ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
2007. Т. 38. ВЫП. 3

PATH-INTEGRAL APPROACH FOR SUPERINTEGRABLE POTENTIALS ON SPACES OF NONCONSTANT CURVATURE: I. DARBOUX SPACES $D_{\rm I}$ AND $D_{\rm II}$

C. Grosche[1], *G. S. Pogosyan*[2,3], *A. N. Sissakian*[2]

[1]II. Institute für Theoretische Physik, Universität Hamburg, Germany
[2]Joint Institute for Nuclear Research, Dubna
[3]Departamento de Matematicas, CUCEI, Universidad de Guadalajara, Jalisco, Mexico

In this paper the Feynman path-integral technique is applied for superintegrable potentials on two-dimensional spaces of nonconstant curvature: these spaces are Darboux spaces $D_{\rm I}$ and $D_{\rm II}$, respectively. On $D_{\rm I}$ there are three and on $D_{\rm II}$ four such potentials, respectively. We are able to evaluate the path integral in the most of the separating coordinate systems, leading to expressions for the Green functions, the discrete and continuous wave functions, and the discrete energy spectra. In some cases, however, the discrete spectrum cannot be stated explicitly, because it is either determined by a transcendental equation involving parabolic cylinder functions (Darboux space I), or by a higher order polynomial equation. The solutions on $D_{\rm I}$, in particular, show that superintegrable systems are not necessarily degenerate. We can also show how the limiting cases of flat space (constant zero curvature) and the two-dimensional hyperboloid (constant negative curvature) emerge.

В представленной статье фейнмановская техника интегрирования по путям применяется для суперинтегрируемых потенциалов на двумерных пространствах переменной кривизны: эти пространства являются пространствами Дарбу $D_{\rm I}$ и $D_{\rm II}$ соответственно. На $D_{\rm I}$ существуют три, а на $D_{\rm II}$ — четыре таких потенциала. Мы можем вычислить интеграл по путям в самой выделенной системе координат, приводящей к выражениям для функций Грина, волновым функциям дискретного и непрерывного спектров, а также к дискретному энергетическому спектру. В некоторых случаях, однако, дискретный спектр нельзя вычислить явно, так как он или определяется трансцендентным уравнением, включающим параболические цилиндрические функции (пространство Дарбу I), или полиномиальным уравнением более высокого порядка. Решения на $D_{\rm I}$, в частности, показывают, как появляются граничные случаи плоского пространства (постоянная кривизны равна нулю) и двумерного гиперболоида.

PACS: 03.65.Db

1. INTRODUCTION

General Overview and Recent Work. In the last years, an enormous amount of work has been archived on solving path integrals in quantum mechanics exactly, and on the application of the path-integral method in various branches of mathematical physics; many of them have been compiled in our publication [22]. In [13] one of us has discussed path-integral representations of the free motion in two and three dimensions for Euclidean space, pseudo-Euclidean space, spheres, and hyperboloids. In these studies, the goal was to find all path-integral representations for the coordinate systems [24, 44–47] in which the Schrödinger equation (respectively the path integral) allows separation of variables. Paper [22] was aimed explicitly to give the best to our knowledge list of up-to-date explicitly known path-integral solutions.

In the present work we extend our studies of superintegrable potentials to spaces of nonconstant curvature, i.e., Darboux spaces, by means of the path-integral method. In the following sections we discuss two Darboux spaces: we set up the Lagrangian, the Hamiltonian, the quantum operator, and formulate and solve (if possible) the corresponding path integral. We also discuss some of the limiting cases of the Darboux spaces, i.e., where we obtain a space of constant (zero or negative) curvature. In the case of $D_{\rm I}$, there is no limiting case, because we have no free parameter in the metric to choose from.

In the recent publication one of us [15] has applied the path-integral technique [7, 22, 38, 49] to the quantum motion on two-dimensional spaces of nonconstant curvature, called Darboux spaces, $D_{\rm I}$–$D_{\rm IV}$, respectively. These spaces have been introduced by Kalnins et al. [27, 28]. They can be embedded in three-dimensional spaces which can be either of Euclidean or Minkowskian type, respectively. Then the Darboux spaces consist of surfaces, which are also called surfaces of revolution [4]. In two dimensions Darboux spaces of nonconstant curvature can be constructed as follows. One takes, for instance, two-dimensional Euclidean space and takes for the metric a superintegrable potential in its simplest form in radial coordinates. For the Coulomb potential $1/r$ one obtains a metric $\propto r$, which gives the Darboux space $D_{\rm I}$, for the radial potential $b - a/r^2$ one obtains a metric $\propto (b - a/r^2)$, i.e., the Darboux space $D_{\rm II}$, etc. The case of two dimensions is especially simple, because one obtains always a conformally flat space. This method of constructing new spaces was first discussed by Koenigs [40].

Superintegrable Potentials. The intention of [27, 28] was, however, not only to construct new spaces, and to study their properties, but another equally important motivation was to find the corresponding superintegrable potentials. The notion of superintegrable systems was introduced by Winternitz and co-workers in [9, 52], Wojciechowski [53], and was further developed later on also by Evans [6]. Superintegrable potentials have the property of additional constants of motion: The simplest of the cases of the only conserved quantity is the energy, gives usually a chaotic system [22]; in order that a physical system is just integrable it requires d constants of motion, where $2d$ denotes the number of degrees of freedom. In two dimensions one obtains in total three functional independent constants of motion and in three dimensions one has four (minimal superintegrable) and five (maximal superintegrable) functional independent constants of motion. Well-known examples are the Coulomb potential with its Lenz–Runge vector and the harmonic oscillator with its quadrupole moment.

Moreover, the existence of an additional conserved quantity in (maximally) superintegrable potentials leads in classical mechanics to the fact that the orbits of a particle in such a potential are closed: Kepler ellipses are stable and do not «rotate». In quantum mechanics it follows that the spectrum is usually degenerate. A perturbation of the pure Newtonian potential causes the Kepler ellipses to rotate (Mercury's or Moon's perihelion rotation), and in quantum mechanics degeneration is lost, respectively.

Another feature of superintegrable potentials is that the corresponding equations in classical and quantum mechanics separate in more than one coordinate system. (However, whereas from the separability in more than one coordinate system the superintegrability and the existence of additional constants of motion follow, a system with additional constants of motion may not be easily separable.) It turns out that the Coulomb potential in three dimensions separates in spherical, conical, parabolic, and prolate-spheroidal coordinates [42]. Even the relativistic Dirac–Coulomb possesses some of this symmetry by the conservation of the Johnson–Lippmann operator which reduces in the nonrelativistic limit to the Lenz–Runge vector [37].

In the previous publications [18–21] we have studied superintegrable potentials in two and three dimensions in Euclidean space, on spheres and on hyperboloids. We restricted ourselves to real spaces and omitted their corresponding complex extensions [25, 31, 33, 34]. Let us also note that by integrating out ignorable coordinates (i.e., variables which have plane waves, respectively circular waves as solutions of the Schrödinger equation) one can obtain more complicated space interacting systems on spaces with constant curvature: the interaction has the form of a superintegrable potential. One example is the Hermitian hyperbolic space [3, 14] where one can find superintegrable potentials on the hyperboloid [29]. The connection of superintegrability and the polynomial solutions was studied (e.g., in [30]) in connection with contractions of Lie algebras (e.g., in [23, 32, 48]), where the various limiting cases from spaces of positive or negative constant curvature to zero curvature were investigated.

In this first paper on superintegrable potentials on Darboux spaces we discuss only the Darboux spaces D_{I} and D_{II}. The superintegrable potentials on the other two Darboux spaces D_{III} and D_{IV} will be discussed in a forthcoming publication.

The paper is organized as follows: In Sec. 2 Section we treat the superintegrable potentials on Darboux space D_{I}. There are three of them, the third consisting of a constant divided by the metric term which makes the potential almost trivial. The common feature of the first two potentials is that the energy eigenvalues are determined by a transcendental equation involving parabolic cylinder functions. For the third (trivial) potential no bound states can be found.

In Sec. 3 the superintegrable potentials on D_{II} are discussed. There are three nontrivial and one trivial potentials. For the first potential we obtain a quadratic equation for the energy levels, and they show an oscillator-like behavior. An exact solution can be found only in the (u, v) system. This is very similar to the Holt potential in two-dimensional Euclidean space.

The second superintegrable potential on $D_{\rm II}$ is exactly solvable in two coordinate systems. Here, we also find a quadratic equation for the energy levels. V_2 is similar to the singular oscillator in two-dimensional Euclidean space.

The third superintegrable potential has a relation to the Coulomb potential in two-dimensional Euclidean space. The energy levels are determined by an equation of eighth order in E which cannot be solved in general. For a special case, however, we find a Coulomb-like behavior of the energy levels.

The fourth potential is a constant times the metric term, and is therefore trivial. As for $D_{\rm I}$, this potential is included for completeness.

Section 4 contains a discussion of our results and an outlook for the remaining two Darboux spaces $D_{\rm III}$ and $D_{\rm IV}$.

Introducing Darboux Spaces. Kalnins et al. [27, 28] denoted four types of two-dimensional spaces of nonconstant curvature, labeled by $D_{\rm I}$–$D_{\rm IV}$, which are called Darboux spaces [40]. In terms of the infinitesimal distance they are described by (the coordinates (u, v) will be called the (u, v) system; the (x, y) system in turn can be called light-cone coordinates):

$$\begin{aligned}
\text{(I)}\quad ds^2 &= (x+y)dx\,dy,\\
&= 2u(du^2+dv^2), \qquad (x=u+iv, y=u-iv),
\end{aligned} \tag{1.1}$$

$$\begin{aligned}
\text{(II)}\quad ds^2 &= \left(\frac{a}{(x-y)^2}+b\right)dx\,dy,\\
&= \frac{bu^2-a}{u^2}(du^2+dv^2),\ \left(x=\frac{1}{2}(v+iu),\ y=\frac{1}{2}(v-iu)\right),
\end{aligned} \tag{1.2}$$

$$\begin{aligned}
\text{(III)}\ ds^2 &= \left(a\,e^{-(x+y)/2}+b\,e^{-x-y}\right)dxdy,\\
&= e^{-2u}(b+a\,e^{u})(du^2+dv^2),\ (x=u-\mathrm{i}\,v, y=u+iv),
\end{aligned} \tag{1.3}$$

$$\begin{aligned}
\text{(IV)}\ ds^2 &= -\frac{a\left(e^{(x-y)/2}+e^{(y-x)/2}\right)+b}{\left(e^{(x-y)/2}-e^{(y-x)/2}\right)^2}dx\,dy,\\
&= \left(\frac{a_+}{\sin^2 u}+\frac{a_-}{\cos^2 u}\right)(du^2+dv^2),\ (x=u+iv,\ y=u-iv),
\end{aligned} \tag{1.4}$$

a and b are additional (real) parameters ($a_\pm=(a\pm 2b)/4$). Kalnins et al. [27, 28] studied not only the solution of the free motion, but also emphasized on the superintegrable systems in these spaces. They found appropriate coordinate systems, and we will consider all of them. In the majority of the cases we will be able to find a solution, however in some cases this will be impossible due to a quartic anharmonicity of the problem in question.

2. SUPERINTEGRABLE POTENTIALS ON DARBOUX SPACE $D_{\rm I}$

We start with Darboux space $D_{\rm I}$ and consider the following coordinate systems:

$$\begin{aligned}
&((u,v)\text{ system:}) && x = u + iv,\ y = u - iv\ (u \geqslant a), && (2.1)\\
&(\text{Rotated } (r,q) \text{ coordinates:}) && u = r\cos\vartheta + q\sin\vartheta, &&\\
& && v = -r\sin\vartheta + q\cos\vartheta\ \ (\theta \in [0,\pi]), && (2.2)\\
&(\text{Displaced parabolic:}) && u = \frac{1}{2}(\xi^2 - \eta^2) + c,\ v = \xi\eta &&\\
& && (\xi \in \mathbb{R},\ \eta > 0,\ c > 0). && (2.3)
\end{aligned}$$

The infinitesimal distance, i.e., the metric is given by

$$\begin{aligned}
ds^2 &= 2u(du^2 + dv^2), && (2.4)\\
(\text{Rotated } (r,q) \text{ coordinates:}) &= 2(r\cos\vartheta + q\sin\vartheta)(dr^2 + dq^2), && (2.5)\\
(\text{Displaced parabolic:}) &= (\xi^2 - \eta^2 + 2c)(\xi^2 + \eta^2)(d\xi^2 + d\eta^2). && (2.6)
\end{aligned}$$

The Gaussian curvature in a space with metric $ds^2 = g(u,v)(du^2 + dv^2)$ is given by ($g = \det g(u,v)$)

$$G = -\frac{1}{2g}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\ln g. \tag{2.7}$$

Equation (2.7) will be used to discuss shortly the curvature properties of the Darboux spaces, including their limiting cases of constant curvature.

We find, e.g., in the (u,v) system for the Gaussian curvature

$$G = \frac{1}{u^4}. \tag{2.8}$$

There is no further parameter in the metric, therefore this space is of nonconstant curvature throughout for all $u > a$ with a being some real constant $a > 0$. However, $D_{\rm I}$ can be embedded in a three-dimensional Euclidean space. It can then be visualized as an infinite surface (similar to one sheet of a double-sheeted hyberboloid) with a circular hole at the bottom*. The constant a may be taken as $a = 1/2$. In order to set up the path integral formulation we follow our canonical procedure as presented in [22]. The free Lagrangian and Hamiltonian are given by, respectively:

$$\mathcal{L}(u,\dot u,v,\dot v) = mu(\dot u^2 + \dot v^2) - V(u,v),\ \mathcal{H}(u,p_u,v,p_v) = \frac{1}{4mu}(p_u^2 + p_v^2) + V(u,v), \tag{2.9}$$

* *Kalnins E.* Private communication.

and we must require $u > a$ for some $a > 0$, and $\varphi \in [0, 2\pi]$ can be considered as a cyclic variable [28]. The canonical momenta are

$$p_u = \frac{\hbar}{i}\left(\frac{\partial}{\partial u} + \frac{1}{2u}\right), \qquad p_v = \frac{\hbar}{i}\frac{\partial}{\partial v}, \tag{2.10}$$

and for the quantum Hamiltonian we find

$$H = -\frac{\hbar^2}{2m}\frac{1}{2u}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + V(u,v) = \frac{1}{2m}\frac{1}{\sqrt{2u}}(p_u^2 + p_v^2)\frac{1}{\sqrt{2u}} + V(u,v). \tag{2.11}$$

We formulate the path integral (ignoring the half-space constraint for the time being):

$$K(u'', u', v'', v'; T) = \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^N \prod_{j=1}^{N-1}\int 2u_j du_j\, dv_j \times$$
$$\times \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[m\widehat{u_j}(\Delta^2 u_j + \Delta^2 v_j) - V(u_j, v_j)\right]\right\} = \tag{2.12}$$

$$= \int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t) 2u \exp\left\{\frac{i}{\hbar}\int_0^T\left[mu(\dot{u}^2 + \dot{v}^2) - V(u,v)\right]dt\right\}, \tag{2.13}$$

where $u_j = u(t_j)$, $\Delta u_j = u_j - u_{j-1}$, $\epsilon = T/N$, $\widehat{u_j} = \sqrt{u_{j-1}u_j}$. We have displayed the path integral in our product-lattice definition, which will be used throughout this paper [22]. Due to this lattice definition of the path integral, we have no additional $\hbar^2$ potential because the dimension of the space of nonconstant curvature equals 2, c.f. [22].

