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УДК 539.142

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A semiclassical theory of linear response in finite Fermi systems, based on the Vlasov equation, and its applications to the study of isoscalar vibrations in heavy nuclei are reviewed. It is argued that the Vlasov equation can be used to study the response of small quantum systems like (heavy) nuclei in regimes for which the finite size of the system is more important than the collisions between constituents. This requires solving the linearized Vlasov equation for finite systems, however, in this case the problem of choosing appropriate boundary conditions for the fluctuations of the phase-space density is nontrivial. Calculations of the isoscalar response functions performed by using different boundary conditions, corresponding to fixed and moving nuclear surface, are compared for different multipoles and it is found that, in a sharp-surface model, the moving-surface boundary conditions give better agreement with experiment. The semiclassical strength functions given by this theory are strikingly similar to the results of analogous quantum calculations, in spite of the fact that shell effects are not included in the theory. This happens because of a well-known close relation between classical trajectories and shell structure.

В данном обзоре рассматривается полуклассическая теория линейного отклика конечных ферми-систем, которая основана на уравнении Власова, и ее применения к изучению изоскалярных колебаний тяжелых ядер. Показано, что уравнение Власова может быть использовано для изучения отклика небольших квантовых систем, таких как (тяжелые) ядра, в тех случаях, когда конечный размер системы является более важным, чем столкновения между ее элементами. Для этого необходимо решить линеаризованное уравнение Власова для конечных систем. Однако в этом случае возникает вопрос о выборе подходящих граничных условий на флуктуации плотности фазового пространства. Проведено сравнение расчетов изоскалярных мультипольных функций отклика, которые выполнены с помощью разных граничных условий, соответствующих фиксированной и подвижной ядерной поверхности. Установлено, что в модели с резкой поверхностью граничные условия на подвижной поверхности дают лучшее согласие с экспериментом. Полуклассические силовые функции, полученные в предложенной теории, поразительно похожи на найденные в аналогичных квантовых расчетах, несмотря на то, что оболочечные эффекты не учитываются в данной теории. Это объясняется наличием тесной взаимосвязи между классическими траекториями и оболочечной структурой.

INTRODUCTION

The Landau kinetic equation for Fermi liquids [1,2] contains some important differences when compared to the classical Boltzmann equation for a dilute gas, one of them is the presence of an effective mean-field term. Thus, in Landau's approach, at least a part of the force which is exerted on a particle by the other constituents of a many-body system can be approximated by an effective mean field. Another important feature of the Landau kinetic equation is the introduction of an effective mass for quasiparticles, however here we neglect the difference between bare and effective mass. Landau's kinetic equation allows also for a collision term which, nonetheless, in some cases may be neglected (see, for example, Sec. 4 of [3]). When this is done, we are dealing with a collisionless kinetic equation in the mean field approximation. Such an approximation had been considered long ago by Jeans (who gave the credit to Boltzmann) in the context of stellar dynamics [4] and later by Vlasov in that of plasma physics [5] . Here we follow the use that has become common both in plasma and in nuclear physics and refer to the collisionless kinetic equation in the mean field approximation as the Vlasov equation (see [6] for a discussion of historical priorities).

Kirzhnitz and collaborators [7] extended the approach to nonhomogeneous systems and used it to study the possibility of collective excitations in the electron cloud of heavy atoms. They pointed out that the main difficulty arising in finite systems concerns the boundary conditions to be imposed on the fluctuations of the phase-space density. Moreover, these authors also derived an interesting expression for the polarization propagator determining the linear response of these systems. Unfortunately, the practical usefulness of this expression is limited to rather special systems in which the constituents move along closed orbits.

Another attempt to study the dynamic response of inhomogeneous Fermi systems was limited to one-dimensional problems [8].

Bertsch [9] argued that the Vlasov equation could be used as a starting point for a semiclassical theory of giant resonances in heavy nuclei. He pointed out that, in spite of being a classical equation of motion, this equation would not violate the Pauli principle, at least in a semiclassical sense. This is a consequence of the Liouville theorem. When applying this method to nuclei, however, one is faced with the problem of finite-size effects since a nucleon close to the Fermi surface is more likely to reach the nuclear surface than to suffer a violent collision with another nucleon. Therefore finite-size effects become more important than the collision integral and also in the case of nuclei it is reasonable to study the kinetic equation in the mean-field (or Vlasov) approximation, at least as a first step.

Some remarkable progress on this problem has been made in the field of galactic dynamics: Polyacenko and Shukhman [10] solved the linearized Vlasov equation for finite spherical systems in their study of the stability of collisionless stellar systems. In this context one of the main problems is that of determining a stable equilibrium distribution of particles (stars). Small deviations from the equilibrium distribution are characterized by eigenfrequencies that are purely imaginary in the case of unstable systems. Their approach has found several applications in the field [11–13].

A similar solution of the linearized Vlasov equation for nuclear response has been derived independently in [14]. This solution turns out to agree with that of [10] and gives a reasonable description of giant resonances in heavy nuclei [15].

Abrosimov, Di Toro and Strutinsky [16] used the same approach within a sharp-surface model in which the nuclear mean field is approximated by a square-well potential and used also different (moving-surface) boundary conditions in order to extend the approach of [14] to low-energy surface vibrations in heavy nuclei.

This paper is a review of work done in the last ten years, based on the approaches of [14] and [16]. In Sec. 1, both approaches are recalled, while in Sec. 2 several applications of the theory to the study of isoscalar vibrations of different multipolarity (monopole, dipole, quadrupole, octupole) in heavy nuclei are discussed. Finally, in Sec. 3, conclusions are drawn. The two Appendices contain some more technical material on the moving-surface response functions and on the Fourier coefficients that replace the quantum matrix elements in our semiclassical approach.

1. REMINDER OF FORMALISM

1.1. Smooth Surface. In our semiclassical approach we assume that a (heavy) spherical nucleus in its ground state can be described by the following equilibrium phase-space distribution:

$$f_0(\mathbf{r},\mathbf{p}) = \frac{4}{(2\pi\hbar)^3}\vartheta(\epsilon_F - h_0(\mathbf{r},\mathbf{p})) = F(h_0), \tag{1}$$

where ϑ is the step function; ϵ_F is the Fermi energy, while

$$h_0(\mathbf{r},\mathbf{p}) = \frac{p^2}{2m} + V_0(r) = \epsilon \tag{2}$$

is the quasiparticle energy and the equilibrium mean field $V_0(r)$ is assumed to be spherical. In principle the mean field should be determined self-consistently as (Hartree approximation)

$$V_0(\mathbf{r}) = \int d\mathbf{r}' v(\mathbf{r},\mathbf{r}')\varrho_0(\mathbf{r}'), \tag{3}$$

where $v(\mathbf{r},\mathbf{r}')$ is the effective interaction between quasiparticles and

$$\varrho_0(\mathbf{r}) = \int d\mathbf{p} f_0(\mathbf{r},\mathbf{p}) \tag{4}$$

is the equilibrium density of the nucleus (for the sake of simplicity we do not take into account explicitly the spin and isospin degrees of freedom since this would only complicate the formalism without posing any new conceptual difficulty; the statistical weight 4 in Eq. (1) accounts for these degrees of freedom). In practice we shall use instead a phenomenological equilibrium mean field, which can be either of the Saxon–Woods shape or even a simple square-well potential of radius $R = 1.2A^{1/3}$ fm. For the square-well approximation, several expressions can be evaluated analytically and this is one of the merits of our simplified approach. Note that, contrary to the standard Fermi liquid theory, our equilibrium distribution (1) depends also on the space coordinate $\mathbf{r}$.

Next we assume that at time $t = 0$ the system is subject to an external driving field of the kind

$$V^{\rm ext}(\mathbf{r}, t) = \beta\delta(t)Q(\mathbf{r}). \tag{5}$$

Here β is a parameter determining the intensity of the external force, which is applied only for a very short time around $t = 0$, as described by the Dirac δ function, and $Q(\mathbf{r})$ gives the space dependence of the external field. Typically we shall be interested in the multipole response of order L, for which*

$$Q(\mathbf{r}) = r^L Y_{LM}(\hat{\mathbf{r}}). \tag{6}$$

The response of the system to an external force is described by the fluctuation of the phase-space density defined by

$$f(\mathbf{r}, \mathbf{p}, t) = f_0(\mathbf{r}, \mathbf{p}) + \delta f(\mathbf{r}, \mathbf{p}, t) \tag{7}$$

or, equivalently, by its time Fourier transform

$$\delta f(\mathbf{r}, \mathbf{p}, \omega) = \int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\omega t}\delta f(\mathbf{r}, \mathbf{p}, t). \tag{8}$$

Since $\delta f(\mathbf{r}, \mathbf{p}, t)$ vanishes for $t < 0$, we can suppose that ω has a vanishingly small imaginary part $i\varepsilon$ to ensure the convergence of this integral when $t \to +\infty$.

The perturbed system is described by a time-dependent phase-space density satisfying the mean-field (or Vlasov) kinetic equation

$$\frac{\partial f}{\partial t} + \{f, h\} = 0, \tag{9}$$

where the braces are Poisson brackets. The time-dependent Hamiltonian h is given by

$$h(\mathbf{r}, \mathbf{p}, t) = h_0(\mathbf{r}, \mathbf{p}) + \delta h(\mathbf{r}, t), \tag{10}$$

with

$$\delta h(\mathbf{r}, t) = V^{\rm ext}(\mathbf{r}, t) + \delta V^{\rm int}(\mathbf{r}, t). \tag{11}$$

*For compression modes we'll be interested also in $Q(\mathbf{r}) = r^{L+2}Y_{LM}(\hat{\mathbf{r}})$.

