum Zeno effect, QZE).

The QZE has been discussed for many physical systems including atomic physics [2, 3] and mesoscopic physics [4, 5], and has been even proposed as a way to control decoherence for effective quantum computations [6]. Recently, however, it has been found [7, 8] that under some conditions the repeated observations could speed up the decay of the quantum system (the quantum anti-Zeno effect). The anti-Zeno effect has been further analyzed in [9–11].

While there exist experiments [12, 13] demonstrating the perturbed evolution of a coherent dynamics, the demonstration of the QZE for an unstable system with exponential decay, as originally proposed in [1], has long been an open question. Only recently, both Zeno and anti-Zeno effects have been observed in the experiment [14].

In order to analyze the short-time behaviour of an unstable system, we use here the Friedrichs model [15], which is very appropriate for the discussion of the particle decay and for the description of dressed unstable states [16, 17]. The analytical structure of the N-level Friedrichs model has been widely discussed, see, e. g., [18, 19] and references therein.

1. N-LEVEL FRIEDRICHS MODEL

The Hamiltonian of the Friedrichs model [15] generalized to N level is

$$H = H_0 + \lambda V, \tag{1}$$

where

$$\begin{aligned} H_0 &= \sum_{k=1}^{N} \omega_k |k\rangle\langle k| + \int_0^\infty d\omega\, \omega |\omega\rangle\langle\omega|, \\ V &= \sum_{k=1}^{N} \int_0^\infty d\omega \hat{f}_k(\omega) \left(|k\rangle\langle\omega| + |\omega\rangle\langle k|\right). \end{aligned} \tag{2}$$

Here, $|k\rangle$ represent states of the discrete spectrum with the energy ω_k, $\omega_k > 0$. The vectors $|\omega\rangle$ represent states of the continuous spectrum with the energy ω; $\hat{f}_k(\omega)$ are the form factors for the transitions between the discrete and the continuous spectrum, and λ is the coupling parameter. The vacuum energy is chosen to be zero. The states $|k\rangle$ and $|\omega\rangle$ form a complete orthonormal basis:

$$\langle k|k'\rangle = \delta_{kk'}, \quad \langle\omega|\omega'\rangle = \delta(\omega - \omega'), \quad \langle\omega|k\rangle = 0, \quad k, k' = 1, \ldots, N, \tag{3}$$

$$\sum_{k=1}^{N} |k\rangle\langle k| + \int_0^\infty d\omega\, |\omega\rangle\langle\omega| = I, \tag{4}$$

where $\delta_{kk'}$ is the Kronecker symbol; $\delta(\omega - \omega')$ is Dirac's delta function, and I is the unity operator. The Hamiltonian H_0 has the continuous spectrum on the interval $[0, \infty)$ and the discrete spectrum $\omega_1, \ldots, \omega_k$ embedded in the continuous spectrum. As the interaction λV is switched on, the eigenstates $|k\rangle$ become resonances of H as in the case of the one-level Friedrichs model [15].

The total evolution leads to the decay of an initial unstable state

$$\Phi = \sum_k \alpha_k |k\rangle, \quad \langle\Phi|\Phi\rangle = 1. \tag{5}$$

Decay is described by the survival probability $p(t)$ to find, after time t, the initial state evolving according to the evolution $\exp(-iHt)$ in the same state [3]:

$$p(t) \equiv |\langle\Phi|\, \mathrm{e}^{-iHt}|\Phi\rangle|^2 = |A(t)|^2, \tag{6}$$

where $A(t)$ is the survival amplitude. The survival amplitude can be explicitly expressed in terms of the form factors $\hat{f}_k(\omega)$ [19].