According to [27, 28] we introduce the following three integrals of motion in $D_{\rm I}$. They are

$$\left.\begin{aligned} K &= p_v, \\ X_1 &= p_u p_v - \frac{v}{2u}(p_u^2 + p_v^2), \\ X_2 &= p_v(vp_u - up_v) - \frac{v^2}{4u}(p_u^2 + p_v^2). \end{aligned}\right\} \tag{2.14}$$

They satisfy the relation

$$4\tilde{\mathcal{H}}_0 X_2 + X_1^2 + K^4 = 0. \tag{2.15}$$

(Let us note that by $\tilde{\mathcal{H}}_0$ the classical Hamiltonian without the $1/2m$ factor is meant. Keeping this factor is no problem, however, in the present form the algebra has a simpler showing.) These operators satisfy the Poisson algebra relations

$$\{K, X_1\} = 2\tilde{\mathcal{H}}_0, \qquad \{K, X_2\} = -X_1, \qquad \{X_1, X_2\} = 2K^3. \tag{2.16}$$

The quantum analogues are given by

$$\left.\begin{aligned} \widehat{K} &= \partial_v, \\ \widehat{X}_1 &= \partial_u\partial_v - \frac{v}{2u}(\partial_u^2 + \partial_v^2), \\ \widehat{X}_2 &= \frac{1}{2}\{\partial_v, v\partial_u - u\partial_v\} - \frac{v^2}{4u}(\partial_u^2 + \partial_v^2), \end{aligned}\right\} \tag{2.17}$$

where $\{\cdot,\cdot\}$ is the anticommutator. These operators satisfy the commutation relations

$$[\widehat{K}, \widehat{X}_1] = -2\widehat{H}_0, \qquad [\widehat{K}, \widehat{X}_2] = X_1, \qquad [\widehat{X}_1, \widehat{X}_2] = 2\widehat{K}^3, \tag{2.18}$$

with the operator relation

$$4\widehat{H}_0\widehat{X}_2 + \widehat{X}_1^2 + \widehat{K}^4 = 0. \tag{2.19}$$

The operators K, X_1, X_2 can be used to characterize the separating coordinate systems on $D_{\rm I}$, as indicated in Table 1.

Table 1. **Constants of motion in space** $D_{\rm I}$

Metric	Constants of motion	Coordinate system
$2u(du^2 + dv^2)$	K^2	(u, v) system
$2(r\cos\vartheta + q\sin\vartheta)(dr^2 + dq^2)$	X_1	(r, q) system
$(\xi^2 - \eta^2 + 2c)(\xi^2 + \eta^2)(d\xi^2 + d\eta^2)$	X_2	Parabolic

Let us note again that we do omit here factors of i, $\hbar$, and $1/2m$ for the sake of simplicity. H_0 therefore is the quantum Hamiltonian without the usual $-\hbar^2/2m$. However, in the tables with the constants of motion, these factors are meant to be included. In the remaining Darboux spaces this notation, as long as the algebra is concerned, will be used for the sake of simplicity in the same way.

For the operators which characterize separation of variables in the (r, q) systems and parabolic coordinates, respectively, we introduce

$$\Lambda_1 = \frac{1}{q\sin\theta + r\cos\theta}\left(q\sin\theta\frac{\partial^2}{\partial r^2} - r\cos\theta\frac{\partial^2}{\partial q^2}\right) = -\sin 2\theta X_1 - \cos 2\theta K^2, \tag{2.20}$$

$$\Lambda_2 = \frac{1}{\xi^4-\eta^4}\left(\eta^4\frac{\partial^2}{\partial\xi^2}+\xi^4\frac{\partial^2}{\partial\eta^2}\right)+\frac{4c\xi^2\eta^2}{\xi^2-\eta^2} =$$
$$= -\frac{\partial}{\partial u}-2v\frac{\partial}{\partial u}\frac{\partial}{\partial v}+2(u-c)\frac{\partial^2}{\partial v^2}+\frac{v^2}{2(u-c)}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)+\frac{2cv^2}{(u-c)}. \tag{2.21}$$

These two operators describe the general case. Special cases for Λ_1 are:

- $\theta=\pi/4$, we have $\Lambda_1=-X_1$ (symmetric case),
- $\theta=\pi/2$, we have $\Lambda_1=K^2$,

and for $c=0$ we have $\Lambda_2=-2X_2$.

Now we consider the following potentials on $D_{\rm I}$ (following [28], an additional fourth potential is according to [4]):

$$V_1(u,v) = \frac{1}{2u}\left[\frac{m}{2}\omega^2(4u^2+v^2)+\kappa+\frac{\lambda^2-\frac{1}{4}}{2mv^2}\right], \tag{2.22}$$

$$V_2(u,v) = \frac{1}{2u}\left[\frac{m}{2}\omega^2(u^2+v^2)+\kappa_1+\kappa_2 v\right], \tag{2.23}$$

$$V_3(u,v) = \frac{1}{2u}\frac{\hbar^2 v_0^2}{2m}, \tag{2.24}$$

$$V_4(u,v) = \frac{1}{2u}\left[\frac{a_0}{\sqrt{u-iv}}+a_1+a_2u+a_3\frac{4u-2iv}{\sqrt{u-iv}}\right]. \tag{2.25}$$

Table 2. **Separation of variables for the superintegrable potentials on $D_{\rm I}$**

Potential	Constants of motion	Separating coordinate system
V_1	$R_1=X_2-\frac{m}{2}\omega^2\frac{v^4}{4u}-\frac{\kappa_2}{2}\frac{v^2}{u}-\frac{\hbar^2}{4m}\left(\lambda^2-\frac{1}{4}\right)\frac{4u^2+v^2}{uv^2}$ $R_2=K^2+\frac{m}{2}\omega^2v^2+\frac{\hbar^2}{m}\frac{\lambda^2-1/4}{v^2}$	(u,v) system Parabolic
V_2	$R_1=X_1-\frac{\kappa_1 v}{u}+\frac{\kappa_2(u^2-v^2)}{u}+\frac{m}{2}\omega^2\frac{v(u^2-v^2)}{u}$ $R_2=K^2+2\kappa_2v+m\omega^2v^2$	(u,v) system (r,q) system
V_3	$R_1=X_1-\frac{\hbar^2v_0^2}{2m}\frac{v}{u}$ $R_2=X_2-\frac{\hbar^2v_0^2}{4m}\frac{v^2}{u}$ $R_3=K$	(u,v) system (r,q) system Parabolic

In Table 2 we have summarized some properties of three of these potentials. Actually, V_3 can be considered as a special case either of V_1 or V_2, respectively. The fourth potential separates, for instance, in parabolic coordinates ($c = 0$) and then has the (complex) form

$$V_4(\xi,\eta) = \\ = \frac{1}{\xi^4-\eta^4}\left[\sqrt{2}a_0(\xi+i\eta)+a_1(\xi^2+\eta^2)+\frac{a_2}{2}(\xi^4-\eta^4)+2^{3/2}a_3(\xi^3-i\eta^3)\right]. \tag{2.26}$$

However, this is not tractable and we will not discuss this potential any further.

2.1. The Superintegrable Potential V_1 on $D_{\rm I}$. We start with the potential V_1 in $D_{\rm I}$. V_1 is separable in the (u,v) system and in parabolic coordinates. However, only in the (u,v) system a closed solution can be found. We state for V_1 in the respective coordinate systems

$$V_1(u,v) = \frac{1}{2u}\left[\frac{m}{2}\omega^2(4u^2+v^2)+\kappa+\frac{\lambda^2-\frac{1}{4}}{2mv^2}\right], \tag{2.27}$$

$$= \frac{1}{2u(\xi^2+\eta^2)}\left[\frac{m}{2}\omega^2(\xi^6+\eta^6)+2m\omega^2(\xi^4-\eta^4)+ \right. \\ \left. +(2m\omega^2c^2+\kappa)(\xi^2+\eta^2)+\hbar^2\frac{\lambda^2}{2m}\left(\frac{1}{\xi^2}+\frac{1}{\eta^2}\right)\right]. \tag{2.28}$$

The separation procedure in the space-time transformation gives additional terms according to $-E[(\xi^4-\eta^4)+2c(\xi^2+\eta^2)]$ in the respective Lagrangian. Although symmetric in ξ and η the involvement of quartic and sextic terms make any further evaluation impossible in parabolic coordinates.

The same observations are valid in the case of a Coulomb-like potential on $D_{\rm I}$, which can be put into the form (including already the proper energy term)

$$V_E(u,v) = -\frac{1}{u}\frac{\alpha}{\sqrt{u^2+v^2}}+E, \tag{2.29}$$

which yields after a space-time transformation, with unshifted ($c = 0$) parabolic coordinates

$$V_E(u,v) \to -2\alpha(\xi^2-\eta^2)+E(\xi^4-\eta^4), \tag{2.30}$$

and is not tractable either. In particular, the metric term $2u$ spoils any further investigation. There exist some attempts in the literature to treat such potential systems, and these studies go with the name «quasi-exactly solvable potentials» in the sense of Turbiner [50] and Ushveridze [51]. In fact, sextic oscillators with a centrifugal barrier and quartic hyperbolic and trigonometric ones can be considered, and they are very similar in their structure as, for instance, in (2.28). One can find particular solutions, provided the parameters in the quasi-exactly solvable potentials fulfill special conditions. Furthermore, well-defined expressions for the wave functions and for the energy spectrum can indeed be found if only quadratic, sextic, and particular centrifugal terms are present. The wave functions then have the form of $\Psi(x) \propto P(x^4) \times \mathrm{e}^{-\alpha x^4}$, with a polynomial P. However, quasi-exactly solvable potentials have the feature that only a *finite* number of bound states can be calculated (usually the ground state and some excited states). Another important observation is due to [35, 41]: The authors found quasi-exactly solvable potentials that emerge from dimensional reduction from two- and three-dimensional complex homogeneous spaces. The sextic potential in the Hamiltonian (2.28) is exactly of that type.

This observation now opens an interpretation of two-dimensional systems with higher anharmonic terms. Let us assume that we have a two-dimensional superintegrable potential system. This system has additional constants of motion, respectively observables, and there are in total three of them (including the energy). Let us assume further that we choose an example which is separable in at least two coordinate systems, say, in Cartesian and parabolic coordinates (i.e., a system which is similar to that one described in (2.28), and we can omit the metric term for simplicity).

Writing down the Schrödinger equation of potentials like this, one obtains a coupled system of differential equations in ξ and η, respectively, which are functionally identical. Their difference is that they are defined on another domain in the complex plane [35]. If one looks now for bound state solutions, i.e., solutions which can be written in terms of polynomials and which are therefore square-integrable, one finds a quantization condition for the energy E. Because the potential is assumed to be separable in Cartesian coordinates we already know the energy levels, E_n. The second separation constant λ of the system of coupled differential equations in ξ and η can then be expressed in terms of E_n, i.e., $\lambda_n = f(E_n)$. The wave functions of the bound state solutions are determined by three-term recursion relations, terminating to give polynomials. However, they cannot be solved to give explicit formulas for the polynomials.

Now we can return to the quasi-exactly solvable potentials. We take one of the two coupled differential equations and rename the variable $\xi \to x \in \mathbb{R}$, say. This one-dimensional quasi-exactly solvable potential «remembers» its origin from a two-dimensional superintegrable potential: The subset of wave functions which can be explicitly found corresponds to the case where one of the coupling constants corresponds in a simple way with the energy levels of the superintegrable potential labeled by n, and the emerging energy levels of the quasi-exactly solvable potential are determined by the separation constant λ_n of the coupled system of differential equations. This feature is common to all quasi-exactly solvable potentials, and even more, one is able to construct quasi-exactly solvable potentials from superintegrable potentials in two, three, etc., dimensions. They are of power-like behavior, or powers of trigonometric, hyperbolic, and elliptic functions.

However, there does not exist a theory of the corresponding wave functions, which are determined by terminating three-term recursion relations for the bound states and nonterminating three-term recursion relations for the scattering states. In comparison to the (confluent) hypergeometric functions little is known about expansion and addition theorems (with the exception of Mathieu and spheroidal wave functions in flat space [43]). In some few cases, an interbasis expansion is known to switch from, say, Hermite polynomials to these new wave functions [35].

Summarizing, we are not able to treat systems with the structure of (2.28), and similar with powers of trigonometric and hyperbolic functions any further.

2.1.1. Separation of V_1 in the (u,v) System. We insert V_1 in (2.13) and obtain

$$\begin{aligned} K^{(V_1)}(u'',u',v'',v';T) &= \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int\limits_{v(t')=v'}^{v(t'')=v''} \mathcal{D}v(t) 2u \times \\ \times \exp\left\{ \frac{i}{\hbar}\int\limits_0^T \left[mu(\dot{u}^2+\dot{v}^2) - \frac{1}{2u}\left(\frac{m}{2}\omega^2(4u^2+v^2)+\kappa+\frac{\lambda^2-\frac{1}{4}}{2mv^2}\right)\right] dt \right\} &= \\ = \sqrt{v'v''}\sum_{n=0}^{\infty} \Psi_n^{(\mathrm{RHO},\lambda)}(v'')\Psi_n^{(\mathrm{RHO},\lambda)}(v') K_n^{(V_1)}(u'',u';T), \quad (2.31) \end{aligned}$$

with the path integral $K_n(T)$ given by

$$\begin{aligned} K_n^{(V_1)}(u'',u';T) &= (4u'u'')^{1/4} \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t)\sqrt{2u} \times \\ &\times \exp\left\{ \frac{i}{\hbar}\int\limits_0^T \left[mu\dot{u}^2 - \frac{1}{2u}\Big(m\omega^2u^2+\kappa\Big) - \frac{E_n}{2u}\right] dt \right\}, \quad (2.32) \end{aligned}$$

with $E_n = \hbar\omega(2n+\lambda+1)$ and we have inserted the path integral solution for the radial harmonic oscillator (RHO) with parameter λ and the variable $v>0$. If v is more restricted, say, v is an angular variable, additional boundary conditions must be imposed. However, we continue with the case $v > 0$. The wave functions for the radial harmonic oscillator $V(r) = \frac{m}{2}\omega^2 - \frac{\hbar^2}{2m}\frac{\lambda^2 - 1/4}{r^2}$ have the form

$$\Psi_n^{(\mathrm{RHO},\lambda)}(r) = \sqrt{\frac{2m}{\hbar}\frac{n!}{\Gamma(n+\lambda+1)}}\, r\left(\frac{m\omega}{\hbar}r\right)^{\lambda/2} \exp\left(-\frac{m\omega}{2\hbar}r^2\right) L_n^{(\lambda)}\left(\frac{m\omega}{\hbar}r^2\right). \tag{2.33}$$

The $L_n^{(\lambda)}(z)$ are Laguerre polynomials [10].