This expression shows explicitly that the extra force acting on a particle in the perturbed system has two components: one due to the external driving field $V_{\rm ext}$ and an additional one due to the change in the interaction with the surrounding particles. This last term is given by

$$\delta V^{\rm int}(\mathbf{r},t) = \int d\mathbf{r}' v(\mathbf{r},\mathbf{r}') \int d\mathbf{p}' \delta f(\mathbf{r}',\mathbf{p}',t). \tag{12}$$

If the external force is sufficiently weak, the density fluctuation induced by it is small and we can consider only terms that are linear in β. In this case the fluctuation $\delta f(\mathbf{r},\mathbf{p},t)$ satisfies the linearized Vlasov equation

$$\frac{\partial\,\delta f}{\partial t} + \{\delta f, h_0\} + \{f_0, \delta h\} = 0, \tag{13}$$

or

$$\frac{\partial\,\delta f}{\partial t} + \{\delta f, h_0\} = -F'(h_0)\{h_0, \delta h\}. \tag{14}$$

From the mathematical point of view, if the $(\mathbf{r},\mathbf{p})$ variables are used, this equation is a seven-dimensional differential equation (actually an integro-differential equation because of Eq. (12)) containing partial derivatives with respect to time and to the six variables r_i and p_i. The time derivative can be eliminated by using the Fourier transform (8), while the properties of Poisson brackets suggest that some simplification might be achieved simply by making a change of variables to generalized coordinates and momenta. The new coordinates should be chosen in such a way to include the maximum number of constants of motion of the unperturbed Hamiltonian h_0 so that the corresponding variable will not contribute to the Poisson bracket. For motion in a central force field a convenient set of generalized coordinates is $(\epsilon, \lambda, r, \alpha, \beta, \gamma)$, where ϵ is the particle energy (2); λ, the magnitude of its angular momentum; r, the radial coordinate, and α, β, γ are the Euler angles associated with the rotation of the frame of Cartesian coordinates necessary to align the z axis of the lab frame to the particle angular momentum $\boldsymbol{\lambda}$ and the y axis of the lab frame with the $\mathbf{r}$ vector specifying the instantaneous position of a particle with respect to the force centre (see Fig. 1).

Since four of the six new coordinates are constants of the motion, Eq. (14) simplifies considerably and becomes

$$\frac{\partial\,\delta f}{\partial t} + \frac{\partial\,\delta f}{\partial r}\dot{r} + \frac{\partial\,\delta f}{\partial \gamma}\dot{\gamma} = -F'(\epsilon)\left[\frac{\partial\,\delta h}{\partial r}\dot{r} + \frac{\partial\,\delta h}{\partial \gamma}\dot{\gamma}\right]. \tag{15}$$

By using

$$\dot{\gamma} = \frac{\lambda}{mr^2}, \tag{16}$$

$$\dot{r} = \pm v_r(\epsilon, \lambda, r), \tag{17}$$

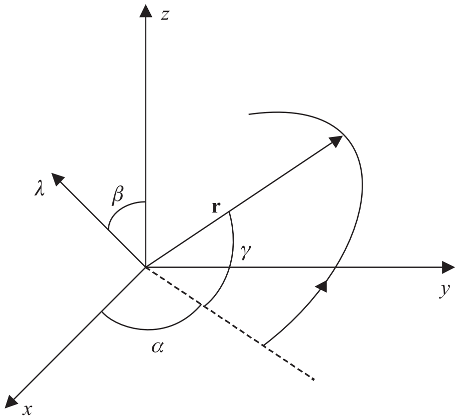

Fig. 1. Angular elements of the orbit. For a particle moving in a central force field, the angles α and β are constant

with

$$v_r(\epsilon, \lambda, r) = \sqrt{\frac{2}{m}\left[\epsilon - V_0(r) - \frac{\lambda^2}{2mr^2}\right]} \tag{18}$$

the magnitude of the radial velocity, we obtain

$$-i\omega\delta f \pm v_r(\epsilon, \lambda, r)\frac{\partial\,\delta f}{\partial r} + \frac{\lambda}{mr^2}\frac{\partial\,\delta f}{\partial\gamma} = F'(\epsilon)\left[\pm v_r(\epsilon, \lambda, r)\frac{\partial\,\delta h}{\partial r} + \frac{\lambda}{mr^2}\frac{\partial\,\delta h}{\partial\gamma}\right] \tag{19}$$

for the Fourier-transformed linearized Vlasov equation.

This equation still contains partial derivatives with respect to the two time-dependent variables r and γ, but the derivative with respect to γ can be eliminated by means of an appropriate partial-wave expansion. The usual partial-wave expansion

$$\delta f(\mathbf{r}, \mathbf{p}, \omega) = \sum_{LM} \delta f_{LM}(r, \mathbf{p}, \omega) Y_{LM}(\vartheta, \varphi), \tag{20}$$

where (ϑ, φ) are the polar angles of the vector $\mathbf{r}$, is a first step in this direction, but it does not solve the problem since the coefficients $\delta f_{LM}(r, \mathbf{p}, \omega)$ still depend on the direction of the vector $\mathbf{p}$, however the following well-known transformation property of the spherical harmonics $Y_{LM}(\vartheta, \varphi)$ under the rotation specified by the Euler angles α, β, γ can help us (see, e.g., ([17, p. 28])*:

$$Y_{LM}(\vartheta, \varphi) = \sum_{N=-L}^{L} \left(\mathcal{D}^L_{MN}(\alpha\beta, \gamma)\right)^* Y_{LN}(\vartheta', \varphi'). \tag{21}$$

In the new reference frame the particle is on the y axis, so $\vartheta' = \pi/2$ and $\varphi' = \pi/2$ and the only time-dependent angle on the right-hand side of Eq. (21) is γ. The functions $\mathcal{D}^L_{MN}(\alpha\beta, \gamma)$ are the coefficients of the rotation matrices and their explicit γ dependence, of the kind $\mathrm{e}^{-iN\gamma}$, can be exploited to eliminate the γ derivative in Eq. (19), giving the following one-dimensional equations:

$$\frac{\partial}{\partial r}\delta f^{L\pm}_{MN} \mp A_N \delta f^{L\pm}_{MN} = B^{L\pm}_{MN}, \tag{22}$$

with

$$A_N(\epsilon, \lambda, r, \omega) = \frac{i\omega}{v_r(\epsilon, \lambda, r)} - \frac{iN}{v_r(\epsilon, \lambda, r)} \frac{\lambda}{mr^2} \tag{23}$$

and

$$B^{L\pm}_{MN}(\epsilon, \lambda, r, \omega) = F'(\epsilon) \left[\frac{\partial}{\partial r} \pm \frac{iN}{v_r(\epsilon, \lambda, r)} \frac{\lambda}{mr^2}\right] [\beta Q_{LM}(r) + \delta V^{\mathrm{int}}_{LM}(r, \omega)], \tag{24}$$

for the coefficients $\delta f^{L\pm}_{MN}(\epsilon, \lambda, r, \omega)$ of the expansion**

$$\delta f(\mathbf{r}, \mathbf{p}, \omega) = \sum_{LMN} [\delta f^{L+}_{MN}(\epsilon, \lambda, r, \omega)\theta(p_r) + \delta f^{L-}_{MN}(\epsilon, \lambda, r, \omega)\theta(-p_r)] \times \\ \times \left(\mathcal{D}^L_{MN}(\alpha\beta, \gamma)\right)^* Y_{LN}\left(\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{25}$$

The functions $Q_{LM}(r)$ and $\delta V^{\mathrm{int}}_{LM}(r, \omega)$ in Eq. (24) are the coefficients of a multipole expansion similar to (20) for the external driving field and for the induced mean-field fluctuation, respectively. In Eq. (25) instead, the θ functions are the usual step function $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ otherwise, while p_r is the radial component of the particle momentum: $p_r = \pm m v_r$.

Thus, by making an appropriate change of variables (and by taking the time Fourier transform), the initial seven-dimensional differential equation (14) has been reduced to the system of two (coupled) one-dimensional differential equations (22). These two equations, involving the distributions δf^+ and δf^-

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*In this rather technical aspect the derivation of [14] differs by that of [10], which is less straightforward (cf. also [11]).

**Note that, contrary to what is done in Ref. 14, here the factor $Y_{LN}(\pi/2, \pi/2)$ is not included in our definition of the coefficients $\delta f^{L\pm}_{MN}$.

of particles with both signs of the radial velocity, are coupled by the mean-field fluctuation, as shown explicitly by the following expression:

$$\delta V_{LM}^{\rm int}(r,\omega) = = \frac{8\pi^2}{2L+1}\sum_{N=-L}^{L}\left|Y_{LN}\left(\frac{\pi}{2},\frac{\pi}{2}\right)\right|^2\int d\epsilon\int d\lambda\lambda\int\frac{dr'}{v_r(\epsilon,\lambda,r')}v_L(r,r')\times \times[\delta f_{MN}^{L+}(\epsilon,\lambda,r',\omega)+\delta f_{MN}^{L-}(\epsilon,\lambda,r',\omega)]. \quad (26)$$

For the most effective interactions $v_L(r,r')$, because of this coupling, the solution of Eq. (22) can only be given in implicit form, however, an explicit solution can be obtained if we neglect the term $\delta V_{LM}^{\rm int}(r,\omega)$ in Eq. (24). Following Ref. 14, we refer to this as the zero-order approximation and recall here the solution details.