In order to calculate the short-time behaviour for the system (2), we will use the Taylor expansion of the survival probability. We shall assume here the existence of all necessary matrix elements, and denote $\langle\cdot\rangle = \langle\Phi| \cdot |\Phi\rangle$. Then we find

$$\begin{aligned} p(t) = \langle \mathrm{e}^{-iHt}\rangle = 1 - t^2 \left(\langle H^2\rangle - \langle H\rangle^2\right) + \\ + t^4 \left(\frac{1}{4}\langle H^2\rangle^2 + \frac{1}{12}\langle H^4\rangle - \frac{1}{3}\langle H\rangle\langle H^3\rangle\right) + O(t^6) = 1 - \frac{t^2}{t_a^2} + \frac{t^4}{t_b^4} + O(t^6). \end{aligned} \tag{7}$$

The expressions for the times t_a and t_b can be deduced using the special structure of the potential V (2):

$$\frac{1}{t_a^2} = \sum_k |\alpha_k|^2 \omega_k^2 - \left(\sum_k |\alpha_k|^2 \omega_k \right)^2 + \lambda^2 \Lambda^2 R_1, \tag{8}$$

$$\frac{1}{t_b^4} = \frac{1}{4}\left(\sum_k |\alpha_k|^2 \omega_k^2 \right)^2 + \frac{1}{12}\sum_k |\alpha_k|^2 \omega_k^4 - \frac{1}{3}\sum_k |\alpha_k|^2 \omega_k \sum_k |\alpha_k|^2 \omega_k^3 +$$
$$+ \lambda^2\Lambda^2 \left(\frac{1}{2} R_1 \sum_k |\alpha_k|^2 \omega_k^2 + \frac{1}{12} R_3 - \frac{1}{3} R_2 \sum_k |\alpha_k|^2 \omega_k \right) + \lambda^4 \Lambda^4 \left(\frac{1}{12} R_4 + \frac{1}{4} R_1^2 \right), \tag{9}$$

where

$$R_1 = \sum_{ik} \alpha_i \alpha_k^* F_{ik}^0, \tag{10}$$

$$R_2 = \sum_{ik} \alpha_i \alpha_k^* \big((\omega_i + \omega_k) F_{ik}^0 + \Lambda F_{ik}^1 \big), \tag{11}$$

$$R_3 = \sum_{ik} \alpha_i \alpha_k^* \big((\omega_i^2 + \omega_i \omega_k + \omega_k^2) F_{ik}^0 + (\omega_i + \omega_k) \Lambda F_{ik}^1 + \Lambda^2 F_{ik}^2 \big), \tag{12}$$

$$R_4 = \sum_{ik} \alpha_i \alpha_k^* ((F^0)^2)_{ik}. \tag{13}$$

Here,

$$F_{ik}^p = \int_0^\infty dx\, x^p f_i(x) f_k(x),$$

where the dimensionless form factor $f_k(x)$ is expressed as

$$f_k(x) = \frac{1}{\sqrt{\Lambda}} \hat{f}_k(\Lambda x),$$

and the parameter Λ has the dimension of energy.

2. ZENO EFFECT AND ZENO TIME

The probability that the state Φ after N equally spaced measurements during the time interval $[0, T]$ has not decayed, is given by [1]: $p_N(T) = p^N(T/N)$. We are interested in the behaviour of $p_N(T)$ as $N \to \infty$ or, equally, when the time interval between the measurements $\tau = T/N$ goes to zero:

$$\lim_{\tau \to 0} p_N(T) = \lim_{\tau \to 0} p(\tau)^{T/\tau} = \begin{cases} 0, & \text{when} \quad p'(0) = -\infty, \\ \mathrm{e}^{-cT}, & \text{when} \quad p'(0) = -c, \\ 1, & \text{when} \quad p'(0) = 0. \end{cases} \tag{14}$$

The results (14) are found in case of continuously ongoing measurements during the entire time interval $[0, T]$. Obviously, this is an idealization. In practice, we have a manifestation of the

Zeno effect, if the probability $p_N(T)$ increases as the time interval τ between measurements decreases. Formula (14) may be accepted as an approximation for a short time interval $\tau \lesssim t_b$. For longer times we cannot use the Taylor expansion, therefore, Eq. (14) is not valid.