At the next step we perform a space-time transformation in (2.32) by eliminating the term $2u$ in the metric. This gives in the usual way

$$G_n^{(V_1)}(u'', u'; E) = \int_0^\infty ds'' \exp\left[\frac{i}{\hbar}\left(\frac{E^2}{2m\omega^2} - \kappa - E_n\right)s''\right] K_n^{(V_1)}(u'', u'; s''), \tag{2.34}$$

with the transformed path integral given by

$$K_n^{(V_1)}(u'', u'; s'') = = \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{u}^2 - \frac{m}{2}(2\omega)^2\left(u - \frac{E}{m\omega}\right)^2\right]ds'\right\}. \tag{2.35}$$

This path integral of a shifted harmonic oscillator with frequency 2ω can be solved. The corresponding Green function has the form

$$G_u^{(V_1)}(E; u'', u'; \mathcal{E}) = \sqrt{\frac{m}{2\pi\hbar^3\omega}}\,\Gamma\left(\frac{1}{2} - \frac{\mathcal{E}}{2\hbar\omega}\right) D_{-\frac{1}{2}+\mathcal{E}/2\hbar\omega} \times \times \left(\sqrt{\frac{4m\omega}{\hbar}}\,\tilde{u}_>\right) D_{-\frac{1}{2}+\mathcal{E}/2\hbar\omega}\left(-\sqrt{\frac{4m\omega}{\hbar}}\,\tilde{u}_<\right). \tag{2.36}$$

Here, the $D_\nu(z)$ are parabolic cylinder functions [10] and $\tilde{u} = u - E/2m\omega^2$. For the evaluation of the s'' integration we use the involution formula

$$G(u'', u', v'', v'; E) = \frac{\hbar}{2\pi i}\int \mathrm{d}\,\mathcal{E} G_v(E; v'', v'; \mathcal{E})\; G_u(E; u'', u'; -\mathcal{E}) \tag{2.37}$$

to obtain

$$K^{(V_1)}(u'', u', v'', v'; T) = \\ = \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \, \mathrm{e}^{-iET/\hbar} \sqrt{v'v''} \sum_{n=0}^{\infty} \Psi^{(\mathrm{RHO},\lambda)}(v'')\Psi^{(\mathrm{RHO},\lambda)}(v') \times \\ \times G_u^{(V_1)}\left[E; u'', u'; \left(\frac{E^2}{2m\omega^2} - \kappa - E_n\right)\right]. \quad (2.38)$$

Solution without Boundary Condition. Let us first solve the potential problem V_1 on D_{I} without any boundary condition on the variables. In this case the path integral in the variable u is just a path integral for a shifted harmonic oscillator with wave functions given by $\Psi_l^{(\mathrm{HO})}(\tilde{u})$ with $\tilde{u} = u - E/m\omega^2$. The wave functions for the harmonic oscillator (HO) are given by the well-known form in terms of Hermite polynomials

$$\Psi_n^{(\mathrm{HO})}(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) \exp\left(-\frac{m\omega}{2\hbar}x^2\right). \quad (2.39)$$

Evaluating the Green function $G_u^{(V_1)}$ we obtain the solution

$$K_{\mathrm{discr}}^{(V_1)}(u'', u', v'', v'; T) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \sqrt{\frac{m\omega^2}{2E_{ln}}\, v'v''} \, \mathrm{e}^{-\mathrm{i}\,E_{ln}T/\hbar} \times \\ \times \Psi_n^{(\mathrm{RHO},\lambda)}(v'')\Psi_n^{(\mathrm{RHO},\lambda)}(v')\Psi_l^{(\mathrm{HO})}(\tilde{u}'')\Psi_l^{(\mathrm{HO})}(\tilde{u}'), \quad (2.40)$$

$$E_{ln} = \pm\sqrt{m\hbar\omega^3(2l + 2n + 2 + \lambda) + 2m\omega^2\kappa}. \quad (2.41)$$

The spectrum is degenerate in n and l, as it is known for superintegrable potentials. However, this «solution» is seriously flawed. If we calculate the norm of the wave functions, we see immediately that the norm is proportional to the energy E_n, which in the negative-sign case is negative, and it follows that the Hilbert space is not properly defined. In the positive-sign case the norm would be positive, however, the corresponding configuration space cannot be extended to $u \to -\infty$, and this does not make sense either.

Solution with Boundary Condition. Due to the coordinate singularity for $u = 0$ we must impose some boundary condition. The simplest way to incorporate such a boundary condition is to require that the wave functions vanish at $u = 0$, or generally the motion in the variable u takes place only in the half-space $u > a$.

By exploiting the Dirichlet boundary conditions [12] at $u = a$ we therefore get

$$G^{(V_1)}_{(x=a)}(u'', u', v'', v'; E) = \sqrt{v'v''} \sum_{n=0}^{\infty} \Psi_n^{(\mathrm{RHO},\lambda)}(v'') \Psi_n^{(\mathrm{RHO},\lambda)}(v'') \times$$
$$\times \left\{ G_n^{(V_1)}(u'', u'; E) - \frac{G_n^{(V_1)}(u'', a; E) G_n^{(V_1)}(a, u'; E)}{G_n^{(V_1)}(a, a; E)} \right\}. \quad (2.42)$$

This Green function cannot be evaluated further. However, we can determine bound states by the poles of (2.42) and obtain the quantization condition

$$D_{\nu_{l,n}} \left[2\sqrt{\frac{m\omega}{\hbar}} \left(a - \frac{E_{l,n}}{m\omega^2} \right) \right] = 0, \quad (2.43)$$

$$\nu_{l,n} = -\frac{1}{2} + \frac{1}{2\omega\hbar} \left(\frac{E_{l,n}^2}{m\omega^2} - \kappa - \hbar\omega(2n + \lambda + 1) \right). \quad (2.44)$$

According to [28] the asymptotic behavior of the energy eigenvalues is in accordance with (2.41) for high-level states. The wave functions can be obtained by taking the residuum of the curly-bracket expression in (2.42).

Our last quantization condition, however, rises a problem. It is not obvious for us how to determine the degeneracy of the energy values which is usually typical for superintegrable systems. The solution (2.41) has this degeneracy but the boundary conditions are not fulfilled and the Hilbert space is not properly defined either. For solution (2.44) it is just the other way round. In the original paper [28] this issue was not addressed any further.

We can see from the quantization condition (2.44) that for each value of the number n a set of energy levels $E_{l,n}$ follows, i.e., a set $E_{l,0}, E_{l,1}, \ldots$ There is no possibility of finding that a level from the set $n = 0$ is equal to one level of the set $n = 1$, for example, $E_{l_a,0} = E_{l_b,1}$ for some numbers l_a, l_b. Therefore, we find that the degeneracy of the energy levels is lost. The usual lore in the study of superintegrable systems is that the statements that a potential is superintegrable and that the spectrum of such a potential is degenerate are equivalent. Indeed, from the Sturm–Liouville theory for differential equations, i.e., in our case the quantum Hamiltonian, it follows that degeneracy implies superintegrability, i.e., additional constants of motion. However, this statement is not valid for the other way round, and the present examples of potentials on Darboux space D_{I} serve as counter examples for such an attempt.

If we look at (2.42), we see that the «lost» degeneracy is due to the boundary condition for the Green function and the wave functions, respectively, for some $u > a > 0$. For $u = 0$ the curvature of the space becomes infinite and a wave function at the coordinate origin does not make sense. Depending whether the Darboux space D_{I} is embedded in three-dimensional space with definite or indefinite metric further determines the parameter a, c.f. [28]. For a positive-definite metric, v is an angle with $v \in [0, 2\pi)$, the two-dimensional surface making up $D_{\rm I}$ has a definite boundary and it follows $a = 1/2$. For a negative-definite metric the boundary turns out to be constrained by $a = 0$. In fact, it is impossible to extend the surface beyond $u < 0$, and all values from 0 to ∞ are definitely excluded. We will see that the same property holds for the potential V_2.

2.2. The Superintegrable Potential V_2 on $D_{\rm I}$. Next, we consider the potential V_2 on $D_{\rm I}$. First, we state the potential in the separating coordinate systems. We have

$$V_2(u,v) = \frac{1}{2u}\left[\frac{m}{2}\omega^2(u^2+v^2)+\kappa_1+\kappa_2 v\right], \tag{2.45}$$

$$= \frac{1}{2u}\left[\frac{m}{2}\omega^2(r^2+q^2)+\kappa_1+\kappa_2(q\cos\vartheta - r\sin\vartheta)\right]. \tag{2.46}$$

2.2.1. Separation of V_2 in the (u,v) System. We proceed in a similar way as before and obtain

$$K^{(V_2)}(u'',u',v'',v';T) = \int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)2u\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[mu(\dot u^2+\dot v^2)-\frac{1}{2u}\left(\frac{m}{2}\omega^2(u^2+v^2)+\kappa_1++\kappa_2 v\right)\right]dt\right\} =$$
$$= \sum_{n=0}^{\infty}\Psi_n^{(\rm HO)}(\tilde v'')\Psi_n^{(\rm HO)}(\tilde v')K_n^{(V_2)}(u'',u';T), \tag{2.47}$$

where $\Psi_n^{(\rm HO)}$ are the wave functions of a shifted harmonic oscillator with $\tilde v = v + \kappa_2/m\omega$. Note that we have to require $v \in \mathbb{R}$, otherwise for v cyclic, the complicated boundary conditions have to be imposed on the solution in v. The remaining path integral in the variable u has the form

$$K_n^{(V_2)}(u'',u';T) = (4u'u'')^{1/4}\int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\sqrt{2u}\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[mu\dot u^2-\frac{1}{2u}\left(\frac{m}{2}\omega^2u^2+\kappa_1+\hbar\left(n+\frac{1}{2}\right)-\frac{\kappa_2^2}{2m\omega^2}\right)\right]dt\right\}. \tag{2.48}$$

This gives in the usual way

$$G_n^{(V_2)}(u'',u';E) = \\ = \int_0^\infty ds'' \exp\left[-\frac{i}{\hbar}s''\left(\kappa_1+\hbar\omega\left(n+\frac{1}{2}\right)-\frac{\kappa_2^2}{2m\omega^2}\right)\right]K_n^{(V_2)}(u'',u';s''), \tag{2.49}$$

with the transformed path integral given by ($\tilde{u} = u - 2E/m\omega^2$)

$$K_n^{(V_2)}(u'',u';s'') = \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}(\dot{u}^2-\omega^2u^2)+2Eu\right]ds'\right\} = \\ = \mathrm{e}^{2is''E/m\omega^2\hbar} \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s)\exp\left[\frac{im}{2\hbar}\int_0^{s''}\frac{m}{2}(\dot{\tilde{u}}^2-\omega^2\tilde{u}^2)ds'\right]. \tag{2.50}$$

Solution without Boundary Condition. This is again a path integral for a shifted harmonic oscillator, and first we ignore the boundary condition for the wave functions in the variable u for $u = 0$, say, we obtain the solution:

$$K_{\text{discr}}^{(V_2)}(u'',u',v'',v';T) = \sum_{n=0}^\infty\sum_{l=0}^\infty \sqrt{\frac{m\omega^2}{4E_{ln}}v'v''}\ \mathrm{e}^{-iE_{ln}T/\hbar} \times \\ \times \Psi_n^{(\mathrm{HO})}(\tilde{v}'')\Psi_n^{(\mathrm{HO})}(\tilde{v}')\Psi_n^{(\mathrm{HO})}(\tilde{u}'')\Psi_n^{(\mathrm{HO})}(\tilde{u}'), \tag{2.51}$$

$$E_{ln} = \pm\sqrt{\frac{m\hbar\omega^2}{2}\left(l+n+1+\kappa_1-\frac{k_2^2}{2m\omega^2}\right)}. \tag{2.52}$$

This spectrum exhibits degeneracy, however the norm is again proportional to the energy, which is negative, and therefore the Hilbert space is not properly defined.

Solution with Boundary Condition. If we now take into account the boundary condition for some $u = a$ such that the wave function vanishes for $u = a$, we obtain in a similar manner as in the previous subsection:

$$G_{(x=a)}^{(V_2)}(u'',u',v'',v';E) = \sqrt{v'v''}\sum_{n=0}^\infty \Psi_n^{(\mathrm{HO})}(\tilde{v}'')\Psi_n^{(\mathrm{HO})}(\tilde{v}')\times \\ \times\left\{G_n^{(V_2)}(u'',u';E) - \frac{G_n^{(V_2)}(u'',a;E)G_n^{(V_2)}(a,u';E)}{G_n^{(V_2)}(a,a;E)}\right\}, \tag{2.53}$$

with the Green function $G_n^{(V_2)}(E)$ given by

$$G_u^{(V_2)}(E;u'',u';\mathcal{E}) = \sqrt{\frac{m}{2\pi\hbar^3\omega}}\,\Gamma\bigg(\frac{1}{2}-\frac{\mathcal{E}}{2\hbar\omega}\bigg)\times \\ \times D_{-\frac{1}{2}+\mathcal{E}/2\hbar\omega}\Bigg(\sqrt{\frac{4m\omega}{\hbar}}\,\tilde{u}_>\Bigg) D_{-\frac{1}{2}+\mathcal{E}/2\hbar\omega}\Bigg(-\sqrt{\frac{4m\omega}{\hbar}}\,\tilde{u}_<\Bigg), \quad (2.54)$$

$$\mathcal{E} = \frac{2E^2+\kappa_2^2/2}{m\omega^2} - \kappa_1 - \hbar\omega\bigg(n+\frac{1}{2}\bigg). \quad (2.55)$$

Bound states can be determined by the quantization condition

$$D_{\nu_{l,n}}\Bigg[\sqrt{\frac{2m\omega}{\hbar}}\bigg(a-\frac{2E_{l,n}}{m\omega^2}\bigg)\Bigg] = 0, \quad (2.56)$$

$$\nu_{l,n} = -\frac{1}{2} + \frac{1}{\omega\hbar}\bigg(\frac{2E_{ln}^2+\kappa_2^2/2}{m\omega^2} - \kappa_1 - \hbar\omega\bigg(n+\frac{1}{2}\bigg). \quad (2.57)$$

Again, degeneracy in the quantum numbers n and l is lost. According to [28] the asymptotic behavior of the energy eigenvalues (2.57) is in accordance with (2.52). The wave functions can be obtained by taking the residuum of the curly-bracket expression in (2.55).