In order to solve Eq. (22), we must first specify the boundary conditions satisfied by $\delta f^+(r)$ and $\delta f^-(r)$ at the turning points r_1 and r_2. The boundary conditions used in [14] were:

$$\delta f_{MN}^{L+}(\epsilon,\lambda,r_1,\omega)=\delta f_{MN}^{L-}(\epsilon,\lambda,r_1,\omega), \quad (27)$$

$$\delta f_{MN}^{L+}(\epsilon,\lambda,r_2,\omega)=\delta f_{MN}^{L-}(\epsilon,\lambda,r_2,\omega), \quad (28)$$

their physical meaning is that, at the turning points the radial motion of the particles simply reverses. For the square-well potential, the condition at the outer turning point implies a mirror reflection of the nucleons on the static equilibrium nuclear surface. As we shall see, in the sharp-surface case there are reasons for modifying this boundary condition, however for the moment we assume a diffused surface and determine the solution of Eq. (22) by using the boundary conditions (27) and (28). The solution can be written as (in slightly simplified notation)

$$\delta f_{MN}^{L\pm}(\epsilon,\lambda,r,\omega)={\rm e}^{\pm i\phi_N(r,\omega)}\left[\int_{r_1}^{r}dr'B_{MN}^{L\pm}(r')\,{\rm e}^{\mp i\phi_N(r',\omega)}+C_\pm(\epsilon,\lambda,\omega)\right], \quad (29)$$

with

$$\phi_N(r,\omega)=-i\int_{r_1}^{r}dyA_N(y)=\omega\tau(r)-N\gamma(r), \quad (30)$$

$$\tau(r)=\int_{r_1}^{r}dr'\frac{1}{v_r(\epsilon,\lambda,r')}, \quad (31)$$

$$\gamma(r)=\int_{r_1}^{r}dr'\frac{\lambda}{mr'^2}\frac{1}{v_r(\epsilon,\lambda,r')}. \quad (32)$$

The functions $C_{\pm}(\epsilon,\lambda,\omega)$ play the role of integration constants and are determined by the boundary conditions. The inner boundary condition (27) implies

$$C_- = C_+, \tag{33}$$

while the outer boundary condition (28) implies

$$\begin{aligned} &\mathrm{e}^{i\phi_N(r_2,\omega)}\left[\int_{r_1}^{r_2} dr' B_{MN}^{L+}(r')\,\mathrm{e}^{-i\phi_N(r',\omega)} \;+\; C_+\right] = \\ &\qquad = \mathrm{e}^{-i\phi_N(r_2,\omega)}\left[\int_{r_1}^{r_2} dr' B_{MN}^{L-}(r')\,\mathrm{e}^{i\phi_N(r',\omega)} \;+\; C_+\right]. \end{aligned} \tag{34}$$

Defining

$$D_{\pm}(\epsilon,\lambda,\omega) = \int_{r_1}^{r_2} dr' B_{MN}^{L\pm}(r')\,\mathrm{e}^{-i\phi_N(r',\omega)} \tag{35}$$

gives

$$\mathrm{e}^{2i\phi_N(r_2,\omega)}[D_+ \;+\; C_+] = D_- \;+ C_+, \tag{36}$$

that is,

$$C_+(\epsilon,\lambda,\omega) = \frac{\mathrm{e}^{2i\phi_N(r_2,\omega)}D_+ - D_-}{1-\mathrm{e}^{2i\phi_N(r_2,\omega)}} = C_-(\epsilon,\lambda,\omega). \tag{37}$$

The most interesting property of the solution (29) is its pole structure in the complex-ω plane, which is entirely determined by the poles of the functions $C_{\pm}(\epsilon,\lambda,\omega)$, that is, by the vanishing of the denominator in Eq. (37). This happens whenever $2\phi_N(r_2,\omega) = n\,2\pi$, with integer n. This is the point where the finite size of the system plays a crucial role since the eigenfrequencies of the density fluctuations in the zero-order approximation are determined by the condition

$$\omega[2\tau(r_2)] - N[2\gamma(r_2)] = n\,2\pi, \tag{38}$$

and the period of radial motion $T(\epsilon,\lambda) = 2\tau(r_2)$ depends on the size of the system. For motion in a central potential, these eigenfrequencies are characterized by the two integers n and N:

$$\omega_{nN}(\epsilon,\lambda) = n\omega_0 + N\omega_{\varphi}, \tag{39}$$

with

$$\omega_0(\epsilon,\lambda) = \frac{2\pi}{T(\epsilon,\lambda)}, \tag{40}$$

$$\omega_{\varphi}(\epsilon,\lambda) = \frac{\gamma(r_2)}{\tau(r_2)} \tag{41}$$

being the frequencies of radial and angular motion for a particle with energy ϵ and angular momentum magnitude λ. The eigenfrequencies (39) can be related to the *difference* of single-particle energy levels ϵ_{nl} in a central potential since, for large quantum numbers ([18, p. 579]),

$$\epsilon_{n'l'} - \epsilon_{n''l''} \approx (n' - n'')\frac{\partial \epsilon}{\partial n} + (l' - l'')\frac{\partial \epsilon}{\partial l} = \hbar\omega_{n'-n''\,l'-l''}. \tag{42}$$

Thus, the integer n can be interpreted as the difference between radial quantum numbers in a single-particle excitation; and the integer N, as the difference between the corresponding orbital quantum numbers.

The time-dependent density fluctuations can be obtained from the functions $\delta f^{L\pm}_{MN}(\epsilon, \lambda, r, \omega)$ by contour integration in the complex-ω plane. For the zero-order fluctuations we obtain:

$$\delta f^{0L\pm}_{MN}(\epsilon, \lambda, r, t) = 0 \quad \text{for} \quad t < 0, \tag{43}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, \mathrm{e}^{-i\omega t}\delta f^{0L\pm}_{MN}(\epsilon, \lambda, r, \omega) \quad \text{for} \quad t > 0, \tag{44}$$

with

$$\delta f^{0L\pm}_{MN}(\epsilon, \lambda, r, \omega) = -\beta F'(\epsilon)\sum_{n=-\infty}^{\infty}\omega_{nN}\,\mathrm{e}^{\pm i\phi_{nN}(r)}\frac{Q_{nN}(\epsilon, \lambda)}{\omega - \omega_{nN} + i\varepsilon}, \tag{45}$$

$$\phi_{nN}(r) = \omega_{nN}\tau(r) - N\gamma(r) \tag{46}$$

and

$$Q_{nN}(\epsilon, \lambda) = \frac{1}{\tau(r_2)}\int_{r_1}^{r_2} dr \frac{Q_{LM}(r)}{v_r(\epsilon, \lambda, r)}\cos[\phi_{nN}(r)]. \tag{47}$$

This result is obtained from Eq. (29) by closing the integration path in the lower part of the complex-ω plane in the integral (44). The coefficients (47) correspond to the classical limit of the quantum matrix elements of the external field (5) [19].

Thus we have seen that, for a spherical nucleus, the linearized Vlasov equation can be solved explicitly in the approximation in which the mean-field fluctuation is neglected. This zero-order solution can be used as a starting point for solving also the more general problem in which the mean field fluctuation δV^{int} is taken into account. The zero-order solution is most conveniently expressed in terms of a semiclassical propagator (obtained from Eq. (45), see [14]) which is analogous to the quantum particle-hole propagator

$$D^0_L(r, r', \omega) = \frac{8\pi^2}{2L+1}\sum_{n=-\infty}^{+\infty}\sum_{N=-L}^{L}\left|Y_{LN}\left(\frac{\pi}{2}, \frac{\pi}{2}\right)\right|^2 \int d\epsilon F'(\epsilon)\int d\lambda\lambda \times$$
$$\times \frac{1}{T(\epsilon, \lambda)}\frac{\cos[\phi_{nN}(r)]}{r^2 v_r(\epsilon, \lambda, r)}\frac{\omega_{nN}}{\omega_{nN} - (\omega + i\varepsilon)}\frac{\cos[\phi_{nN}(r')]}{r'^2 v_r(\epsilon, \lambda, r')}. \tag{48}$$

Taking into account also the mean-field fluctuation (12) gives the collective response of the system and leads, in general, to an integral equation which is analogous to the RPA equation for the quantum propagators

$$D_L(r,r',\omega) = D_L^0(r,r',\omega) + \int dx x^2 \int dy y^2 D_L^0(r,x,\omega) v_L(x,y) D_L(y,r',\omega). \tag{49}$$

The only differences between the present expression and the quantum result are that here the propagator D^0 is given in semiclassical approximation and, of course, the exchange (Fock) term is missing. The integral equation (49) can be easily solved numerically (the zero-order propagator is actually simpler than suggested by Eq. (48) since, for small values of ω, the infinite sum over n can be approximated with a sum over a few terms around $n = 0$ with very good accuracy). The collective (multipole) response function is then given by

$$\mathcal{R}_L(\omega) = \int dr r^2 \int dr' r'^2 Q_{LM}(r) D_L(r,r',\omega) Q_{LM}(r'), \tag{50}$$

with D_L solution of (49), and the corresponding strength function by

$$S_L(\hbar\omega) = -\frac{1}{\pi} \mathrm{Im}\, \mathcal{R}_L(\hbar\omega) \tag{51}$$

(for a spherical system the response is independent of M).