As one refers in discussions about the Zeno effect to the Taylor expansion (7) of survival probability for small times, and specifically to the second term, we shall define the Zeno time t_Z as corresponding to the region where the second term dominates. Hence, as in paper [11], we introduce the Zeno time t_Z as a natural boundary where the second and third terms have the same amplitude:

$$\frac{t_Z^2}{t_a^2} = \frac{t_Z^4}{t_b^4}, \quad \text{so} \quad t_Z = t_b^2/t_a. \tag{15}$$

Expressions (8), (9) include many different parameters and can hardly be analyzed in general case. We consider here few specific representative cases I–III for the physically motivated weak coupling model [11] with

$$\lambda^2 \ll 1 \quad \text{and} \quad \Lambda \gg \omega. \tag{16}$$

I. The decay of one level. In this case, the only level l is initially occupied: $\alpha_l = 1$, $\alpha_k = 0$ for $k \neq l$. Expressions (8), (9) become

$$\frac{1}{t_a^2} = \lambda^2\Lambda^2 F_{ll}^0,$$

$$\frac{1}{t_b^4} = \lambda^2\Lambda^2\left(\frac{1}{12}\omega_l^2 F_{ll}^0 - \frac{1}{6}\Lambda\omega_l F_{ll}^1 + \frac{1}{12}\Lambda^2 F_{ll}^2\right) + \lambda^4\Lambda^4\left(\frac{(F_{ll}^0)^2}{4} + \frac{1}{12}((F^0)^2)_{ll}\right).$$

It is not surprising that the expressions for t_a and t_b practically coincide with those for the one-level Friedrichs model [11]. Therefore, the Zeno time t_Z is also the same:

$$t_Z \sim \frac{1}{\Lambda}\sqrt{\frac{12F_{ll}^0}{F_{ll}^2}}.$$

The only difference in the last term ($\sum_m F_{lm}^0 F_{ml}^0$ instead of $(F_{ll}^0)^2$ for one-level model) does not influence the results for the moderate number of levels N for $\lambda^2 \ll 1$.

II. The completely degenerate case. In this case, all frequencies are identical, $\omega_k = \omega$ for any k. Expressions (8), (9) become

$$\frac{1}{t_a^2} = \lambda^2\Lambda^2\langle F^0\rangle,$$

$$\frac{1}{t_b^4} = \lambda^2\Lambda^2\left(\frac{1}{12}\omega^2\langle F^0\rangle - \frac{1}{6}\Lambda\omega\langle F^1\rangle + \frac{1}{12}\Lambda^2\langle F^2\rangle\right) + \lambda^4\Lambda^4\left(\frac{\langle F^0\rangle^2}{4} + \frac{1}{12}\langle (F^0)^2\rangle\right),$$

where

$$\langle F^p\rangle = \sum_{ik}\alpha_i\alpha_k^* F_{ik}^p.$$

The matrices F^p are the Gramm matrices. For these matrices the following condition is satisfied:

$$\langle F^p\rangle \geq 0, \quad \langle F^p\rangle = 0$$

for some $\tilde{\alpha}$ iff the form factors $f_k(x)$ are linearly dependent:

$$\sum_k m_k f_k(x) \equiv 0. \tag{17}$$

This case resembles case I when all averages $\langle\cdot\rangle$ are separated from zero, e. g., the form factors $f_k(x)$ are linearly independent. The Zeno time is of the same order of magnitude:

$$t_Z \sim \frac{1}{\Lambda}\sqrt{\frac{12\langle F^0\rangle}{\langle F^2\rangle}}.$$

However, the situation may change for special initial conditions Φ. Assuming for the sake of simplicity that the form factors are identical, we have in this approximation

$$\frac{1}{t_b^4} \approx \frac{\lambda^2\Lambda^4}{12}F^2|\alpha|^2, \quad t_Z \approx \frac{1}{\Lambda}\sqrt{\frac{12F^0}{F^2}}, \quad \text{where} \quad \alpha = \sum_k \alpha_k. \tag{18}$$

We can now see that the Zeno time t_Z is independent of the initial conditions. However, both t_a and t_b increase to the infinity when α goes to zero. This means that the state Φ with $\alpha = 0$ does not decay, while the relation between t_a and t_b is unchanged. The same is true when the form factors are not identical but linearly dependent: there also exists a nondecaying state $\tilde{\Phi}$ (17). In this sense, the Zeno time definition (15) may not work for the N-level model and should be modified. We would like to notice that this problem does not exist for the one-level model analyzed in [11].