2.2.2. Separation of V_2 in the (r,q) System. In order to set up the path integral formulation we follow our canonical procedure. The Lagrangian and Hamiltonian are given by, respectively:

$$\mathcal{L}(r,\dot{r},q,\dot{q}) = m(r\cos\vartheta + q\sin\vartheta)(\dot{r}^2+\dot{q}^2) - V(r,q), \quad (2.58)$$

$$\mathcal{H}(r,p_r,q,p_q) = \frac{1}{4m(r\cos\vartheta+q\sin\vartheta)}(p_r^2+p_q^2) + V(r,q). \quad (2.59)$$

The canonical momenta are

$$p_r = \frac{\hbar}{i}\bigg(\frac{\partial}{\partial r} + \frac{\cos\vartheta}{2(r\cos\vartheta+q\sin\vartheta)}\bigg), \quad (2.60)$$

$$p_q = \frac{\hbar}{i}\bigg(\frac{\partial}{\partial q} + \frac{\sin\vartheta}{2(r\cos\vartheta+q\sin\vartheta)}\bigg). \quad (2.61)$$

The quantum Hamiltonian has the form

$$H = -\frac{\hbar^2}{2m}\frac{1}{2(r\cos\vartheta+q\sin\vartheta)}\bigg(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial q^2}\bigg) + V(r,q) = \quad (2.62)$$

$$= \frac{1}{2m}\frac{1}{\sqrt{2(r\cos\vartheta+q\sin\vartheta)}}(p_r^2+p_q^2)\frac{1}{\sqrt{2(r\cos\vartheta+q\sin\vartheta)}} + V(r,q). \quad (2.63)$$

Using the representation (2.46) we write down the path integral for V_2 in the rotated (r,q)-coordinate system, and obtain

$$K(r'',r',q'',q';T)=\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\int\limits_{q(t')=q'}^{q(t'')=q''}\mathcal{D}q(t)2(r\cos\vartheta+q\sin\vartheta)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[m(r\cos\vartheta+q\sin\vartheta)(\dot r^2+\dot q^2)-\right.\right.$$
$$\left.\left.-\frac{1}{2u}\left(\frac{m}{2}\omega^2(r^2+q^2)+\kappa_1+\kappa_2(q\cos\vartheta-r\sin\vartheta)\right)\right]dt\right\}. \quad (2.64)$$

Performing a space-time transformation in the usual way gives

$$G(r'',r',q'',q';E)=\int\limits_0^\infty ds''\,\mathrm{e}^{-is''\kappa_1/\hbar}\,K(r'',r',q'',q';s''), \quad (2.65)$$

with the transformed path integral $K(s'')$ given by

$$K(r'',r',q'',q';s'')=\int\limits_{r(0)=r'}^{r(s'')=r''}\mathcal{D}r(s)\int\limits_{q(0)=q'}^{q(s'')=q''}\mathcal{D}q(s)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\Big(\dot r^2+\dot q^2-\omega^2(r^2+q^2)\Big)+2E(r\cos\vartheta+q\sin\vartheta)-\right.\right.$$
$$\left.\left.-\kappa_2(-r\sin\vartheta+q\cos\vartheta)\right]ds\right\}=\exp\left[\frac{i}{\hbar}\left(\frac{4E^2+\kappa_2^2}{2m\omega^2}-\kappa_1\right)s''\right]\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\Big(\dot r^2+\dot q^2\Big)-\frac{m}{2}\omega^2(\tilde r^2+\tilde q^2)\right]ds\right\}=$$
$$=\exp\left[\frac{i}{\hbar}\left(\frac{4E^2+\kappa_2^2}{2m\omega^2}-\hbar\omega\left(n+\frac{1}{2}\right)-\kappa_1\right)s''\right]\sum_{n=0}^\infty\Psi_n^{(\mathrm{HO})}(\tilde q'')\Psi_n^{(\mathrm{HO})}(\tilde q')\times$$
$$\times\int\limits_{r(0)=r'}^{r(s'')=r''}\mathcal{D}r(s)\exp\left[\frac{im}{2\hbar}\int\limits_0^{s''}(\dot r^2-\omega^2\tilde r^2)ds\right] \quad (2.66)$$

($\tilde{r} = r - (2E\cos\vartheta + \kappa_2 \sin\vartheta)/m\omega^2$, $\tilde{q} = q - (2E\sin\vartheta - \kappa_2\cos\vartheta)/m\omega^2$). Here, we have inserted the path-integral solution for the shifted harmonic oscillator in the variable q.

Solution with Boundary Condition. For the path integral for the shifted harmonic oscillator in the variable r we now take care that the variable u is defined only in the half-space $u \geqslant a$. Setting, for instance, in the definition of the (r, q) system $\vartheta = 0$ yields $r = u$ and $q = v$. For $\vartheta = \pi/2$ the roles of r and q are reversed. In the view of the previous paragraph of V_2 in the (u, v) system, we impose on the Green function in r the boundary condition $r \geqslant a$ and obtain in this limiting case for the bound states the quantization condition

$$D_{\nu_{l,n}}\left[\sqrt{\frac{2m\omega}{\hbar}}\left(a - \frac{2E_{l,n}}{m\omega^2}\right)\right] = 0, \tag{2.67}$$

$$\nu_n = -\frac{1}{2} + \frac{1}{\omega\hbar}\left(\frac{2E_{ln}^2 + \kappa_2^2/2}{m\omega^2} - \kappa_1 - \hbar\omega\left(n + \frac{1}{2}\right)\right). \tag{2.68}$$

This is the result of (2.57). The quantization conditions of (2.57) and (2.68) are identical as it should be.

2.3. The Superintegrable Potential V_3 on $D_{\rm I}$. Next, we consider the potential V_3 on $D_{\rm I}$. First, we state the potential in the separating coordinate systems. We have

$$\begin{aligned} V_3(u,v) &= \frac{1}{2u}\frac{\hbar^2 v_0^2}{2m}, & (2.69)\\ &= \frac{1}{\xi^2 - \eta^2 + 2c}\frac{\hbar^2 v_0^2}{2m}, & (2.70)\\ &= \frac{1}{2(r\cos\vartheta + q\sin\vartheta)}\frac{\hbar^2 v_0^2}{2m}. & (2.71)\end{aligned}$$

This potential can be considered as a special case either of V_1 or V_2, respectively. However, it has an additional conserved quantum number, i.e., $K = p_v$. Therefore we will sketch only the solution in the (u, v) system. Proceeding in the usual way, we obtain for the path integral (assuming v cyclic):

$$K(u'', u', v'', v'; T) = \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int\limits_{v(t')=v'}^{v(t'')=v''} \mathcal{D}v(t) 2u \times$$

$$\times \exp\left\{\frac{i}{\hbar}\int_0^T \left[mu(\dot{u}^2 + \dot{v}^2) - \frac{1}{2u}\frac{\hbar^2 v_0^2}{2m}\right] dt\right\} = \tag{2.72}$$

$$= (4u'u'')^{1/4} \sum_{l=0}^{\infty} \frac{\mathrm{e}^{il(v''-v')}}{2\pi} \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t)\sqrt{2u}\times$$

$$\times \exp\left\{ \frac{i}{\hbar} \int\limits_0^T \left[mu\dot{u}^2 - \frac{1}{2u}\frac{\hbar^2}{2m}(l^2 + v_0^2) \right] dt \right\}. \quad (2.73)$$

We observe that the only effect is the change in the quantum number l in comparison to the $v_0 = 0$ case. Using the solution of [15] we get for the corresponding Green function

$$G(u'', u', v'', v'; E) = \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{il(v''-v')}}{2\pi} \frac{4m}{3\hbar} \left[\left(u' - \frac{\tilde{l}^2\hbar^2}{4mE} \right) \left(u'' - \frac{\tilde{l}^2\hbar^2}{4mE} \right) \right]^{1/2} \times$$

$$\times \left[\tilde{I}_{1/3}\left(u_< - \frac{\tilde{l}^2\hbar^2}{4mE} \right) \tilde{K}_{1/3}\left(u_> - \frac{\tilde{l}^2\hbar^2}{4mE} \right) - \right.$$

$$\left. - \frac{\tilde{I}_{1/3}\left(a - \frac{\tilde{l}^2\hbar^2}{4mE} \right)}{\tilde{K}_{1/3}\left(a - \frac{\tilde{l}^2\hbar^2}{4mE} \right)} \tilde{K}_{1/3}\left(u' - \frac{\tilde{l}^2\hbar^2}{4mE} \right) \tilde{K}_{1/3}\left(u'' - \frac{\tilde{l}^2\hbar^2}{4mE} \right) \right]. \quad (2.74)$$

$\tilde{I}_\nu(z)$ denotes

$$\tilde{I}_\nu(z) = I_\nu\left(\frac{4\sqrt{-mE}}{3\hbar} z^{3/2} \right),$$

with $\tilde{K}_\nu(z)$ similarly, and $\tilde{l}^2 = l^2 + v_0^2$. Due to the relation satisfied by the Airy function [1, 10], $K_{\pm 1/3}(\zeta) = \pi\sqrt{3/z}\,\mathrm{Ai}(z)$, $z = (3\zeta/2)^{2/3}$, and the observation that for $E < 0$ the argument of $\mathrm{Ai}(z)$ is always greater than zero, we infer that in this case there are no bound states. For $E > 0$ there is no real bound state solution, either. This concludes the discussion.

3. SUPERINTEGRABLE POTENTIALS ON DARBOUX SPACE D_{II}

In this section we consider superintegrable potentials on the Darboux space D_{II} (1.2). The following four coordinate systems separate the Schrödinger equation for the free motion:

$$((u,v)\text{-system:})\quad x=\frac{1}{2}(v+iu),\quad y=\frac{1}{2}(v-iu), \tag{3.1}$$

$$(\text{Polar:})\quad u=\varrho\cos\vartheta,\ v=\varrho\sin\vartheta\ \left(\varrho>0,\vartheta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right), \tag{3.2}$$

$$(\text{Parabolic:})\quad u=\xi\eta,\ v=\frac{1}{2}(\xi^2-\eta^2)\qquad(\xi>0,\eta>0), \tag{3.3}$$

$$(\text{Elliptic:})\quad u=d\cosh\omega\cos\varphi,\ v=d\sinh\omega\sin\varphi\ \left(\omega>0,\varphi\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right). \tag{3.4}$$

$2d$ is the interfocal distance in the elliptic system. For convenience we also display in the following the special case of the parameters $a=-1$ and $b=1$ [27]. The infinitesimal distance is given in these four cases (note that the metric gives us the additional requirement $u>0$):

$$ds^2=\frac{bu^2-a}{u^2}(du^2+dv^2), \tag{3.5}$$

$$(\text{Polar:})=\frac{b\varrho^2\cos^2\vartheta-a}{\varrho^2\cos^2\vartheta}(d\varrho^2+\varrho^2d\vartheta^2), \tag{3.6}$$

$$\begin{aligned}(\text{Parabolic:})&=\frac{b\xi^2\eta^2-a}{\xi^2\eta^2}(\xi^2+\eta^2)(d\xi^2+d\eta^2)=\\&=\left[\left(b\xi^2-\frac{a}{\xi^2}\right)+\left(b\eta^2-\frac{a}{\eta^2}\right)\right](d\xi^2+d\eta^2),\end{aligned} \tag{3.7}$$

$$\begin{aligned}(\text{Elliptic:})&=\frac{bd^2\cosh^2\omega\cos^2\varphi-a}{\cosh^2\omega\cos^2\varphi}(\cosh^2\omega-\cos^2\varphi)(d\omega^2+d\varphi^2)=\\&=\left[\left(bd^2\cosh^2\omega+\frac{a}{\cosh^2\omega}\right)-\right.\\&-\left.\left(bd^2\cos^2\varphi+\frac{a}{\cos^2\varphi}\right)\right](d\omega^2+d\varphi^2).\end{aligned} \tag{3.8}$$

We can see that the case $a=-1$, $b=0$ leads to the case of the Poincaré upper half-plane $u>0$ endowed with the metric (3.5) [13], i.e., the two-dimensional hyperboloid $\Lambda^{(2)}$ in horicyclic coordinates. The parabolic case corresponds to the semicircular-parabolic system and the elliptic case to the elliptic–parabolic system on the two-dimensional hyperboloid. On the other hand, the case $a=0$, $b=1$ just gives the usual two-dimensional Euclidean plane with its four coordinate systems which allow separation of variables of the Laplace–Beltrami equation, i.e., the Cartesian, polar, parabolic, and elliptic system. Hence, the Darboux space II contains as special cases a space of constant zero curvature (Euclidean plane) and a space of constant negative curvature (the hyperbolic plane). This includes the emerging of coordinate systems in a flat space from curved spaces.

Table 3. **Constants of motion and limiting cases of coordinate systems on $D_{\rm II}$**

Metric	Constant of motion	$D_{\rm II}$	$\Lambda^{(2)}$ $(a=-1,$ $b=0)$	E_2 $(a=0,$ $b=1)$
$\frac{bu^2-a}{u^2}(du^2+dv^2)$	K^2	(u,v) system	Horicyclic	Cartesian
$\frac{b\varrho^2\cos^2\vartheta-a}{\varrho^2\cos^2\vartheta}(d\varrho^2+d\vartheta^2)$	X_2	Polar	Equidistant	Polar
$\frac{b\xi^2\eta^2-a}{\xi^2\eta^2}(\xi^2+\eta^2)(d\xi^2+d\eta^2)$	X_1	Parabolic	Semicircular parabolic	Parabolic
$\frac{bd^2\cosh^2\omega\cos^2\varphi-a}{\cosh^2\omega\cos^2\varphi}\times$ $\times(\cosh^2\omega-\cos^2\varphi)(d\omega^2+d^2\varphi^2)$	$X_2+d^2K^2$	Elliptic	Elliptic–parabolic	Elliptic

We find for the Gaussian curvature in the (u,v) system

$$G=\frac{a(a-3bu^2)}{(a-2bu^2)^3}. \tag{3.9}$$

For $b=0$ we have $G=1/a$ which is indeed a space of constant curvature, and the quantity a measures the curvature. In particular, for the unit-two-dimensional hyperboloid we have $G=1/a$, with $a=-1$ as the special case of $\Lambda^{(2)}$. In the following we will assume that $a<0$ in order to assure the positive definiteness of the metric (1.2).

The following constants of motion (see Table 3) are introduced on $D_{\rm II}$ (without potential):

$$K = p_v, \tag{3.10}$$

$$X_1 = \frac{2v(p_v^2-u^2p_u^2)}{bu^2-a}+2up_up_v, \tag{3.11}$$

$$X_2 = \frac{(v^2-u^4)p_v^2+u^2(1-v^2)p_u^2}{bu^2-a}+2uvp_up_v. \tag{3.12}$$

They satisfy the Poisson algebra relations

$$\{K,X_1\}=2(K^2-\tilde{\mathcal{H}}_0),\qquad \{K,X_2\}=X_1,\qquad \{X_1,X_2\}=4KX_2 \tag{3.13}$$

and the relation

$$X_1^2-4K^2X_2+4\tilde{\mathcal{H}}_0X_2-4\tilde{\mathcal{H}}_0^2=0. \tag{3.14}$$

The quantum analogues have the form (again with i, $\hbar$, $2m$)

$$\begin{aligned} K &= \partial_v, &(3.15)\\ X_1 &= \frac{2v}{bu^2-a}(\partial_v^2 - u^2\partial_u^2) + 2u\partial_u\partial_v, &(3.16)\\ X_2 &= \frac{1}{bu^2-a}\Big[(v^2-u^4)\partial_v^2 + u^2(1-v^2)\partial_u^2\Big] + \\ &\quad + 2uv\partial_u\partial_v + u\partial_u + v\partial_v - \frac{1}{4} &(3.17)\end{aligned}$$

and satisfy the operator relation ($\widehat{H}_0$ — the Hamiltonian operator, $\{,\}$ — the anticommutator)

$$\widehat{X}_1^2 - 2\{\widehat{K}^2, \widehat{X}_2\} + 4\widehat{H}_0\widehat{X}_2 - 4\widehat{H}_0^2 + 4\widehat{K}^2 = 0 \qquad (3.18)$$

and the commutation relations

$$[\widehat{K}, \widehat{X}_1] = 2(\widehat{K}^2 - \widehat{H}_0), \;\; [\widehat{K}, \widehat{X}_2] = \widehat{X}_1, \;\; [\widehat{X}_1, \widehat{X}_2] = 2\{\widehat{K}, \widehat{X}_2\}. \qquad (3.19)$$

We consider the following potentials on $D_{\rm II}$:

$$V_1(u,v) = \frac{bu^2-a}{u^2}\left[\frac{m}{2}\omega^2(u^2+4v^2) + k_1 v + \frac{\hbar^2}{2m}\frac{k_2^2-\frac{1}{4}}{u^2}\right], \qquad (3.20)$$

$$V_2(u,v) = \frac{bu^2-a}{u^2}\left[\frac{m}{2}\omega^2(u^2+v^2) + \frac{\hbar^2}{2m}\left(\frac{k_1^2-\frac{1}{4}}{u^2} + \frac{k_2^2-\frac{1}{4}}{v^2}\right)\right], \qquad (3.21)$$

$$\begin{aligned} V_3(u,v) &= \frac{bu^2-a}{u^2}\frac{2m}{\sqrt{u^2+v^2}}\left[-\alpha + \frac{\hbar^2}{2m}\left(\frac{k_1^2-\frac{1}{4}}{\sqrt{u^2+v^2}+v} + \right.\right. \\ &\quad \left.\left. + \frac{k_2^2-\frac{1}{4}}{\sqrt{u^2+v^2}-v}\right)\right], &(3.22)\\ V_4(u,v) &= \frac{bu^2-a}{u^2}\frac{\hbar^2}{2m}v_0^2. &(3.23)\end{aligned}$$

In Table 4 we have listed the properties of these potentials (the coordinate systems where an explicit path-integral evaluation is possible are underlined).