However there is a special case in which also the collective solution of the linearized Vlasov equation can be obtained explicitly. This happens if the interaction between particles is supposed to be of the separable multipole–multipole type:

$$v(\mathbf{r},\mathbf{r}') = \sum_{LM} \kappa_L r^L r'^L Y_{LM}(\hat{r}) Y^*_{LM}(\hat{r}') \tag{52}$$

and the external field is also of the multipole type (6). In this case Eqs. (49) and (50) give immediately

$$\mathcal{R}_L(\omega) = \frac{\mathcal{R}_L^0(\omega)}{1 - \kappa_L \mathcal{R}_L^0(\omega)}, \tag{53}$$

with the zero-order response function $\mathcal{R}_L^0(\omega)$ given by

$$\mathcal{R}_L^0(\omega) = \frac{1}{\beta} \frac{8\pi^2}{2L+1} \sum_{N=-L}^{L} \left| Y_{LN}\left(\frac{\pi}{2},\frac{\pi}{2}\right)\right|^2 \int d\epsilon \int d\lambda \lambda \times$$
$$\times \int_{r_1}^{r_2} dr \frac{Q_{LM}(r)}{v_r(\epsilon,\lambda,r)} [\delta f_{MN}^{0L+}(\epsilon,\lambda,r,\omega) + \delta f_{MN}^{0L-}(\epsilon,\lambda,r,\omega)]. \tag{54}$$

Moreover the fluctuation δf can also be obtained in an explicit form, like δf^0. For the separable interaction (52), Eq. (26) for the mean-field fluctuation gives

$$\delta V_{LM}^{\rm int}(r,\omega) = \beta\kappa_L r^L \mathcal{R}_L(\omega), \tag{55}$$

while Eq. (24) gives

$$B_{MN}^{L\pm}(\epsilon,\lambda,r,\omega) = F'(\epsilon)\left[\frac{\partial}{\partial r} \pm \frac{iN}{v_r(\epsilon,\lambda,r)}\frac{\lambda}{mr^2}\right][\beta r^L + \beta\kappa_L r^L \mathcal{R}_L(\omega)], \tag{56}$$

and, from Eq. (22), we get $\delta f_{MN}^{L\pm}(\epsilon,\lambda,r,\omega) = \delta f_{MN}^{0L\pm}(\epsilon,\lambda,r,\omega)[1+\kappa_L\mathcal{R}_L(\omega)]$ or, by using Eq. (53),

$$\delta f_{MN}^{L\pm}(\epsilon,\lambda,r,\omega) = \frac{\delta f_{MN}^{0L\pm}(\epsilon,\lambda,r,\omega)}{1-\kappa_L\mathcal{R}_L^0(\omega)}. \tag{57}$$

1.2. Action-Angle Variables. Up to now we have assumed a spherically symmetric equilibrium mean field, however the method outlined here for the solution of the linearized Vlasov equation is valid also for a wider class of physical systems. This method, which is based on the use of generalized coordinate and momenta in order to simplify the Vlasov equation, can actually be used for all systems which are described by an integrable equilibrium Hamiltonian $h_0(\mathbf{r},\mathbf{p})$. Such a Hamiltonian includes also some deformed systems [20]. For integrable systems, it is convenient to introduce action-angle variables $(\mathbf{I},\mathbf{\Phi})$ instead of $(\mathbf{r},\mathbf{p})$, since in this case the action variables I_α are constants of the motion, while the angle variables Φ_α are linear functions of time (see, for example, Ref. [21, p. 457]). An important property of these variables is that the motion is periodic in the angle variables with period 2π. Consequently the field felt by a particle that is moving along a trajectory determined by an integrable Hamiltonian can be Fourier expanded as

$$V^{\rm ext}(\mathbf{r},\omega) = \beta\sum_{\mathbf{n}} Q_{\mathbf{n}}(\mathbf{I})\,{\rm e}^{i\mathbf{n}\cdot\mathbf{\Phi}}, \tag{58}$$

and

$$\delta V^{\rm int}(\mathbf{r},\omega) = \sum_{\mathbf{n}} \delta V_{\mathbf{n}}(\mathbf{I},\omega)\,{\rm e}^{i\mathbf{n}\cdot\mathbf{\Phi}}, \tag{59}$$

where $\mathbf{n}$ is a three-dimensional vector with integer components.

The phase-space-density fluctuation can also be expanded in the same way:

$$\delta f(\mathbf{r},\mathbf{p},\omega) = \sum_{\mathbf{n}} \delta f_{\mathbf{n}}(\mathbf{I},\omega)\,{\rm e}^{i\mathbf{n}\cdot\mathbf{\Phi}}, \tag{60}$$

and the linearized Vlasov equation gives the following equation for the coefficients $\delta f_{\mathbf{n}}(\mathbf{I},\omega)$:

$$\delta f_{\mathbf{n}}(\mathbf{I},\omega) = F'(\epsilon)\Big[\beta Q_{\mathbf{n}}(\mathbf{I}) + \delta V^{\rm int}_{\mathbf{n}}(\mathbf{I},\omega)\Big]\frac{\mathbf{n}\cdot\boldsymbol{\omega}}{\mathbf{n}\cdot\boldsymbol{\omega} - (\omega + i\varepsilon)}, \tag{61}$$

where the vector $\boldsymbol{\omega}$ has components

$$\omega_\alpha = \frac{\partial h_0(\mathbf{I})}{\partial I_\alpha}\,. \tag{62}$$

Again, Eq. (61) gives only an implicit solution of the Vlasov equation, since the mean-field fluctuation $\delta V^{\rm int}$ depends on δf. If the term $\delta V^{\rm int}$ is neglected, then Eq. (61) is an explicit solution that, in the case of spherical systems, agrees with the zero-order solution (45). Generally speaking, the vector $\boldsymbol{\omega}$ has three components, but in spherical systems there are only two basic eigenfrequencies: ω_0 and ω_φ. This is because spherical systems are over-integrable and this implies that one of the angle variables is also a constant of the motion. The coefficients $Q_{\mathbf{n}}(\mathbf{I})$ in Eq. (61) are given by

$$Q_{\mathbf{n}}(\mathbf{I}) = \frac{1}{2\pi^3}\int d\boldsymbol{\Phi}\, \mathrm{e}^{-i\mathbf{n}\cdot\boldsymbol{\Phi}} Q(\mathbf{r}) \tag{63}$$

and correspond to the quantum matrix elements of the operator $Q(\mathbf{r})$.

1.3. Sharp and Moving Surface. With the aim of establishing a link between the present microscopic theory of nuclear excitations and the macroscopic description given by the liquid-drop model (see, e.g., [18, Appendix 6A]) the authors of [16] studied in detail the model in which the equilibrium mean field is approximated by a square-well potential: $V_0(r) = -V_0\vartheta(R-r)$. They noticed that in this case the boundary condition (28) corresponds to a mirror reflection of nucleons when they reach the static nuclear surface at $r = R$. With that boundary condition the sharp-surface model allows only for compressional excitations, while a liquid drop has both surface and compression modes. They argued that, in order to allow also for a microscopic description of surface modes, in the sharp-surface case, the boundary condition (28) should be modified and they proposed to replace it with the following (moving-surface) boundary condition:

$$\delta\tilde{f}^{L+}_{MN}(\epsilon,\lambda,R,\omega) = \delta\tilde{f}^{L-}_{MN}(\epsilon,\lambda,R,\omega) + F'(\epsilon)2mv_r(\epsilon,\lambda,R)i\omega\delta R_{LM}(\omega). \tag{64}$$

Whenever the same symbol is used, we put a tilde over the quantities evaluated with the moving-surface boundary condition (64), to distinguish them from the corresponding quantities satisfying the fixed-surface boundary condition (28). The physical picture behind this new boundary condition still corresponds to a mirror reflection of nucleons at the nuclear surface $r = R$, but in a reference frame that is moving in the radial direction at a speed $\dot{R}(\vartheta, \varphi, t)$. The radial momentum transferred by the moving surface to the impinging nucleon will differ from that occurring in a collision with the static surface and this modifies the momentum distribution of nucleons according to (64), in the first approximation. Formally, in this approach the nuclear surface is allowed to vibrate (for $t > 0$) according to the usual liquid-drop model expression

$$R(\vartheta, \varphi, t) = R + \sum_{LM} \delta R_{LM}(t) Y_{LM}(\vartheta, \varphi), \tag{65}$$

giving

$$\dot{R}(\vartheta, \varphi, t) = \sum_{LM} \delta \dot{R}_{LM}(t) Y_{LM}(\vartheta, \varphi) \tag{66}$$

for the surface speed and

$$\dot{R}(\vartheta, \varphi, \omega) = \sum_{LM} -i\omega \delta R_{LM}(\omega) Y_{LM}(\vartheta, \varphi) \tag{67}$$

for its time Fourier transform, thus leading to the extra term in Eq. (64).