III. N-level model with one different level. In this case, energy of one level differs from the others: $\omega_k = \omega$, $k = 1, \ldots, N-1$, $\omega_N = \omega + \Delta$, and the form factors are identical: $f_k(x) = f(x)$. Then we have for the time t_a

$$\frac{1}{t_a^2} = \Delta^2(|\alpha_N|^2 - |\alpha_N|^4) + \lambda^2\Lambda^2F^0|\alpha|^2.$$

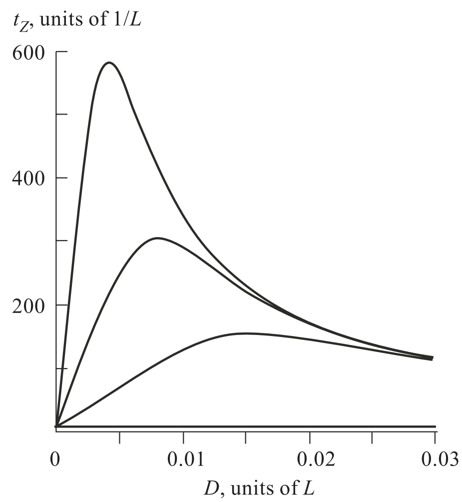
We first analyze the time t_b on condition (16):

$$\frac{1}{t_b^4} = \frac{\Delta^4}{12}(|\alpha_N|^2 - |\alpha_N|^4) + \frac{\lambda^2\Lambda^4}{12}F^2|\alpha|^2.$$

Both these times and the Zeno time depend on the initial vector. One can easily see that the Zeno time always has a maximum as the function of energy difference Δ. We can easily estimate the position and the value of this maximum:

$$t_Z\left(\frac{\Delta}{\Lambda} \approx \sqrt{\lambda}\sqrt[4]{\frac{F^2|\alpha|^2}{|\alpha_N|^2 - |\alpha_N|^4}}\right) \approx \sqrt[4]{\frac{36(|\alpha_N|^2 - |\alpha_N|^4)}{F^2|\alpha|^2}}\frac{1}{\Lambda\sqrt{\lambda}}. \tag{19}$$

One can see that the Zeno time is increased by the factor $\sim 1/\sqrt{\lambda}$ with respect to the one-level model. When conditions (16) are not satisfied, the prolongation of the Zeno time still takes place. We illustrate this prolongation in the figure using exact expression (9) for the time t_b.



The Zeno time t_Z as the function of the energy difference Δ for two-level Friedrichs model. The parameters of the model are: $\Lambda = 8.498 \cdot 10^{18}$ s^{-1}, $\omega = 1.55 \cdot 10^{16}$ s^{-1}, $\lambda^2 = 6.43 \cdot 10^{-9}$. From above, the curves correspond to the initial condition Φ: $(\alpha_1, \alpha_2) = (1, -0.6)$, $(1, 1)$, $(1, 0.1)$ and $(1, 0)$, respectively

We use the parameters of the model associated with the hydrogen atom [2]. One can see that for this system the maximum (19) cannot be reached. However, for small Δ we can find

$$t_Z\left(\frac{\Delta}{\Lambda}\right) \approx t_Z(0)\left(1 + \frac{|\alpha_N|^2 - |\alpha_N|^4}{2\lambda^2 F^0 |\alpha|^2}\left(\frac{\Delta}{\Lambda}\right)^2\right) > t_Z(0), \tag{20}$$

therefore, a prolongation always takes place.

CONCLUSION

We have analyzed the short-time behaviour for the N-level Friedrichs model. Compared to the one-level model, there exists one important difference: the speed of the decay depends on initial conditions. As a result, there exist situations when the decay is considerably slowed down or even stopped. By choosing proper parameters of the system, the Zeno time can also be considerably extended.

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