Table 4. **Separation of variables for the superintegrable potentials on $D_{\rm II}$**

Potential	Constants of motion	Separating coordinate system
V_1	$R_1 = X_1 + m\omega^2 v\left(u^2+\frac{u^2+4v^2}{bu^2-a}\right)+$ $+\frac{k_1}{2}\left(u^2+\frac{4v^2}{bu^2-a}\right)-\hbar^2\frac{k_2^2-\frac{1}{4}}{m}\frac{v}{bu^2-a}$ $R_2 = K^2+2m\omega^2v^2+k_1v$	(u,v) system Parabolic
V_2	$R_1 = X_2+$ $+\frac{u^2+v^2}{bu^2-a}\left[\frac{m}{2}\omega^2(u^2+v^2)-\frac{\hbar^2}{2m}\left(k_1^2-\frac{1}{4}-(k_2^2-\frac{1}{4})\frac{u^2}{v^2}\right)\right]$ $R_2 = K^2+\frac{m}{2}\omega^2v^2+\frac{\hbar^2}{2m}\frac{k_2^2-\frac{1}{4}}{v^2}$	(u,v) system Polar Elliptic
V_3	$R_1 = X_1+$ $+\frac{-\alpha\xi^2\left(\eta^4+1\right)+\frac{\hbar^2}{2m}\left(k_1^2-\frac{1}{4}\right)\left(\eta^4+1\right)-\frac{\hbar^2}{2m}\left(k_2^2-\frac{1}{4}\right)\left(\xi^4+1\right)}{(b\xi^2\eta^2-a)\left(\xi^2+\eta^2\right)}$ $R_2 = X_2-$ $-\frac{\alpha(\xi^2+\eta^2)+\frac{\hbar^2}{2m}\left(k_1^2-\frac{1}{4}\right)\left(\xi^4-1\right)+\frac{\hbar^2}{2m}\left(k_2^2-\frac{1}{4}\right)\left(\xi^4-1\right)}{4(b\xi^2\eta^2-a)}$	Polar Parabolic Displaced elliptic
V_4	$R_1 = X_1+\frac{\hbar^2v_0^2}{m}\frac{v}{bu^2-a}$ $R_2 = X_2+\frac{\hbar^2v_0^2}{2m}\frac{u^2+v^2}{bu^2-a}$ $R_3 = K = p_v$	(u,v) system Polar Parabolic Elliptic

3.1. The Superintegrable Potential V_1 on $D_{\rm II}$. We state the potential V_1 in the respective coordinate systems:

$$V_1(u,v) = \frac{bu^2-a}{u^2}\left[\frac{m}{2}\omega^2(u^2+4v^2)+k_1v+\frac{\hbar^2}{2m}\frac{k_2^2-\frac{1}{4}}{u^2}\right] = \tag{3.24}$$

$$= \frac{bu^2-a}{u^2}\frac{1}{\xi^2+\eta^2}\left[\frac{m}{2}\omega^2(\xi^6+\eta^6)-\frac{k_1}{2}(\xi^4-\eta^4)-\hbar^2\frac{k_1^2-\frac{1}{4}}{2m}\left(\frac{1}{\xi^2}+\frac{1}{\eta^2}\right)\right]. \tag{3.25}$$

In a flat space, the corresponding potential is known as the Holt potential [18]. It consists of a radial harmonic oscillator in one variable (here in the variable u), and a harmonic oscillator plus a linear term in the second variable (here in the variable v). There is an analogue of this potential on the two-dimensional hyperboloid [20], which separates in horicyclic and semicircular parabolic coordinates, the limiting cases of the (u,v) system and the parabolic coordinates, respectively.

3.1.1. Separation of V_1 in the (u,v) System. We start with the (u,v)-coordinate system. We formulate the classical Lagrangian and Hamiltonian, respectively:

$$\mathcal{L}(u,\dot{u},v,\dot{v}) = \frac{m}{2}\frac{bu^2-a}{u^2}(\dot{u}^2+\dot{v}^2) - V(u,v), \tag{3.26}$$

$$\mathcal{H}(u,p_u,v,p_v) = \frac{1}{2m}\frac{u^2}{bu^2-a}(p_u^2+p_v^2) + V(u,v). \tag{3.27}$$

The canonical momenta are

$$p_u = \frac{\hbar}{i}\left(\frac{\partial}{\partial u} + \frac{bu}{bu^2-a} - \frac{1}{u}\right), \quad p_v = \frac{\hbar}{i}\frac{\partial}{\partial v}. \tag{3.28}$$

The quantum Hamiltonian has the form

$$H = -\frac{\hbar^2}{2m}\frac{u^2}{bu^2-a}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) + V(u,v) = \tag{3.29}$$

$$= \frac{1}{2m}\frac{u}{\sqrt{bu^2-a}}(p_u^2+p_v^2)\frac{u}{\sqrt{bu^2-a}} + V(u,v). \tag{3.30}$$

Therefore the path integral for V_1 in the (u,v) system has the following form:

$$K^{(V_1)}(u'',u',v'',v';T) = \int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)\frac{bu^2-a}{u^2}\exp\times$$
$$\times\left(\frac{i}{\hbar}\int_0^T\left\{\frac{m}{2f}(\dot{u}^2+\dot{v}^2) - f\left[\frac{m}{2}\omega^2(u^2+4v^2)+k_1v+\frac{\hbar^2}{2m}\frac{k_2^2-\frac{1}{4}}{u^2}\right]\right\}dt\right), \tag{3.31}$$

and we have abbreviated $f = u^2/(bu^2-a)$. First, we separate the v-path integration according to

$$\int\limits_{v(t')=v'}^{v(t'')=v''} \mathcal{D}v(t) \exp\left\{\frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2}\dot{v}^2 - \left(\frac{m}{2}\omega^2 v^2 + k_1 v\right)\right] dt\right\} =$$

$$= \sum_{n=0}^{\infty} \sqrt{\frac{2m\omega}{\pi\hbar}} \frac{1}{2^n n!} \exp\left[-\frac{m\omega}{\hbar}(\tilde{v}'^2 + \tilde{v}''^2)\right] \times$$

$$\times H_n\left(\sqrt{\frac{2m\omega}{\hbar}}\,\tilde{v}'\right) H_n\left(\sqrt{\frac{2m\omega}{\hbar}}\,\tilde{v}''\right) \mathrm{e}^{-iE_n T/\hbar}, \quad (3.32)$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{k_1^2}{8m\omega^2}, \quad (3.33)$$

with $\tilde{v} = v + k_1/4m\omega$, which is the solution for the shifted harmonic oscillator. Writing for short the wave functions of the shifted harmonic oscillator by $\Psi_n^{(\mathrm{HO})}$, we thus obtain:

$$K^{(V_1)}(u'', u', v'', v'; T) = \sum_{n=0}^{\infty} \Psi_n^{(\mathrm{HO})}(\tilde{v}')\Psi_n^{(\mathrm{HO})}(\tilde{v}'') K_n^{(V_1)}(u'', u'; T), \quad (3.34)$$

$$K_n^{(V_1)}(u'', u'; T) = \left[f(u')f(u'')\right]^{-1/4} \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \sqrt{\frac{bu^2 - a}{u^2}} \times$$

$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2f}\dot{u}^2 - f\left(\frac{m}{2}\omega^2 u^2 + \frac{\hbar^2}{2m}\frac{k_2^2 - \frac{1}{4}}{u^2}\right) + E_n\right] dt\right\}. \quad (3.35)$$

We obtain in the usual way by means of a space-time transformation

$$G_n^{(V_1)}(u'', u'; E) = \int\limits_0^{\infty} K_n^{(V_1)}(u'', u'; s'') \exp\left[\frac{i}{\hbar}(bE - E_n)s''\right] \quad (3.36)$$

with the transformed path integral given by

$$K_n^{(V_1)}(u'', u'; s'')r = \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \times$$

$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''} \left[\frac{m}{2}(\dot{u}^2 - \omega^2 u^2) - \frac{\hbar^2}{2m}\frac{k_2^2 + 2maE/\hbar^2 - \frac{1}{4}}{u^2}\right] ds\right\} = \quad (3.37)$$

$$
= \frac{m\omega\sqrt{u'u''}}{i\hbar\sin\omega s''}\exp\left[-\frac{m\omega}{2i\hbar}(u'^2+u''^2)\cot\omega s''\right]I_\lambda\left(\frac{m\omega u'u''}{i\hbar\sin\omega s''}\right) =
$$

$$
= \sum_{l=0}^{\infty}\Psi_l^{(\mathrm{RHO},\lambda)}(u')\Psi_l^{(\mathrm{RHO},\lambda)}(u'')\ \mathrm{e}^{is''\omega(2l+\lambda+1)}. \quad (3.38)
$$

Alternatively we have for the Green function ($\lambda^2 = k_2^2 + 2maE/\hbar^2$)

$$
G_n^{(V_1)}(u'',u';E) = \frac{\Gamma\left[\frac{1}{2}\left(1+\lambda-\frac{1}{\hbar\omega}(bE-E_n)\right)\right]}{\hbar\omega\sqrt{u'u''}\,\Gamma(1+\lambda)}\times
$$

$$
\times W_{\frac{bE-E_n}{2\hbar\omega},\frac{\lambda}{2}}\left(\frac{m\omega}{\hbar}u_>^2\right)M_{\frac{bE-E_n}{2\hbar\omega},\frac{\lambda}{2}}\left(\frac{m\omega}{\hbar}u_<^2\right). \quad (3.39)
$$

The $W_{\mu,\nu}(z)$ are Whittaker functions [10]. We can either evaluate the s'' integration or analyze the poles of the Green function. The latter gives the poles in terms of the poles of the Γ function yielding the quantization condition for the bound states E_{ln}:

$$
\frac{1}{2}\left(1+\lambda+\frac{E_n-bE_{ln}}{\hbar\omega}\right) = -l, \quad (3.40)
$$

which is equivalent to

$$
\hbar\omega\left(2l+n+\frac{3}{2}+\sqrt{k_2^2+\frac{2maE_{ln}}{\hbar^2}}\right)+\frac{k_1^2}{8m\omega^2}-bE_{ln}=0. \quad (3.41)
$$

Let us analyze this equation in more detail. We obtain similar equations for the other potentials, and the present case serves as a standard example for those which come later. Let us note that the specific form of the discrete spectrum and the corresponding wave functions depend on the special choice of the parameters a and b and the special space of revolution one considers. For instance, the plus respectively the minus sign in the square-root expression below may be allowed giving positive normed states for some cases, and for others the minus sign may be allowed. Similarly, the radicand of the square root can become negative and we may obtain semibound states.

The quadratic equation in E_{ln} gives ($\epsilon_{ln} = (2l+n+3/2)$)

$$
E_{ln} = \frac{\hbar\omega\epsilon_{ln}}{b}+\frac{k_1^2}{8mb\omega^2}+
$$

$$
+\frac{am\omega^2}{b}\pm\frac{1}{b^2}\sqrt{a^2m^2\omega^4+b^2\omega^2\hbar^2k_2^2+2abm\hbar\omega^3\epsilon_{ln}+\frac{ab}{4}k_1^2}, \quad (3.42)
$$

$$(l, n \to \infty) \simeq \frac{\hbar\omega}{b}\left(2l + n + \frac{3}{2}\right) + \frac{k_1^2}{8mb\omega^2} + \frac{a}{b}m\omega^2 + O(\sqrt{\epsilon_{ln}}), \quad (3.43)$$

$$(a = -1, b = 1) =$$
$$= \hbar\omega\epsilon_{ln} + \frac{k_1^2}{8m\omega^2} - m\omega^2 \pm \sqrt{m^2\omega^4 + \omega^2\hbar^2k_2^2 - 2m\hbar\omega^3\epsilon_{ln} - \frac{k_1^2}{4}}. \quad (3.44)$$

In the latter (special) case this gives bound states for $m^2\omega^4 + \hbar^2\omega^2\hbar^2k_2^2 - 2m\hbar\omega^3\epsilon_{ln} - k_1^2/4 \geqslant 0$, i.e., the number of levels is determined by

$$2l + n \leqslant \frac{\hbar k_2^2}{2m\omega} + \frac{m\omega}{2\hbar} - \frac{k_1^2}{8m\hbar\omega^3} - \frac{3}{2}, \quad (3.45)$$

otherwise we may have semibound states, that is, bound states with energy $\Re(E_{ln})$ and with a decay width $\Im(E_{ln})$. They are located in the continuous spectrum. In particular, we have a ground state

$$E_{00} = \frac{3\hbar\omega}{2b} + \frac{k_1^2}{8mb\omega^2} + \frac{am\omega^2}{b} \pm \frac{1}{b^2}\sqrt{a^2m^2\omega^4 + b^2\omega^2\hbar^2k_2^2 + 3abm\hbar\omega^3 + \frac{ab}{4}k_1^2}. \quad (3.46)$$

Note that if the radicand of the square root equals the upper bound of the energy levels, for the case $ab < 1$ we get

$$E_{\text{upper-bound}} = \frac{b\hbar^2k_2^2}{2|ab|m} + \frac{ma\omega^2}{|ab|}\left(\frac{1}{2b} - 1\right) = \quad (3.47)$$

$$= \frac{\hbar^2k_2^2}{2m} - \frac{m}{2}\omega^2 \qquad (a = -1, \quad b = 1). \quad (3.48)$$

The spectrum is similar to the spectrum of the Holt potential: Flat Euclidean space corresponds to $a = 0$, then (3.42) is identical with the result of [18].

Note that different energy spectra emerge depending on the signs of the parameters a and b. For both parameters positive, the discrete spectrum cannot be simultaneously located in the continuous spectrum. For b negative, the properties of the space D_{II} must be further analyzed if a discrete spectrum with negative infinite values is allowed (which is the case for the single-sheeted hyperboloid).