The new collective coordinates $\delta R_{LM}(\omega)$ are still to be determined. In [16] this has been done by recalling that in the liquid-drop model a change in the curvature radius of the surface results in a change of pressure given by (see Eq. (6A-57) of [18])

$$\delta \mathcal{P}(R, \theta, \varphi, \omega) = \sum_{LM} C_L \frac{\delta R_{LM}(\omega)}{R^4} Y_{LM}(\theta, \varphi), \tag{68}$$

with the restoring force parameter C_L that can be related to the phenomenologically determined surface tension parameter $\sigma \approx 1\ \mathrm{MeV} \cdot \mathrm{fm}^{-2}$. If the Coulomb repulsion between protons is neglected, this relation is simply

$$C_L = \sigma R^2 (L-1)(L+2), \tag{69}$$

while taking into account also the Coulomb interaction gives an additional contribution to C_L (see [18, p. 660]). The pressure fluctuation (68) can also be related to the appropriate component of the pressure tensor (generalized to Fermi liquids, see [22, Sec. 74])

$$\delta \mathcal{P}(R, \theta, \varphi, \omega) = \int d\mathbf{p} m v_r^2 \big(\delta \tilde{f}(\mathbf{R}, \mathbf{p}, \omega) - F'(\epsilon) \delta \tilde{V}^{\mathrm{int}}(\mathbf{R}, \omega) \big). \tag{70}$$

By equating the pressure fluctuations given by Eqs. (68) and (70), the collective coordinates $\delta R_{LM}(\omega)$ can be related to the phase-space-density fluctuation:

$$\delta R_{LM}(\omega) = \frac{8\pi^2}{2L+1}\frac{R^2}{C_L}\sum_{N=-L}^{L}\left|Y_{LN}(\frac{\pi}{2},\frac{\pi}{2})\right|^2 \int d\epsilon \int d\lambda\lambda m v_r(\epsilon,\lambda,R)\times$$
$$\times\left[\delta\tilde{f}^{L+}_{MN}(\epsilon,\lambda,R,\omega)+\delta\tilde{f}^{L-}_{MN}(\epsilon,\lambda,R,\omega)-2F'(\epsilon)\delta\tilde{V}^{\rm int}_{LM}(R,\omega)\right]. \quad (71)$$

The internal part of the mean-field fluctuation $\delta\tilde{V}^{\rm int}_{LM}(r,\omega)$ in the moving-surface case will be specified better in Appendix A. This can be done most easily by assuming a separable interaction of the kind (52). We have already seen that such an effective interaction leads to simple analytical expressions for the multipole response function and for the solution of the linearized Vlasov equation with fixed-surface boundary conditions. The same happens also in the moving-surface case, although the final expressions are somewhat more involved. Since the explicit derivation of the moving-surface multipole response function is rather lengthy, we report here only the final result

$$\tilde{\mathcal{R}}_L(\omega) = \mathcal{R}_L(\omega) -$$
$$-\frac{1}{1-\kappa_L\mathcal{R}^0_L(\omega)}\frac{\left[\chi^0_L(\omega)+\frac{3}{4\pi}A\kappa_L R^L\mathcal{R}^0_L(\omega)\right]^2}{[C_L-\chi_L(\omega)][1-\kappa_L\mathcal{R}^0_L(\omega)]+\kappa_L\left[\chi^0_L(\omega)+\frac{3}{4\pi}AR^L\right]^2}, \quad (72)$$

and outline it in Appendix A. The response function $\mathcal{R}_L(\omega)$ in the equation above is still that given by Eq. (53), while the functions $\chi_L(\omega)$ and $\chi^0_L(\omega)$ are defined as*

$$\chi_L(\omega) = \frac{8\pi^2}{2L+1}R^2\sum_{N=-L}^{L}\left|Y_{LN}\left(\frac{\pi}{2},\frac{\pi}{2}\right)\right|^2\times$$
$$\times\int d\epsilon\int d\lambda\lambda 2F'(\epsilon)\cot\left[\phi_N(R,\omega)\right][m v_r(\epsilon,\lambda,R)]^2\omega, \quad (73)$$

and

$$\chi^0_L(\omega) = \frac{1}{\beta}\frac{8\pi^2}{2L+1}R\sum_{N=-L}^{L}\left|Y_{LN}\left(\frac{\pi}{2},\frac{\pi}{2}\right)\right|^2\times$$
$$\times\int d\epsilon\int d\lambda\lambda m v_r(\epsilon,\lambda,R)\left[\delta f^{0L+}_{MN}(\epsilon,\lambda,R,\omega)+\delta f^{0L-}_{MN}(\epsilon,\lambda,R,\omega)\right]. \quad (74)$$

*In Refs. 23, 24, a different normalization of χ^0_L has been used.

2. ISOSCALAR EXCITATIONS IN HEAVY NUCLEI

In the remaining part of this paper we review some applications of the theory outlined in the first part to the study of isoscalar nuclear response. Our approximation for the mean field (a square-well potential) is not particularly realistic since it neglects the surface diffusion and the assumed residual interaction of the multipole–multipole type is also rather special, however these approximations have the advantage of leading to simple analytical formulae for the nuclear response functions. Our semiclassical approach shows that some features of the nuclear response that are usually ascribed to quantum effects can be understood in terms of classical concepts like nucleon trajectories. The correspondence between shell effects (not included in our treatment) and the properties of classical trajectories helps one to shed a new light on the nuclear response at low energy.

Our starting point is the zero-order response function (54). By using the explicit expression (45) of the phase-space-density fluctuations, with $F'(\epsilon) = -\frac{4}{(2\pi)^3}\delta(\epsilon_F - \epsilon)$, this equation gives

$$\mathcal{R}_L^0(s) = \frac{9A}{8\pi}\frac{R^{2L}}{\epsilon_F}\sum_{N=-L}^{L} C_{LN}^2 \sum_{n=-\infty}^{+\infty}\int_0^1 dx x^2 \frac{s_{nN}(x)}{s - s_{nN}(x) + i\varepsilon}\left(\frac{Q_{nN}^{(L)}(x)}{R^L}\right)^2, \tag{75}$$

with $x = \sin\alpha$, $\cos\alpha = \lambda/(p_F R)$, $s = \omega/(v_F/R)$, $v_F = p_F/m$,

$$s_{nN}(x) = \frac{n\pi + N\alpha(x)}{x}, \tag{76}$$

$$Q_{nN}^{(k)}(x) = \frac{2}{T}\int_{r_1}^{R} dr \frac{r^k}{v_r(\epsilon_F, \lambda, r)}\cos\left[\phi_{nN}(r)\right], \tag{77}$$

and

$$C_{LN}^2 = \frac{4\pi}{2L+1}\left|Y_{LN}\left(\frac{\pi}{2}, \frac{\pi}{2}\right)\right|^2. \tag{78}$$

Instead of the frequency ω, we have introduced the dimensionless parameter s; and instead of the particle angular momentum λ, the parameter $x = \sin\alpha$. The Fourier coefficients (77) can be easily evaluated explicitly, the expressions needed here are grouped together in Appendix B. In terms of the new dimensionless variables, the auxiliary functions χ_L^0 and χ_L appearing in Eq. (72) read

$$\chi_L^0(s) = \frac{9A}{4\pi}R^L \sum_{nN} C_{LN}^2 \int_0^1 dx x^2 s_{nN}(x)\frac{(-)^n Q_{nN}^{(L)}(x)/R^L}{s + i\varepsilon - s_{nN}(x)} \tag{79}$$

and

$$\chi_L(s) = -\frac{9A}{2\pi}\epsilon_F s \sum_{nN} C_{LN}^2 \int_0^1 dx x^2 \frac{1}{s + i\varepsilon - s_{nN}(x)}. \tag{80}$$

In the last equation we have used the pole expansion of the cotangent:

$$\cot z = \sum_{n=-\infty}^{+\infty} \frac{1}{z - n\pi}.$$

2.1. Monopole Response. This channel corresponds to a compression mode. For these modes the radial dependence of the external field is not given by Eq. (6), but rather by

$$Q(\mathbf{r}) = r^{L+2} Y_{LM}(\hat{\mathbf{r}}). \tag{81}$$

As a consequence the zero-order response function (75) will involve the Fourier coefficients $Q_{nN}^{(L+2)}(x)$ instead of $Q_{nN}^{(L)}(x)$ and the same is true for the auxiliary function $\chi_L^0(s)$ in Eq. (79). Accordingly, the moving-surface response function (72) is also slightly changed to

$$\tilde{\mathcal{R}}_L(\omega) = \mathcal{R}_L(\omega) -$$
$$- \frac{1}{1 - \kappa_L \mathcal{R}_L^0(\omega)} \frac{\left[\chi_L^0(\omega) + \frac{3}{4\pi} A \kappa_L R^{L+2} \mathcal{R}_L^0(\omega)\right]^2}{[C_L - \chi_L(\omega)][1 - \kappa_L \mathcal{R}_L^0(\omega)] + \kappa_L \left[\chi_L^0(\omega) + \frac{3}{4\pi} A R^{L+2}\right]^2}. \tag{82}$$

For $L = 0$ this expression gives the collective moving-surface strength function shown in Fig. 2 (solid curve, $E = \hbar\omega$). The collective strength function of Fig. 2 has been calculated by assuming a residual interaction

$$v_{L=0}(r, r') = \kappa_{L=0} r^2 r'^2, \tag{83}$$

with a value of $\kappa_{L=0} = -2 \cdot 10^{-4}$ MeV $\cdot$ fm^{-4}. This parameter has been determined by fitting the experimental position of the giant monopole resonance in ^{208}Pb.

The dotted curve instead shows the zero-order strength function (proportional to the imaginary part of the response function $\mathcal{R}_L^0(\omega)$, which is similar to the quantum single-particle response function).

Finally, the dashed curve shows the collective moving-surface response given by Eq. (82) with $L = 0$ and $\kappa_L = 0$. If $\kappa_L = 0$, the frequency of the collective monopole vibration is determined by the solution of the equation

$$C_L - \chi_L(\omega) = 0. \tag{84}$$

It has already been pointed out in [16] that this approximation gives a very reasonable description of the position (including the A dependence) of the isoscalar giant monopole resonance in heavy nuclei.