In order to extract the continuous spectrum we consider the dispersion relation [11]

$$I_\lambda(z) = \frac{2}{\pi^2}\int_0^\infty \frac{dp\,\sinh\pi p}{p^2 - \lambda^2}K_{ip}(z). \quad (3.49)$$

This gives

$$G_n^{(V_1)}(u'',u';E) = \sqrt{u'u''}\int_0^\infty \frac{\omega ds''}{i\hbar\sin\omega s''}\times$$
$$\times\exp\left[\frac{i}{\hbar}s''(bE-E_N) - \frac{m\omega}{2i\hbar}(u'^2+u''^2)\cot\omega s''\right] I_\lambda\left(\frac{m\omega u'u''}{i\hbar\sin\omega s''}\right) =$$
$$= \frac{\hbar^2}{\pi^2}\frac{1}{2m\omega\sqrt{u'u''}}\int_0^\infty \frac{dp\,\sinh\pi p}{\frac{\hbar^2}{2m|a|}(p^2+k_2^2)-E}\times$$
$$\times\left|\Gamma\left[\frac{1}{2}(1+ip-\frac{bE-E_n}{\hbar\omega})\right]\right|^2 W_{\frac{bE-E_n}{2\hbar\omega},\frac{ip}{2}}\left(\frac{m\omega}{\hbar}u''^2\right) W_{\frac{bE-E_n}{2\hbar\omega},\frac{ip}{2}}\left(\frac{m\omega}{\hbar}u'^2\right). \tag{3.50}$$

The continuous spectrum has the form

$$E_p = \frac{\hbar^2}{2m|a|}(p^2+k_2^2), \tag{3.51}$$

and the wave functions are

$$\Psi_{pn}(u) = \frac{\hbar}{\pi}\sqrt{\frac{p\sinh\pi p}{2m\omega u}}\Gamma\left[\frac{1}{2}\left(1+ip-\frac{bE-E_n}{\hbar\omega}\right)\right] W_{\frac{bE-E_n}{2\hbar\omega},\frac{ip}{2}}\left(\frac{m\omega}{\hbar}u^2\right). \tag{3.52}$$

Note that for $k_2 = \pm 1/2$, i.e., the radial potential equals zero, we obtain the case from the free motion on $D_{\rm II}$.

Finally, we state the kernel $K^{(V_1)}(T)$ and the Green function $G^{(V_1)}(E)$ which have the form

$$K^{(V_1)}(u'',u',v'',v';T) = \sum_{n=0}^\infty \Psi_n^{(\rm HO)}(\tilde v')\Psi_n^{(\rm HO)}(\tilde v'')\times$$
$$\times\left\{\sum_{l=0}^\infty N_{ln}^2 \Psi_l^{(\rm RHO,\lambda)}(u')\Psi_l^{(\rm RHO,\lambda)}(u'')\ {\rm e}^{-iTE_{ln}/\hbar} + \right.$$
$$\left. + \int_0^\infty dp\Psi_{pn}^*(u'')\Psi_{pn}(u')\ {\rm e}^{-iTE_p/\hbar}\right\}, \tag{3.53}$$

$$G^{(V_1)}(u'', u', v'', v'; E) = \sum_{n,l=0}^{\infty} \Psi_n^{(\mathrm{HO})}(\tilde{v}')\Psi_n^{(\mathrm{HO})}(\tilde{v}'')\times$$

$$\times \frac{\Gamma\left[\frac{1}{2}\left(1+\lambda-\frac{1}{\hbar\omega}(bE-E_n)\right)\right]}{\hbar\omega\sqrt{u'u''}\,\Gamma(1+\lambda)} W_{\frac{bE-E_n}{2\hbar\omega},\frac{\lambda}{2}}\left(\frac{m\omega}{\hbar}u_>^2\right) M_{\frac{bE-E_n}{2\hbar\omega},\frac{\lambda}{2}}\left(\frac{m\omega}{\hbar}u_<^2\right). \tag{3.54}$$

The normalization constant N_{ln} emerges form evaluating the residuum of the Green function (3.39) at the energy E_{ln} as given in (3.42).

3.1.2. Separation of V_1 in Parabolic Coordinates on D_{II}. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(\xi, \dot{\xi}, \eta, \dot{\eta}) = \frac{m}{2}\frac{b\xi^2\eta^2 - a}{\xi^2\eta^2}(\xi^2+\eta^2)(\dot{\xi}^2+\dot{\eta}^2) - V(\xi,\eta), \tag{3.55}$$

$$\mathcal{H}(\xi, p_\xi, \eta, p_\eta) = \frac{m}{2}\frac{\xi^2\eta^2}{b\xi^2\eta^2 - a}\frac{p_\xi^2+p_\eta^2}{\xi^2+\eta^2} + V(\xi,\eta). \tag{3.56}$$

The canonical momenta are given by

$$p_\xi = \frac{\hbar}{i}\left(\frac{\partial}{\partial\xi} + \frac{b\xi + a/\xi^3}{\sqrt{g}}\right), \tag{3.57}$$

$$p_\eta = \frac{\hbar}{i}\left(\frac{\partial}{\partial\eta} + \frac{b\eta + a/\eta^3}{\sqrt{g}}\right). \tag{3.58}$$

The quantum Hamiltonian has the form

$$H = -\frac{\hbar^2}{2m}\left(b\xi^2 + b\eta^2 - \frac{a}{\xi^2} - \frac{a}{\eta^2}\right)^{-1}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) + V(\xi,\eta) = \tag{3.59}$$

$$= \frac{1}{2m}\left(b\xi^2+b\eta^2-\frac{a}{\xi^2}-\frac{a}{\eta^2}\right)^{-1/2}(p_\xi^2+p_\eta^2)\left(b\xi^2+b\eta^2-\frac{a}{\xi^2}-\frac{a}{\eta^2}\right)^{-1/2}+V(\xi,\eta). \tag{3.60}$$

We obtain for the path integral in parabolic coordinates, c.f. (3.25), $(1/f(\xi,\eta) = (b\xi^2\eta^2 - a)/\xi^2\eta^2)$:

$$K^{(V_1)}(\xi'',\xi',\eta'',\eta';T) = \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t) \frac{b\xi^2\eta^2 - a}{\xi^2\eta^2}(\xi^2+\eta^2)\times$$
$$\times \exp\left\{ \frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2f}(\xi^2+\eta^2)(\dot{\xi}^2+\dot{\eta}^2) - \right.\right.$$
$$\left.\left. -f\left(\frac{m}{2}\omega^2(u^2+4v^2) + k_1 v + \frac{\hbar^2}{2m}\frac{k_2^2 - \frac{1}{4}}{u^2} \right) \right] dt \right\}. \quad (3.61)$$

Performing the space-time transformation yields

$$G^{(V_1)}(\xi'',\xi',\eta'',\eta';E) = \int\limits_0^\infty ds'' K^{(V_1)}(\xi'',\xi',\eta'',\eta';s'') \quad (3.62)$$

with the transformed path integral given by

$$K^{(V_1)}(\xi'',\xi',\eta'',\eta';s'') = \int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''} \mathcal{D}\xi(s)\times$$
$$\times \exp\left[\frac{i}{\hbar}\int\limits_0^{s''} \left(\frac{m}{2}(\dot{\xi}^2 - \omega^2\xi^6) - \frac{k_1}{2}\xi^4 - Eb\xi^2 - \frac{\hbar^2}{2m}\frac{k_1^2 + 2maE/\hbar^2 - \frac{1}{4}}{\xi^2} \right) ds \right] \times$$
$$\times \int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''} \mathcal{D}\eta(s)\times$$
$$\times \exp\left[\frac{i}{\hbar}\int\limits_0^{s''} \left(\frac{m}{2}(\dot{\eta}^2 - \omega^2\eta^6) + \frac{k_1}{2}\eta^4 - Eb\eta^2 - \frac{\hbar^2}{2m}\frac{k_1^2 + 2maE/\hbar^2 - \frac{1}{4}}{\eta^2} \right) ds \right]. \quad (3.63)$$

These path integrals are due to the anharmonic terms in ξ and η not tractable, a well-known fact due to its relation to the Holt potential.

3.2. The Superintegrable Potential V_2 on D_{II}. We consider the potential V_2. The corresponding quantum mechanical problem is separable in the (u, v)

system, in polar and elliptic coordinates. First, we state the potential V_2 in the respective coordinate systems:

$$V_2(u,v) = \frac{u^2}{bu^2-a}\left[\frac{m}{2}\omega^2(u^2+v^2)+\frac{\hbar^2}{2m}\left(\frac{k_1^2-\frac{1}{4}}{u^2}+\frac{k_2^2-\frac{1}{4}}{v^2}\right)\right] = \quad (3.64)$$

$$= \frac{\varrho^2\cos^2\vartheta}{b\varrho^2\cos^2\vartheta-a}\left[\frac{m}{2}\omega^2\varrho^2+\frac{\hbar^2}{2m\varrho^2}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\vartheta}+\frac{k_2^2-\frac{1}{4}}{\sin^2\vartheta}\right)\right] = \quad (3.65)$$

$$= \frac{f}{\cosh^2\omega-\cos^2\varphi}\left[\frac{m}{2}d^2\omega^2(\cosh^2\omega\sinh^2\omega+\sin^2\varphi\cos^2\varphi)+ \right.$$
$$\left. +\frac{\hbar^2}{2md^2}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}-\frac{k_1^2-\frac{1}{4}}{\cosh^2\omega}+\frac{k_2^2-\frac{1}{4}}{\sinh^2\omega}\right)\right]. \quad (3.66)$$

The potential V_2 can be interpreted as a two-dimensional oscillator with radial term similarly as its analogue in flat space. Note that a Higgs-like harmonic oscillator on $D_{\rm II}$ could have a form according to (with the limiting case the Higgs oscillator on the hyperboloid)

$$V_{\rm Higgs} = \frac{m}{2}\omega^2\frac{u^2}{bu^2-a}\left(1-\frac{4u^2}{(1+u^2+v^2)^2}\right) = \frac{m}{2}\omega^2\frac{\varrho^2\cos^2\vartheta}{b\varrho^2\cos^2\vartheta-a}\times$$
$$\times\left(1-\frac{4\varrho^2\cos^2\vartheta}{(1+\varrho^2)^2}\right) = \frac{m}{2}\omega^2\left(\frac{b\,{\rm e}^{2\tau_2}}{\cosh^2\tau_1}-a\right)^{-1}\left(1-\frac{1}{\cosh^2\tau_1\cosh^2\tau_2}\right), \quad (3.67)$$

with $\varrho={\rm e}^{\tau_2}$, $\cos\vartheta = 1/\cosh\tau_1$, $\tau_{1,2}$ being equidistant coordinates. The corresponding path integral cannot be solved, and $V_{\rm Higgs}$ is not superintegrable in $D_{\rm II}$ either.

3.2.1. Separation of V_2 in the (u,v) System. We start with the consideration in the (u,v) system, and the path integral has the form

$$K^{(V_2)}(u'',u',v'',v';T) = \int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)\frac{bu^2-a}{u^2}\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2f}(\dot u^2+\dot v^2)-f\frac{m}{2}\omega^2(u^2+v^2)-f\frac{\hbar^2}{2m}\left(\frac{k_1^2-\frac{1}{4}}{u^2}+\frac{k_2^2-\frac{1}{4}}{v^2}\right)\right]dt\right\} = \quad (3.68)$$

$$= \sum_{n=0}^{\infty} \Psi_n^{(\mathrm{RHO},k_2)}(v'')\Psi_n^{(\mathrm{RHO},k_2)}(v')\big[f(u')f(u'')\big]^{-1/4} \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t)\sqrt{\frac{bu^2-a}{u^2}}\times$$

$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2f}\dot{u}^2 - f\left(\frac{m}{2}u^2 + \frac{\hbar^2}{2m}\frac{k_1^2-\frac{1}{4}}{u^2} + E_n\right)\right]dt\right\}, \quad (3.69)$$

where $E_n = \hbar\omega(2n+|k_2|+1)$. Performing a space-time transformation in the usual way yields:

$$G_n^{(V_2)}(u'',u';E) = \int\limits_0^\infty ds''\,\mathrm{e}^{is''(bE-E_n)/\hbar}\,K_n^{(V_2)}(u'',u';s'') \quad (3.70)$$

with the transformed path integral given by

$$K_n^{(V_2)}(u'',u';s'') = $$

$$= \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s)\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot{u}^2-\omega^2u^2) - \frac{\hbar^2}{2m}\frac{\lambda_1^2-\frac{1}{4}}{u^2}\right]ds\right\}, \quad (3.71)$$

where $\lambda_1^2 = k_1^2 + 2maE/\hbar^2$. This path integral has almost the same form as the path integral (3.37), the only difference being another E_n. Thus we can write the solution as follows:

$$K_n^{(V_2)}(u'',u';s'') = \frac{m\omega\sqrt{u'u''}}{i\hbar\sin\omega s''}\exp\left[-\frac{m\omega}{2i\hbar}({u'}^2+{u''}^2)\cot\omega s''\right]I_{\lambda_1}\left(\frac{m\omega u'u''}{i\hbar\sin\omega s''}\right) =$$

$$= \sum_{l=0}^{\infty}\Psi_l^{(\mathrm{RHO},\lambda_1)}(u')\Psi_l^{(\mathrm{RHO},\lambda_1)}(u'')\,\mathrm{e}^{is''\omega(2l+\lambda_1+1)]}, \quad (3.72)$$

and alternatively we have for the Green function

$$G_n^{(V_2)}(u'',u';E) = \frac{\Gamma\left[\frac{1}{2}\left(1+\lambda_1-\frac{1}{\hbar\omega}(bE-E_n)\right)\right]}{\hbar\omega\sqrt{u'u''}\,\Gamma(1+\lambda_1)}\times$$

$$\times W_{\frac{bE-E_n}{2\hbar\omega},\frac{\lambda_1}{2}}\left(\frac{m\omega}{\hbar}u_>^2\right)M_{\frac{bE-E_n}{2\hbar\omega},\frac{\lambda_1}{2}}\left(\frac{m\omega}{\hbar}u_<^2\right). \quad (3.73)$$

We can either evaluate the s'' integration or analyze the poles of the Green function. The latter gives the poles in terms of the poles of the Γ function yielding the quantization condition for the bound states E_{ln}:

$$\frac{1}{2}\left(1+\lambda_1+\frac{E_n-bE_{ln}}{\hbar\omega}\right)=-l, \tag{3.74}$$

which is equivalent to

$$\hbar\omega\left(2l+2n+2+|k_2|+\sqrt{k_1^2+\frac{2maE_{ln}}{\hbar}}\right)-bE_{ln}=0. \tag{3.75}$$

The quadratic equation in E_{ln} gives ($\epsilon_{ln}=(2l+2n+2+|k_2|)$)

$$E_{ln}=\frac{\hbar\omega\epsilon_{ln}}{b}+\frac{a}{b}m\omega^2-\frac{1}{b^2}\sqrt{a^2m^2\omega^4+b^2\hbar^2\omega^2k_1^2+2abm\hbar\omega^3\epsilon_{ln}}, \tag{3.76}$$

$$(a=-1,\, b=1)=\hbar\omega\epsilon_{ln}-m\omega^2-\sqrt{m^2\omega^4+\hbar^2\omega^2k_1^2-2m\omega^2\epsilon_{ln}}, \tag{3.77}$$

$$(l,n\to\infty)\simeq\hbar\omega\epsilon_{ln}-m\omega^2. \tag{3.78}$$

This gives for the special case the bound states for $m^2\omega^4+\hbar^2\omega^2k_1^2-2m\omega^2\epsilon_{ln}\geqslant 0$, otherwise we can infer for semibound states, that is, bound states with energy $\Re(E_{ln})$ and with a decay width $\Im(E_{ln})$. They are located in the continuous spectrum. Again, the limiting case of flat space emerges from $a=0, b=1$

$$E_{ln}=\hbar\omega(2l+2n+|k_1|+|k_2|+2). \tag{3.79}$$

Finally, we state the kernel $K^{(V_2)}(T)$ and the Green function $G^{(V_2)}(E)$ which have the form

$$K^{(V_2)}_{\rm disc}(u'',u',v'',v';T)=\sum_{n,l=0}^{\infty}N_{ln}^2\times \\ \times\,\Psi_n^{({\rm RHO},k_2)}(v')\Psi_n^{({\rm RHO},k_2)}(v'')\Psi_l^{({\rm RHO},\lambda_1)}(u')\Psi_l^{({\rm RHO},\lambda_1)}(u'')\,{\rm e}^{-iTE_{ln}/\hbar}, \tag{3.80}$$

$$G^{(V_2)}(u'',u',v'',v';E)=\sum_{n=0}^{\infty}\Psi_n^{({\rm RHO},k_2)}(v')\Psi_n^{({\rm RHO},k_2)}(v'')\times \\ \times\frac{\Gamma\left[\frac{1}{2}\left(1+\lambda_1-\frac{1}{\hbar\omega}(bE-E_n)\right)\right]}{\hbar\omega\sqrt{u'u''}\,\Gamma(1+\lambda_1)}W_{\frac{bE-E_n}{2\hbar\omega},\frac{\lambda_1}{2}}\left(\frac{m\omega}{\hbar}u_>^2\right)\times \\ \times M_{\frac{bE-E_n}{2\hbar\omega},\frac{\lambda_1}{2}}\left(\frac{m\omega}{\hbar}u_<^2\right). \tag{3.81}$$

The normalization constant N_{ln} emerges from evaluating the residuum of the Green function (3.73) at the energy E_{ln} as given in (3.76). We omit the continuous part of $K^{(V_2)}$ due to its similarity to the case of V_1.

3.2.2. Separation of V_2 in Polar Coordinates. The potential V_2 is also separable in polar coordinates on $D_{\rm II}$. In polar coordinates the classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(r,\dot r,\vartheta,\dot\vartheta) = \frac{m}{2}\left(b-\frac{a}{\varrho^2\cos^2\vartheta}\right)(\dot\varrho^2+\varrho^2\dot\vartheta^2) - V(\varrho,\vartheta), \tag{3.82}$$

$$\mathcal{H}(\varrho,p_\varrho,\vartheta,p_\vartheta) = \frac{1}{2m}\left(b-\frac{a}{\varrho^2\cos^2\vartheta}\right)^{-1}\left(p_\varrho^2+\frac{1}{\varrho^2}p_\vartheta^2\right) + V(\varrho,\vartheta). \tag{3.83}$$

The momentum operators are

$$p_\varrho = \frac{\hbar}{i}\left[\frac{\partial}{\partial\varrho}+\left(\frac{b\varrho\cos^2\vartheta}{b\cos^2\vartheta\varrho^2-a}-\frac{1}{2\varrho}\right)\right], \tag{3.84}$$

$$p_\vartheta = \frac{\hbar}{i}\left[\frac{\partial}{\partial\vartheta}+\left(\tan\vartheta-\frac{b\varrho^2\sin\vartheta\cos\vartheta}{b\varrho^2\cos^2\vartheta-a}\right)\right], \tag{3.85}$$

and the quantum Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m}\left(b-\frac{a}{\varrho^2\cos^2\vartheta}\right)^{-1}\left(\frac{\partial^2}{\partial\varrho^2}+\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\frac{\partial^2}{\partial\vartheta^2}\right)+V(\varrho,\vartheta)= \tag{3.86}$$

$$= \frac{1}{2m}f^{1/2}\left(p_\varrho^2+\frac{1}{\varrho^2}p_\vartheta^2\right)f^{1/2}+V(\varrho,\vartheta)-f\frac{\hbar^2}{8m\varrho^2} \tag{3.87}$$

with the abbreviation $1/f = b - a/\varrho^2\cos^2\vartheta$. Hence, we get for the path integral

$$\begin{aligned}K^{(V_2)}(\varrho'',\varrho',\vartheta'',\vartheta';T) &= \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\varrho\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\left(b-\frac{a}{\varrho^2\cos^2\vartheta}\right)\times\\ &\times\exp\left(\frac{i}{\hbar}\int\limits_0^T\left\{\frac{m}{2f}(\dot\varrho^2+\varrho^2\dot\vartheta^2)-\right.\right.\\ &\left.\left.-f\left[\frac{m}{2}\omega^2\varrho^2+\frac{\hbar^2}{2m\varrho^2}\left(\frac{k_1^2-\frac14}{\cos^2\vartheta}+\frac{k_2^2-\frac14}{\sin^2\vartheta}+\frac14\right)\right]\right\}dt\right).\end{aligned} \tag{3.88}$$

Performing the space time transformation with the function f yields

$$G^{(V_2)}(\varrho'',\varrho',\vartheta'',\vartheta';E) = \int\limits_0^\infty ds''\, \mathrm{e}^{is''bE/\hbar}\, K^{(V_2)}(\varrho'',\varrho',\vartheta'',\vartheta';s'') \tag{3.89}$$

and the transformed path integral given by ($\lambda_1^2 = k_1^2 + 2maE/\hbar^2$)

$$K^{(V_2)}(\varrho'', \varrho', \vartheta'', \vartheta'; s'') = \int\limits_{\varrho(0)=\varrho'}^{\varrho(s'')=\varrho''} \mathcal{D}\varrho(s) \int\limits_{\vartheta(0)=\vartheta'}^{\vartheta(s'')=\vartheta''} \mathcal{D}\vartheta(s)\varrho \times$$

$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot{\varrho}^2 + \varrho^2\dot{\vartheta}^2) - \frac{m}{2}\omega^2\varrho^2 - \right.\right.$$

$$\left.\left. - \frac{\hbar^2}{2m\varrho^2}\left(\frac{\lambda_1^2 - \frac{1}{4}}{\cos^2\vartheta} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\vartheta} + \frac{1}{4}\right)\right] ds\right\} = \quad (3.90)$$

$$= \frac{1}{\sqrt{\varrho'\varrho''}} \sum_{n=0}^{\infty} \Phi_n^{(\lambda_1,k_2)}(\vartheta'')\Phi_n^{(\lambda_1,k_2)}(\vartheta') \times$$

$$\times \int\limits_{\rho(0)=\rho'}^{\rho(s'')=\rho''} \mathcal{D}\rho(s) \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot{\varrho}^2 - \omega^2\varrho^2) - \frac{\hbar^2}{2m}\frac{\lambda_2^2 - \frac{1}{4}}{\varrho^2}\right] ds\right\} = \quad (3.91)$$

$$= \frac{1}{\sqrt{\varrho'\varrho''}} \sum_{n=0}^{\infty} \Phi_n^{(\lambda_1,k_2)}(\vartheta'')\Phi_n^{(\lambda_1,k_2)}(\vartheta') \times$$

$$\times \sum_{l=0}^{\infty} \Psi_l^{(\mathrm{RHO},\lambda_2)}(\varrho'')\Psi_l^{(\mathrm{RHO},\lambda_2)}(\varrho')\, \mathrm{e}^{-is''E_l/\hbar}. \quad (3.92)$$

Here we denote $E_l = \hbar\omega(2l + \lambda_2 + 1)$, and the quantity λ_2 is defined by means of the energy spectrum of the Pöschl–Teller spectrum

$$\frac{\hbar^2}{2m}(2n + 1 + \lambda_1 + |k_2|) = \frac{\hbar^2}{2m}\lambda_2^2. \quad (3.93)$$

The $\Phi_n^{(k_1,k_2)}(\beta)$ are the wave functions of the Pöschl–Teller potential, which are given by [2, 5, 8, 39]

$$V(x) = \frac{\hbar^2}{2m}\left(\frac{\alpha^2 - \frac{1}{4}}{\sin^2 x} + \frac{\beta^2 - \frac{1}{4}}{\cos^2 x}\right), \quad (3.94)$$

$$\Phi_n^{(\alpha,\beta)}(x) = \left[2(\alpha + \beta + 2l + 1)\frac{l!\Gamma(\alpha + \beta + l + 1)}{\Gamma(\alpha + l + 1)\Gamma(\beta + l + 1)}\right]^{1/2} \times$$

$$\times (\sin x)^{\alpha + 1/2}(\cos x)^{\beta + 1/2} P_n^{(\alpha,\beta)}(\cos 2x). \quad (3.95)$$

The $P_n^{(\alpha,\beta)}(z)$ are Gegenbauer polynomials [10]. Performing the s'' integration gives poles in the Green function for

$$\hbar\omega(2l+2n+2+\lambda_1+|k_2|)-bE_{ln}=0. \tag{3.96}$$

This is identical to (3.75), as it should be. Concerning the discrete spectrum we can state the kernel as follows:

$$K_{\rm disc}^{(V_2)}(\varrho'',\varrho',\vartheta'',\vartheta';T)=\frac{1}{\sqrt{\varrho'\varrho''}}\sum_{n=0}^{\infty}\Phi_n^{(\lambda_1,k_2)}(\vartheta'')\Phi_n^{(\lambda_1,k_2)}(\vartheta')\times$$
$$\times\sum_{l=0}^{\infty}N_{ln}^2\Psi_l^{({\rm RHO},\lambda_2)}(\varrho'')\Psi_l^{({\rm RHO},\lambda_2)}(\varrho')\,{\rm e}^{-is''E_{ln}/\hbar}, \tag{3.97}$$

with N_{ln} defined by the residuum of the Green function at the energy E_{ln} as given in (3.76).

3.2.3. Separation of V_2 in Elliptic Coordinates on $D_{\rm II}$. The free classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(\omega,\dot\omega,\varphi,\dot\varphi)=\frac{m}{2}\frac{bd^2\cosh^2\omega\cos^2\varphi-a}{\cosh^2\omega\cos^2\varphi}(\cosh^2\omega-\cos^2\varphi)(\dot\omega^2+\dot\varphi^2)=$$
$$=\frac{m}{2}\left[\left(bd^2\cosh^2\omega+\frac{a}{\cosh^2\omega}\right)-\left(bd^2\cos^2\varphi+\frac{a}{\cos^2\varphi}\right)\right](\dot\omega^2+\dot\varphi^2), \tag{3.98}$$

$$\mathcal{H}(\omega,p_\omega,\varphi,p_\varphi)=\frac{1}{2m}\frac{\cosh^2\omega\cos^2\varphi}{(bd^2\cosh^2\omega\cos^2\varphi-a)(\cosh^2\omega-\cos^2\varphi)}(p_\omega^2+p_\varphi^2). \tag{3.99}$$

In the following we use

$$\sqrt{g}=\frac{bd^2\cosh^2\omega\cos^2\varphi-a}{\cosh^2\omega\cos^2\varphi}(\cosh^2\omega-\cos^2\varphi).$$

For the momentum operators we obtain

$$p_\omega=\frac{\hbar}{i}\left[\frac{\partial}{\partial\omega}+\frac{\tanh\omega}{\sqrt{g}}\left(bd^2\cosh^2\omega-\frac{a}{\cosh^2\omega}\right)\right], \tag{3.100}$$
$$p_\varphi=\frac{\hbar}{i}\left[\frac{\partial}{\partial\varphi}+\frac{\tan\varphi}{\sqrt{g}}\left(bd^2\cos^2\varphi-\frac{a}{\cos^2\varphi}\right)\right]. \tag{3.101}$$

This gives for the quantum Hamiltonian

$$H=-\frac{\hbar^2}{2m}\frac{\cosh^2\omega\cos^2\varphi}{(bd^2\cosh^2\omega\cos^2\varphi-a)(\cosh^2\omega-\cos^2\varphi)}\left(\frac{\partial^2}{\partial\omega^2}+\frac{\partial^2}{\partial\varphi^2}\right)=$$
$$=\frac{1}{2m}\frac{1}{\sqrt[4]{g}}(p_\omega^2+p_\phi^2)\frac{1}{\sqrt[4]{g}}. \tag{3.102}$$

Therefore we obtain for the path integral ($1/f = (bd^2 \cosh^2 \omega \cos^2 \varphi - a)/\cosh^2 \omega \cos^2 \varphi$)

$$K^{(V_2)}(\omega'', \omega', \varphi'', \varphi'; T) = \int\limits_{\omega(t')=\omega'}^{\omega(t'')=\omega''} \mathcal{D}\omega(t) \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) \frac{(\cosh^2 \omega - \cos^2 \varphi)}{f} \times$$
$$\times \exp\left\{ \frac{i}{\hbar} \int\limits_0^T \left[\frac{m}{2f} (\cosh^2 \omega - \cos^2 \varphi)(\dot\omega^2 + \dot\varphi^2) - \right.\right.$$
$$\left.\left. - f \frac{m}{2} \omega^2 (u^2 + v^2) - f \frac{\hbar^2}{2m} \left(\frac{k_1^2 - \frac{1}{4}}{u^2} + \frac{k_2^2 - \frac{1}{4}}{v^2} \right) \right] dt \right\} =$$
$$= \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \, \mathrm{e}^{-iET/\hbar} \int\limits_0^\infty ds'' K^{(V_2)}(\omega'', \omega', \varphi'', \varphi'; s''), \quad (3.103)$$

with the transformed path integral $K^{(V_2)}(\omega'', \omega', \varphi'', \varphi'; s'')$ given by ($a < 0$)

$$K^{(V_2)}(\omega'', \omega', \varphi'', \varphi'; s'') = \int\limits_{\omega(0)=\omega'}^{\omega(s'')=\omega''} \mathcal{D}\omega(s) \exp\left\{ \frac{i}{\hbar} \int\limits_0^{s''} \left[\frac{m}{2} (\dot\omega^2 - \right.\right.$$
$$- d^2 \omega^2 \cosh^2 \omega \sinh^2 \omega) - \frac{\hbar^2}{2m} \left(\frac{-k_1^2 - 2m|a|E/\hbar^2 - \frac{1}{4}}{\cosh^2 \omega} + \frac{k_2^2 - \frac{1}{4}}{\sinh^2 \omega} \right) +$$
$$\left.\left. + Ebd^2 \cosh^2 \omega \right] ds \right\} \int\limits_{\varphi(0)=\varphi'}^{\varphi(s'')=\varphi''} \mathcal{D}\varphi(s) \exp\left\{ \frac{i}{\hbar} \int\limits_0^{s''} \left[\frac{m}{2} (\dot\varphi^2 - d^2 \omega^2 \sin^2 \varphi \cos^2 \varphi) - \right.\right.$$
$$\left.\left. - \frac{\hbar^2}{2m} \left(\frac{k_1^2 - 2m|a|E/\hbar^2 - \frac{1}{4}}{\cos^2 \varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2 \varphi} \right) - Ebd^2 \cos^2 \varphi \right] ds \right\}. \quad (3.104)$$

We leave these path integrals as they are, because they are not tractable.