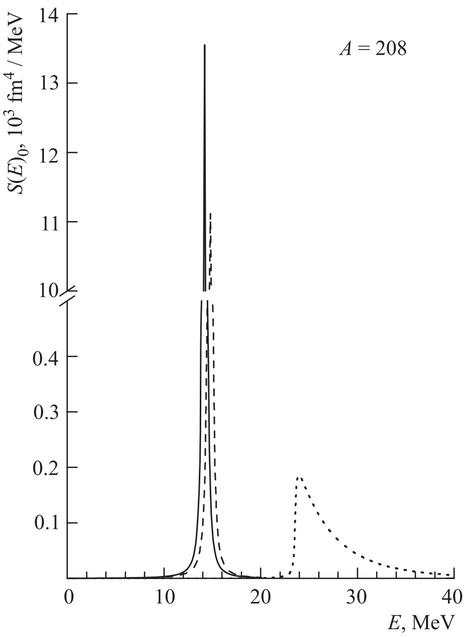

Fig. 2. Monopole strength function in three different approximations: zero-order (dotted curve), collective with finite value of κ_L (solid) and collective with $\kappa_L = 0$ (dashed). Note the change in the vertical scale

Another interesting feature of the monopole response pointed out in [16] is that the zero-order strength function vanishes for $\omega < \pi v_F/R$. As a consequence of this fact, within this model, there is no Landau damping of the collective monopole mode. This absence of Landau damping is in qualitative agreement with the results of analogous quantum calculations [25, 26]. The very small width appearing in Fig. 2 is due to our use of a fnite value of the infinitesimal parameter ε (for numerical reasons, we have used $\varepsilon = 0.1$ MeV).

We have checked numerically that the collective state shown in Fig. 2 exhausts about 99% of the monopole energy-weighted sum rule, which is given by [27]

$$\int_0^\infty dE\, E\, S(E) = \frac{3}{10\pi}\frac{\hbar^2}{m}AR^2. \tag{85}$$

2.2. Dipole Response (Translation and Compression Modes). It is well known that the mean-field approximation violates the translation invariance of the nuclear Hamiltonian and that this results in the appearing of spurious strength in the isoscalar dipole response. Hence, the isoscalar dipole channel, excited by the external field (6) with $L = 1$, is usually not interesting because it should correspond to a simple translation of an unexcited nucleus, while in the mean-field approximation this pure translation is replaced by a spurious excitation of the nuclear centre-of-mass bound by an unphysical force. However the corresponding compression mode, excited by the field

$$Q(\mathbf{r}) = r^3 Y_{1M}(\hat{r}), \tag{86}$$

has received considerable attention because of the possibility of obtaining from it additional information about the compressibility of nuclear matter [28]. Since the external field (86) can excite also the centre of mass, the problem of subtracting the unwanted spurious strength from the corresponding response function has usually been dealt with by using an ingenious trick due to Van Giai and Sagawa [29]. These authors suggested that, instead of studying the response to the external field (86), one should look at the response to an effective external field of the kind

$$Q_{\text{eff}}(\mathbf{r}) = (r^3 - \eta r) Y_{1M}(\hat{r}), \tag{87}$$

where η is a parameter determined by the condition that, under the action of the external force, the centre of mass should remain at rest.

The moving-surface theory of [16] allows for a different approach to the problem of evaluating the intrinsic response associated with the field (86). It is clear from expression (69) of the restoring-force parameter C_L that this parameter vanishes for $L = 1$. This means that in this case there is no restoring force, hence the moving-surface boundary condition seems to be able to readjust the translation symmetry that is broken by the mean-field approximation. This statement can be easily verified by looking at the form taken by the response function (82) when the residual interaction is neglected. If we put $\kappa_L = 0$ in Eq. (82), the moving-surface response function becomes

$$\tilde{\mathcal{R}}^0_{L=1}(\omega) = \mathcal{R}^0_1(\omega) - \frac{[\chi^0_1(\omega)]^2}{[-\chi_1(\omega)]}, \tag{88}$$

or (cf. Eq. (3.1) of Ref. 23)

$$\tilde{\mathcal{R}}^0_{L=1}(\omega) = \frac{3}{4\pi} \frac{A}{m\omega^2}, \tag{89}$$

which has the behavior expected for a free particle. Since this response function has no poles for $\omega \neq 0$, it does not give spurious dissipation at positive ω.

The translation invariance of the model when the residual interaction is taken into account is less obvious, however it has been shown in [23], by using sum rule arguments, that no spurious strength is added to the zero-order intrinsic response even when a residual dipole–dipole interaction is included. Thus we are confident that the present semiclassical theory with moving-surface boundary conditions correctly separates the intrinsic from the centre-of-mass excitations and do not need to use the effective operator (87).

For the sake of simplicity, here we discuss in some detail only the zero-order response, in which the residual interaction is neglected. This approximation corresponds to treating the nucleus as a gas of noninteracting fermions confined to a spherical cavity with perfectly reflecting walls that are allowed to translate freely. The residual interaction changes the compressibility of this nuclear fluid and its effects on the response have been evaluated in Ref. 23.

We first need to give a slight generalization of the zero-order response function (75), by defining the functions

$$\mathcal{R}^0_{L,jk}(s) = \frac{9A}{16\pi}\frac{R^{j+k}}{\epsilon_F}\sum_{N=-L}^{L} C^2_{LN}\times \\ \times \sum_{n=-\infty}^{+\infty}\int_0^1 dx x^2 s_{nN}(x)\frac{(Q^{(j)}_{nN}(x)/R^j)(Q^{(k)}_{nN}(x)/R^k)}{s - s_{nN}(x) + i\varepsilon}. \quad (90)$$

In terms of these new functions, the fixed-surface zero-order response to the external field (86) is determined by the function $\mathcal{R}^0_{1,33}(s)$, while $\mathcal{R}^0_{1,13}(s)$ has a direct physical interpretation as the displacement of the nuclear centre of mass induced by the external field (86) [23]. Similarly $\mathcal{R}^0_{1,11}(s)$ gives the response to the field $rY_{1M}(\hat{r})$.

It has been shown in [23] that the (zero-order) moving-surface response function for the field (86) can be written as

$$\tilde{\mathcal{R}}^0_{1,33}(s) = \tilde{\mathcal{R}}^0_{\rm cm}(s) + \tilde{\mathcal{R}}^0_{\rm intr}(s), \quad (91)$$

with

$$\tilde{\mathcal{R}}^0_{\rm cm}(s) = \frac{3A}{4\pi}\frac{R^6}{2\epsilon_F}\frac{1}{s^2} \quad (92)$$

and

$$\tilde{\mathcal{R}}^0_{\rm intr}(s) = \mathcal{R}^0_{1,33}(s) - \frac{3A}{4\pi}\frac{R^6}{2\epsilon_F}\frac{1}{s^2}\left\{1 - \frac{\left[1 - \frac{1}{2}s^2\frac{\mathcal{R}^0_{1,13}(s)}{\mathcal{M}^1_{13}}\right]^2}{1 - \frac{1}{2}s^2\frac{\mathcal{R}^0_{1,11}(s)}{\mathcal{M}^1_{11}}}\right\}. \quad (93)$$

The moments

$$\mathcal{M}^p_{jk} = \int_0^\infty ds s^p\left[-\frac{1}{\pi}{\rm Im}\,\mathcal{R}^0_{1,jk}(s)\right] \quad (94)$$

are defined in terms of the generalized fixed-surface response functions (90) and they can be easily evaluated, giving $\mathcal{M}^1_{11} = \frac{1}{3}\frac{9A}{16\pi}\frac{R^2}{\epsilon_F}$ and $\mathcal{M}^1_{13} = R^2\mathcal{M}^1_{11}$. An essential property of the intrinsic response function (93) is that its limit for $s \to 0$ is finite, so it has no pole in $\omega = 0$.

A properly normalized and energy-weighted strength function associated with the intrinsic response function (93) is shown in Fig. 3. It is interesting to note

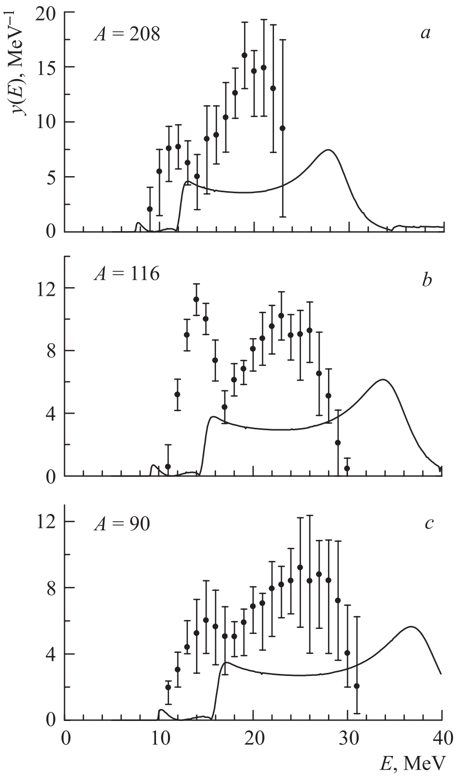

Fig. 3. Comparison of our energy-weighted strength function with data from Ref. 30. The curve shows the strength function in the case of vanishing residual interaction, i.e., for a confined Fermi gas with an incompressibility of $K = 200$ MeV

that the simple model used here qualitatively reproduces the experimental data of [30], in particular the double-peak structure of the dipole compression mode. Clearly at this level we can only hope in a qualitative agreement since the present model has several unrealistic aspects (sharp surface, no residual interaction, etc.).

2.3. Quadrupole Response. Figure 4 shows the quadrupole response function given by Eq. (72) (solid curve) with the strength of the residual quadrupole–quadrupole interaction determined by [31]

$$\kappa_{L=2} = -1.0 \cdot 10^{-3}\ \mathrm{MeV} \cdot \mathrm{fm}^{-4}. \tag{95}$$

The value of this parameter has been fixed by the requirement that the position of the giant quadrupole resonance (GQR) in our hypothetical nucleus of $A = 208$

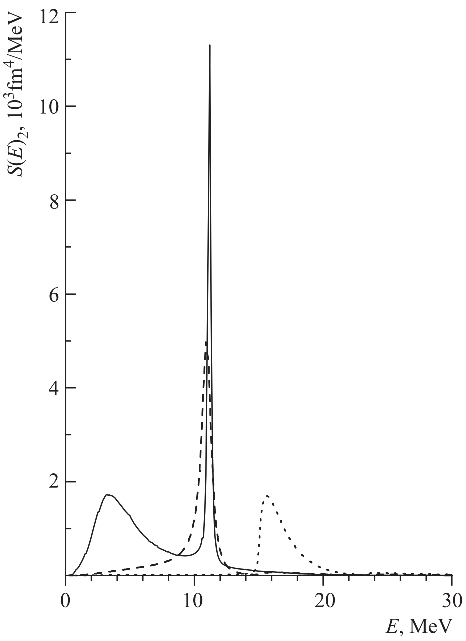

Fig. 4. Quadrupole strength function for a hypothetical nucleus of $A = 208$ nucleons. The solid curve shows the moving-surface response, while the dashed curve gives the fixed-surface response. The dotted curve shows the response in the zero-order approximation

nucleons agrees with the experimental position of the GQR in ^{208}Pb. The obtained value turns out to be about twice that given by the Bohr–Mottelson prescription [18, p. 509]. Taking into account the fact that our equilibrium mean field has a different shape (square well rather than harmonic oscillator) and that we are assuming a semiclassical framework, this kind of agreement looks reasonable. While the position of the GQR is well reproduced by the appropriate value of κ_2, its width is severely underestimated by our theory. This is a well-known limit of all mean-field calculations in which the width is generated only by Landau damping, including of a collision term in our kinetic equation would increase the width of this resonance. It is interesting to compare the moving surface response with the fixed-surface one (dashed curve in Fig. 4): contrary to the fixed-surface response function, the moving surface response displays a low-energy bump whose exact position is determined by the value of the surface tension parameter σ of Eq. (69). Thus the moving-surface theory does reproduce both systematic features of the quadrupole nuclear response that are the GQR and the low-energy surface excitations. Quadrupole response functions calculated for other values of A, corresponding to other medium-heavy spherical nuclei, are qualitatively similar to the case shown in Fig. 4.

2.4. Octupole Response. For $L = 3$, Eq. (72) gives the moving-surface collective octupole response function. A detailed study of this case has been made in Ref. 24. It is interesting to look first at the fixed-surface zero-order response.

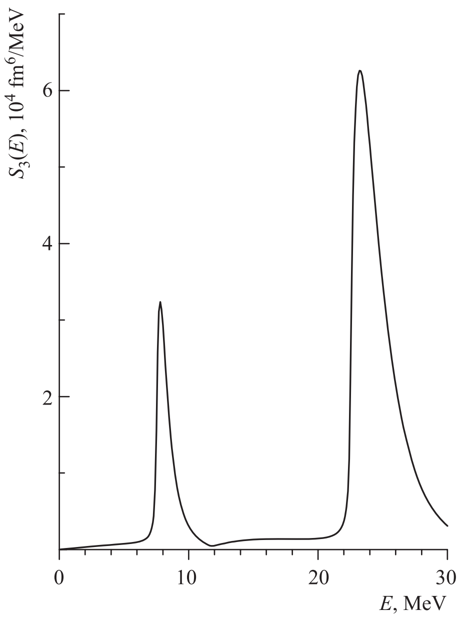

Fig. 5. Semiclassical octupole strength function analogous to quantum single-particle strength function. Calculations are for $A = 208$ nucleons in a square-well potential of radius $R = 1.2\ A^{1/3}$ fm

It can be seen in Fig. 5 that, for $A = 208$, the single-particle octupole strength is concentrated in two regions around 8 and 24 MeV. As pointed out already in [14], in this respect our semiclassical response is strikingly similar to the quantum response, which is concentrated in the $1\hbar\omega$ and $3\hbar\omega$ regions. This concentration of strength is quite remarkable because our static distribution, which is given by Eq. (1), does not include any shell effect, however, because of the close connection between shell structure and classical trajectories expressed by Eq. (42), we still obtain a strength distribution that is very similar to the quantum one.

The collective moving-surface octupole strength function is shown instead in Fig. 6 (solid curve).

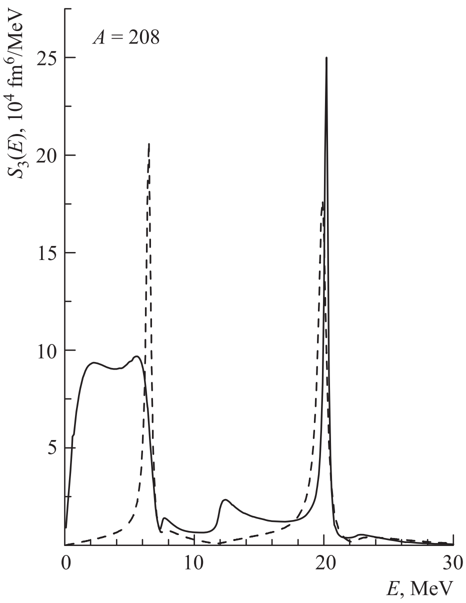

Fig. 6. The octupole strength function given by the moving-surface solution (72) (solid curve) and the corresponding fixed-surface response (dashed curve)

Again we obtain a qualitative agreement with experiment and with the result of analogous quantum calculations. Like for the quadrupole case, agreement with experiment can be obtained with a residual interaction parameter about twice that given by the Bohr–Mottelson prescription. The rather broad double hump on the low-energy side has been interpreted as a superposition of surface vibrations and of the low-energy component of the giant octupole resonance. Within the present semiclassical theory, it can be shown that, for $L = 3$, the parameters $\delta R_{LM}(t)$, describing the octupole surface vibrations, approximately satisfy an equation of motion of the damped-oscillator kind [24]:

$$D_3 \delta \ddot{R}_{3M}(t) + \gamma_3 \delta \dot{R}_{3M}(t) + C_3 \delta R_{3M}(t) = 0. \tag{96}$$

The coefficients D_3, γ_3, and C_3 are easily evaluated, showing that the octupole surface oscillations described by Eq. (96) are overdamped. Another interesting result of this semiclassical analysis concerns the shape stability of heavy spherical nuclei against octupole-type deformation: within the present model the spherical shape is stable.

CONCLUSIONS

The linearized Vlasov equation, that can be seen as a particular case of the Landau kinetic equation for the phase-space-density fluctuations, gives a good qualitative description of the low-energy isoscalar nuclear response of different multipolarities. This collisionless equation, which has been initially derived for other systems, can be applied also to nuclei because in nuclear matter the mean-free-path of nucleons close to the Fermi surface is larger than the typical nuclear dimensions. This fact has two consequences:

• the mean-field approximation is a reasonable one in the study of low-energy nuclear response,

• finite size effects are important and should be taken into account.

Hence, because of the interplay between nucleon mean-free-path and nuclear dimensions, the Vlasov equation can be used to study the nuclear response to a weak driving field of long wavelength. Clearly in different physical situations, like those realized in the collisions of heavy ions of intermediate or high energy, collisions become more important and should be taken into account. Thus in nuclei finite-size effects are more important at low energy, while collisions between nucleons become more and more important with increasing energy.

As pointed out by Kirzhnitz and collaborators [7], in finite systems, the boundary conditions satisfied by the fluctuations of the phase-space density become essential. While in quantum mechanics these boundary conditions are authomatically enforced by the requirement that the wave function of bound states decreases exponentially outside the system, in the semiclassical kinetic-equation approach there is no similar requirement and the boundary conditions satisfied by the phase-space-density fluctuations must be imposed by using some reasonable criterion. Here we have studied the small fluctuations of the phase-space density induced by applying a weak external driving force to spherical nuclei and have assumed a sharp-surface model for the spatial density and mean field in heavy nuclei. Then we have compared the experimental isoscalar strength functions with those calculated by imposing two different kinds of boundary conditions on the phase-space-density fluctuations: fixed- and moving-surface boundary conditions. On the whole we find a better qualitative agreement with experiment for the moving-surface response.

As a final comment, we would like to add that the present semiclassical theory can be applied also to other systems of many fermions in which it is important to take into account the finite size. Atomic clusters and magnetically trapped droplets of fermions are interesting examples. Clearly, quantum calculations are more appropriate, however it might be of some interest to see to what an extent the quantum features stand on a classical «skeleton», and this can be appreciated most clearly within the present approach.

Appendix A
MOVING-SURFACE RESPONSE FUNCTION

In this Appendix we give a few more details on the derivation of the analytical expression (72) for the multipole response function.

The solution $\delta\tilde{f}^{L\pm}_{MN}$ of the linearized Vlasov equation with the boundary condition (64) can be written as (cf. Eq. (29))

$$\delta\tilde{f}^{L\pm}_{MN}(\epsilon,\lambda,r,\omega)=\mathrm{e}^{\pm i\phi_N(r,\omega)}\left[\int_{r_1}^{r}dr'\,\tilde{B}^{L\pm}_{MN}(r')\,\mathrm{e}^{\mp i\phi_N(r',\omega)]}+\tilde{C}_\pm(\epsilon,\lambda,\omega)\right] \tag{97}$$

with the functions $\tilde{B}^{L\pm}_{MN}$ given by an equation similar to (24)

$$\tilde{B}^{L\pm}_{MN}(\epsilon,\lambda,r,\omega)=F'(\epsilon)\left[\frac{\partial}{\partial r}\pm\frac{iN}{v_r(\epsilon,\lambda,r)}\frac{\lambda}{mr^2}\right][\beta Q_{LM}(r)+\delta\tilde{V}^{\rm int}_{LM}(r,\omega)] \tag{98}$$

and the functions $\tilde{C}_\pm$ given by

$$\tilde{C}_\pm(\epsilon,\lambda,\omega)=\frac{\mathrm{e}^{2i\phi_N(R,\omega)}\tilde{D}_+-\tilde{D}_-}{1-\mathrm{e}^{2i\phi_N(R,\omega)}}+ \\ +F'(\epsilon)\frac{1}{\sin[\phi_N(R,\omega)]}mv_r(\epsilon,\lambda,R)\omega\delta R_{LM}(\omega). \tag{99}$$

The mean-field fluctuation $\delta\tilde{V}^{\rm int}_{LM}(r,\omega)$ is a crucial quantity in our calculations. Usually phenomenological models that describe the physical properties of the medium differ mainly in the assumptions made about this term. Our present approach is no exception to this general rule and Eq. (72) has been derived by assuming that

$$\delta\tilde{V}^{\rm int}_{LM}(r,\omega)=\beta\kappa_L r^L\mathcal{R}^V_L(\omega), \tag{100}$$

with

$$\mathcal{R}_L^V(\omega) = \frac{1}{\beta}\int dr r^2 r^L \delta\tilde{\varrho}_{LM}(r,\omega), \tag{101}$$

and

$$\delta\tilde{\varrho}_{LM}(r,\omega) = \frac{8\pi^2}{2L+1}\frac{1}{r^2}\sum_{N=-L}^{L}\left|Y_{LN}\left(\frac{\pi}{2},\frac{\pi}{2}\right)\right|^2 \times \\ \times \int d\epsilon \int d\lambda \frac{\lambda}{v_r(\epsilon,\lambda,r)}\Big[\delta\tilde{f}_{MN}^{L+}(\epsilon,\lambda,r,\omega) + \delta\tilde{f}_{MN}^{L-}(\epsilon,\lambda,r,\omega)\Big]. \tag{102}$$

Function (101) is not the response function (72) because, as discussed in [23], in the moving-surface case the response function should include an additional term in order to take into account the shape changes, thus a more satisfactory definition of the multipole response function in the moving-surface case is (see also [32])

$$\tilde{\mathcal{R}}_L(\omega) = \frac{1}{\beta}\int dr r^2 r^L \delta\bar{\varrho}_{LM}(r,\omega), \tag{103}$$

with

$$\delta\bar{\varrho}_{LM}(r,\omega) = \delta\tilde{\varrho}_{LM}(r,\omega) + \varrho_0\delta(r-R)\delta R_{LM}(\omega), \tag{104}$$

giving

$$\tilde{\mathcal{R}}_L(\omega) = \mathcal{R}_L^V(\omega) + \frac{1}{\beta}R^{L+2}\varrho_0\delta R_{LM}(\omega). \tag{105}$$

The response function (72) corresponds to (105), rather than to (101). The equilibrium density ϱ_0 appearing in Eq. (104) is $\varrho_0 = \frac{2}{3\pi^2}(p_F/\hbar)^3$.

In order to obtain the explicit expression (72) of the response function (105), we need the explicit expressions of the function $\mathcal{R}_L^V(\omega)$ and of the collective coordinates $\delta R_{LM}(\omega)$. For deriving these quantities, the moving-surface solution (97) should be expressed in terms of $\mathcal{R}_L^V(\omega)$ and of $\delta R_{LM}(\omega)$. By replacing the mean-field fluctuation (100) into the quantities $\tilde{B}_{MN}^{L\pm}$ and $\tilde{C}_\pm$, given by Eqs. (98) and (99), and by inserting the resulting expressions into Eq. (97), the fluctuation $\delta\tilde{f}_{MN}^{L\pm}$ can be written as

$$\delta\tilde{f}_{MN}^{L\pm}(\epsilon,\lambda,r,\omega) = \delta f_{MN}^{0L\pm}(\epsilon,\lambda,r,\omega)\left[1 + \kappa_L\mathcal{R}_L^V(\omega)\right] + \\ + F'(\epsilon)\frac{e^{\pm i\Phi_N(r,\omega)}}{\sin\left[\Phi_N(R,\omega)\right]}mv_r(\epsilon,\lambda,R)\omega\delta R_{LM}(\omega), \tag{106}$$

with the zero-order solution $\delta f_{MN}^{0L\pm}$ given by Eq. (45). Now, by inserting the solution (106) into Eqs. (101) and (71), we obtain a system of algebraic equations for the functions $\mathcal{R}_L^V(\omega)$ and $\delta R_{LM}(\omega)$, that can be written as

$$\mathcal{R}_L^V(\omega) = \mathcal{R}_L^0(\omega)\left[1 + \kappa_L\mathcal{R}_L^V(\omega)\right] - \frac{1}{\beta}\Big[\chi_L^0(\omega) + \varrho_0 R^{L+3}\Big]\frac{\delta R_{LM}(\omega)}{R} \tag{107}$$

and

$$\delta R_{LM}(\omega) = \frac{1}{C_L}\Big\{\beta R\chi_L^0(\omega)[1 + \kappa_L\mathcal{R}_L^V(\omega)]+ \\ + \chi_L(\omega)\delta R_{LM}(\omega) + \beta\kappa_L\varrho_0 R^{L+4}\mathcal{R}_L^V(\omega)\Big\}, \quad (108)$$

with the functions $\mathcal{R}_L^0(\omega)$, $\chi_L^0(\omega)$, and $\chi_L(\omega)$ given by Eqs. (54), (79), and (80), respectively. Solving the system (107), (108) gives the explicit expressions of $\mathcal{R}_L^V(\omega)$ and $\delta R_{LM}(\omega)$, which read

$$\mathcal{R}_L^V(\omega) = \mathcal{R}_L(\omega) - \frac{1}{\beta}\frac{\chi_L^0(\omega) + \varrho_0 R^{L+3}}{1 - \kappa_L\mathcal{R}_L^0(\omega)}\frac{\delta R_{LM}(\omega)}{R} \quad (109)$$

and

$$\frac{\delta R_{LM}(\omega)}{R} = \beta\frac{\chi_L^0(\omega) + \kappa_L\varrho_0 R^{L+3}\mathcal{R}_L^0(\omega)}{[C_L - \chi_L(\omega)][1 - \kappa_L\mathcal{R}_L^0(\omega)] + \kappa_L[\chi_L^0(\omega) + \varrho_0 R^{L+3}]^2}\,. \quad (110)$$

The fixed-surface collective response function $\mathcal{R}_L(\omega)$ appearing in Eq. (109) is given by Eq. (53). Finally, by inserting Eqs. (109) and (110) into the response function (105) and taking into account that $\varrho_0 R^3 = \frac{3}{4\pi}A$, we find Eq. (72).

Appendix B
FOURIER COEFFICIENTS

In this Appendix we collect the expressions of the integrals (77) needed here. Since the spherical harmonics in (78) vanish unless N has the same parity as L, we only need the corresponding integrals. Normally we need the coefficients with $k = L$, however for compression modes we also need $k = L+2$. The coefficients involved in the monopole response are

$$\begin{aligned} &L = 0, \quad N = 0, \\ &Q_{n0}^{(0+2)}(x) = \frac{2}{T}\int_{r_1}^{R} dr\frac{r^2}{v_r(\epsilon_F, \lambda, r)}\cos[\phi_{nN}(r)], \\ &= (-)^n R^2\frac{2}{s_{nN}^2(x)} \quad \text{for} \quad n \neq 0, \\ &= R^2\left(1 - \frac{2}{3}x^2\right) \quad \text{for} \quad n = 0, \end{aligned} \quad (111)$$

while for the quadrupole response they are:

$$L = 2, \quad N = 0 \quad \text{same as monopole},$$

and

$$L = 2, \quad N = \pm 2,$$

$$Q^{(2)}_{nN}(x) = (-)^n R^2 \frac{2}{s^2_{nN}(x)} \left(1 + N \frac{\sqrt{1-x^2}}{s_{nN}(x)}\right). \tag{112}$$

In the dipole case we need the coefficients with

$$L = 1, \quad N = \pm 1,$$

$$Q^{(1)}_{nN}(x) = (-)^n R \frac{1}{s^2_{nN}(x)} \tag{113}$$

for translation modes and

$$Q^{(3)}_{nN}(x) = (-)^n R^3 \frac{3}{s^2_{nN}(x)} \left(1 + \frac{4}{3} N \frac{\sqrt{1-x^2}}{s_{nN}(x)} - \frac{2}{s^2_{nN}(x)}\right) \tag{114}$$

for compression modes.

Finally for the octupole response we need:

$$L = 3, \quad N = \pm 1, \pm 3,$$

$$Q^{(3)}_{nN}(x) = (-)^n R^3 \frac{3}{s^2_{nN}(x)} \left(1 + \frac{4}{3} N \frac{\sqrt{1-x^2}}{s_{nN}(x)} - \frac{2}{s^2_{nN}(x)} + \right. \\ \left. + 4(|N|-1) \frac{1-x^2}{s^2_{nN}(x)}\right). \tag{115}$$

For a given nucleus, the integrals $Q^{(k)}_{nN}$ could depend on two variables: the nucleon energy ϵ_F and its angular momentum λ. For the square-well potential however, they display a scaling property and depend only on the variable x. Moreover their A-dependence factorizes because $Q^{(k)}_{nN} \propto R^k$. As a consequence the A-dependence factorizes also in the zero-order propagator (75), that takes the form of an A-dependent factor times a universal propagator.

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