3.3. The Superintegrable Potential V_3 on D_{II}. We consider the potential V_3 and start by expressing V_3 in the respective coordinate systems. We have

$$V_3(u, v) = \frac{f}{\sqrt{u^2 + v^2}} \left[-\alpha + \frac{\hbar^2}{2m} \left(\frac{k_1^2 - \frac{1}{4}}{\sqrt{u^2 + v^2} + v} + \frac{k_2^2 - \frac{1}{4}}{\sqrt{u^2 + v^2} - v} \right) \right] =$$

$$\text{(Polar coordinates:)} \quad = \frac{2f}{\varrho}\left[-\alpha + \frac{\hbar^2}{2m\varrho}\left(\frac{k_1^2 - \frac{1}{4}}{1+\sin\vartheta} + \frac{k_2^2 - \frac{1}{4}}{1-\sin\vartheta}\right)\right],$$

(Transformation: $\cos\vartheta = \sin 2\phi,\ \sin\vartheta = \cos 2\phi,\ \varrho = r^2/2$)

$$= \frac{f}{r^2}\left[-\alpha + \frac{\hbar^2}{2mr^2}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\cos^2\varphi}\right)\right], \quad (3.105)$$

$$\text{(Parabolic coordinates:)} = \frac{2f}{\xi^2+\eta^2}\left[-\alpha + \frac{\hbar^2}{2m}\left(\frac{k_1^2 - \frac{1}{4}}{\xi^2} + \frac{k_2^2 - \frac{1}{4}}{\eta^2}\right)\right], \quad (3.106)$$

$$\text{(Rotated elliptic coordinates:)} = \frac{f}{\sqrt{u^2+v^2}}\frac{1}{\cosh^2\omega' - \cos^2\varphi'}\times$$
$$\times\Bigg[-b'^2\alpha(\cosh^2\omega' - \cos^2\varphi') +$$
$$+\frac{\hbar^2}{2m}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi'} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi'} - \frac{k_1^2 - \frac{1}{4}}{\cosh^2\omega'} + \frac{k_2^2 - \frac{1}{4}}{\sinh^2\omega'}\right)\Bigg]. \quad (3.107)$$

In the last case the rotated elliptic coordinates are given by

$$u = \frac{b'^2}{4}\sinh 2\omega' \sin 2\varphi', \qquad v = \frac{b'^2}{4}(\cosh 2\omega' \cos 2\varphi' + 1). \quad (3.108)$$

Due to the complicated structure of the path integral in rotated elliptic coordinates no closed solution can be stated. We will omit a path integral discussion of V_3 in these coordinates. The potential V_3 can be interpreted as an analogue of the Coulomb potential. Similarly as in flat space and on the two-dimensional hyperboloid it is separable in three coordinate systems, i.e., in spherical, parabolic, and rotated elliptic coordinates (there are no conical coordinates in D_{II}).

3.3.1. Separation of V_3 in Polar Coordinates. We start with the investigation of V_3 in polar coordinates and we immediately switch from the (ρ,ϑ) system to the (r,φ) system. This gives for the path integral:

$$K^{(V_3)}(r'',r',\varphi'',\varphi';T)=\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\times$$
$$\times\left(br^2-\frac{a}{r^2\sin^2\varphi\cos^2\varphi}\right)r\exp\left(\frac{i}{\hbar}\int\limits_0^T\left\{\frac{m}{2f}(\dot r^2+r^2\dot\varphi^2)-\right.\right.$$
$$\left.\left.-f\left[-\alpha+\frac{\hbar^2}{2mr^2}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\cos^2\varphi}-\frac14\right)\right]\right\}dt\right),\quad(3.109)$$

with $1/f=br^2-a/r^2\sin^2\varphi\cos^2\varphi$. Proceeding in the usual way by means of a space time transformation gives

$$G^{(V_3)}(r'',r',\varphi'',\varphi';E)=\int\limits_0^\infty ds''\,\mathrm{e}^{is''\alpha/\hbar}\,K^{(V_3)}(r'',r',\varphi'',\varphi';s'')\qquad(3.110)$$

and the path integral $K^{(V_3)}(s'')$ given by

$$K^{(V_3)}(r'',r',\varphi'',\varphi';s'')=\int\limits_{r(0)=r'}^{r(s'')=r''}\mathcal{D}r(s)\int\limits_{\varphi(0)=\varphi'}^{\varphi(s'')=\varphi''}\mathcal{D}\varphi(s)r\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot r^2+r^2\dot\varphi^2)+Ebr^2-\frac{\hbar^2}{2mr^2}\left(\frac{\lambda_2^2-\frac14}{\sin^2\varphi}+\frac{\lambda_1^2-\frac14}{\cos^2\varphi}-\frac14\right)\right]ds\right\}=$$
$$=\sum_{n=0}^\infty\Phi_n^{(\lambda_1,\lambda_2)}(\varphi'')\Phi_n^{(\lambda_1,\lambda_2)}(\varphi')K_n^{(V_3)}(r'',r',s''),\quad(3.111)$$

with $\lambda_{1,2}^2=k_{1,2}^2+2maE/\hbar^2$. The path integral $K_n^{(V_3)}(s'')$ has the form

$$K_n^{(V_3)}(r'',r';s'') = \frac{1}{\sqrt{r'r''}} \int\limits_{r(0)=r'}^{r(s'')=r''} \mathcal{D}r(s) \exp \times$$

$$\times \left[\frac{i}{\hbar} \int\limits_0^{s''} \left(\frac{m}{2}\dot{r}^2 + Ebr^2 - \frac{\hbar^2}{2m} \frac{\Lambda^2 - \frac{1}{4}}{r^2} \right) ds \right] =$$

$$= \frac{m\omega}{i\hbar \sin \omega s''} \exp\left[-\frac{m\omega}{2i\hbar}({r'}^2 + {r''}^2) \cot \omega s'' \right] I_\Lambda \left(\frac{m\omega r' r''}{i\hbar \sin \omega s''} \right) =$$

$$= \sqrt{-\frac{m}{2Eb}} \frac{\Gamma\left[\frac{1}{2}\left(1 + \Lambda - \frac{\alpha}{\hbar}\sqrt{-\frac{m}{2Eb}}\right)\right]}{\Gamma(1+\Lambda)\sqrt{r'r''}} \times$$

$$\times M_{\frac{\alpha}{2\hbar}\sqrt{-\frac{m}{2Eb}}, \frac{\Lambda}{2}} \left(\frac{m}{\hbar}\sqrt{-\frac{2Eb}{m}} r_<^2 \right) W_{\frac{\alpha}{2\hbar}\sqrt{-\frac{m}{2Eb}}, \frac{\Lambda}{2}} \left(\frac{m}{\hbar}\sqrt{-\frac{2Eb}{m}} r_>^2 \right), \quad (3.112)$$

$$K_{n,\mathrm{disc}}^{(V_3)}(r'',r';s'') = \frac{1}{\sqrt{r'r''}} \sum_{l=0}^{\infty} \Psi_l^{(\mathrm{RHO},\Lambda)}(r'') \Psi_l^{(\mathrm{RHO},\Lambda)}(r')\ \mathrm{e}^{-i\omega(2l+\Lambda+1)s''}. \quad (3.113)$$

This is the usual radial harmonic oscillator solution, and we have set $\Lambda = 2n + \lambda_1 + \lambda_2 + 1$, $\omega^2 = -2Eb/m$. The bound states are determined by the quantization condition

$$2\hbar\omega(l + n + 1) + \frac{\hbar^2 k_1^2}{2m} + \hbar\omega(\lambda_2 + \lambda_3) = 0, \quad (3.114)$$

or alternatively

$$2(l + n + 1) - \frac{\alpha}{\hbar}\sqrt{-\frac{m}{2Eb}} + \sqrt{k_1^2 + \frac{2maE}{\hbar^2}} + \sqrt{k_2^2 + \frac{2maE}{\hbar^2}} = 0. \quad (3.115)$$

For $a = 0$, $b = 1$ we recover the two-dimensional flat space Coulomb spectrum. In general, this is an equation of eighth order in E, where no closed solution can be stated. However, we can study the special case $k_1 = k_2 = 0$, which gives the quantization condition (we also take $a < 0, b > 0$)

$$2(l + n + 1) - \frac{\alpha}{\hbar}\sqrt{\frac{m}{2b}} \frac{1}{\sqrt{-E}} + \frac{2}{\hbar}\sqrt{2m|a|}\sqrt{-E} = 0. \quad (3.116)$$

This is a quadratic equation in E with solution (only one solution is physical, set $N = l + n + 1$)

$$E_{ln} = -\frac{\hbar^2 N^2}{8m|a|}\left(\sqrt{1+\frac{2m\alpha}{\hbar^2 N^2}\sqrt{\frac{|a|}{b}}}-1\right)^2 \simeq \tag{3.117}$$

$$\simeq -\frac{m\alpha^2}{8b\hbar^2 N^2} \qquad (N\to\infty) \tag{3.118}$$

showing a Coulomb behavior.

In order to extract the continuous spectrum we use (3.49) and obtain for the entire kernel

$$G^{(V_3)}(r'',r',\varphi'',\varphi';E) = \frac{\hbar}{\sqrt{r'r''}}\sum_{n=0}^{\infty}\Phi_n^{(\lambda_1,\lambda_2)}(\varphi'')\Phi_n^{(\lambda_1,\lambda_2)}(\varphi')\times$$
$$\times\left\{\sum_{l=0}^{\infty}\frac{N_{ln}^2}{E_{ln}-E}\Psi_l^{(\mathrm{RHO},\Lambda)}(r'')\Psi_l^{(\mathrm{RHO},\Lambda)}(r') + \int_{-\infty}^{\infty}\times\right.$$
$$\left.\times\frac{1}{E_p-E}\Psi_p^{(\mathrm{RHO},\Lambda)\,*}(r'')\Psi_p^{(\mathrm{RHO},\Lambda)}(r')\right\} \tag{3.119}$$

with the discrete energy spectrum as determined by (3.115) and the normalization constant N_{ln} resulting from the residuum in (3.113). The continuous spectrum is given by ($k_<$ denotes the smaller of k_1, k_2)

$$\Psi_p^{(\mathrm{RHO})}(r) = \frac{\mathrm{e}^{\pi p/2}}{\sqrt{\pi}}\frac{\Gamma\left[\frac{1}{2}(1+\Lambda)+ip\right]}{\Gamma(1+\Lambda)}M_{ip/2,\Lambda/2}\left(-\frac{\sqrt{-2mbE}}{\hbar}\,r\right), \tag{3.120}$$

$$E_p = \frac{\hbar^2}{2m}(p^2+k_<^2). \tag{3.121}$$

3.3.2. Separation of V_3 in Parabolic Coordinates. Finally, we consider V_3 in parabolic coordinates. The formulation of the path integral for a potential on D_{II} we know already from V_1. We therefore have ($f = b - a/\xi^2\eta^2$)

$$K^{(V_3)}(\xi'',\xi',\eta'',\eta';T) = \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)\left(b-\frac{a}{\xi^2\eta^2}\right)(\xi^2+\eta^2)\times$$
$$\times\exp\left(\frac{i}{\hbar}\int_0^{\infty}\left\{\frac{m}{2}f(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2)-\frac{1}{f(\xi^2+\eta^2)}\times\right.\right.$$
$$\left.\left.\times\left[-\alpha+\frac{\hbar^2}{2m}\left(\frac{k_1^2-\frac{1}{4}}{\xi^2}+\frac{k_2^2-\frac{1}{4}}{\eta^2}\right)\right]\right\}dt\right), \tag{3.122}$$

$$G^{(V_3)}(\xi'',\xi',\eta'',\eta';E)=\int\limits_0^\infty ds''\,\mathrm{e}^{i\alpha s''/\hbar}\,K^{(V_3)}(\xi'',\xi',\eta'',\eta';s''),\qquad(3.123)$$

with the transformed path integral $K(s'')$ given by

$$\begin{aligned}K^{(V_3)}(\xi'',\xi',\eta'',\eta';s'')=&\int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''}\mathcal{D}\xi(s)\times\\
&\times\exp\left[\frac{i}{\hbar}\int\limits_0^{s''}\left(\frac{m}{2}\dot\xi^2+Eb\xi^2-\frac{\hbar^2}{2m}\frac{\lambda_1^2-\frac14}{\xi^2}\right)ds\right]\times\\
&\times\int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''}\mathcal{D}\eta(s)\exp\left[\frac{i}{\hbar}\int\limits_0^{s''}\left(\frac{m}{2}\dot\eta^2+Eb\eta^2-\frac{\hbar^2}{2m}\frac{\lambda_2^2-\frac14}{\eta^2}\right)ds\right]r\times\\
&\times K^{(V_3)}_{\text{disc}}(\xi'',\xi',\eta'',\eta';s'')=\\
=&\sum_{n_\xi n_\eta=0}^\infty\Psi_{n_\xi}^{(\text{RHO},\lambda_1)}(\xi'')\Psi_{n_\xi}^{(\text{RHO},\lambda_1)}(\xi')\Psi_{n_\eta}^{(\text{RHO},\lambda_2)}(\eta'')\Psi_{n_\eta}^{(\text{RHO},\lambda_2)}(\eta')\times\\
&\times\exp\left[-\frac{i}{\hbar}s''(2n_\xi+2n_\eta+\lambda_1+\lambda_2+2)\hbar\omega\right].\qquad(3.124)\end{aligned}$$

We have inserted for the discrete spectrum the solution of the radial harmonic oscillator in the usual way. Performing the s'' integration gives the same spectrum as in (3.115), as it should be.

The continuous spectrum is extracted in the usual way by means of (3.49) and we obtain:

$$\begin{aligned}&G^{(V_3)}(\xi'',\xi',\eta'',\eta';E)=\\
&=\sum_{n_\xi n_\eta=0}^\infty\frac{\hbar N_{ln}^2}{E_{ln}-E}\Psi_{n_\xi}^{(\text{RHO},\lambda_1)}(\xi'')\Psi_{n_\xi}^{(\text{RHO},\lambda_1)}(\xi')\Psi_{n_\eta}^{(\text{RHO},\lambda_2)}(\eta'')\Psi_{n_\eta}^{(\text{RHO},\lambda_2)}(\eta')+\\
&\qquad+\hbar(\xi'\xi''\eta'\eta'')^{-1/2}\int d\mathcal{E}\int\limits_0^\infty\times\\
&\qquad\times\frac{dp\,p\sinh\pi p}{\frac{\hbar^2}{2m|a|}(p^2+k_<^2)-E}\frac{\left|\Gamma\left[\frac12(1+ip_1-\mathcal{E})\right]\right|^2\left|\Gamma\left[\frac12(1+ip_2-\mathcal{E})\right]\right|^2}{2\pi\tilde p^2}\times\end{aligned}$$

$$\times W_{\mathcal{E}/2,ip/2}\Big(i\tilde{p}_1\xi''^2\Big)W^*_{\mathcal{E}/2,ip/2}\Big(i\tilde{p}_1\xi'^2\Big)W_{\mathcal{E}/2,ip/2}\Big(i\tilde{p}_2\xi''^2\Big)W^*_{\mathcal{E}/2,ip/2}\Big(i\tilde{p}_2\eta'^2\Big) \tag{3.125}$$

($\tilde{p}_{1,2} \equiv \sqrt{b(p^2+k^2_{1,2})/|a|}$), with the discrete energy spectrum as determined by (3.115) and the normalization constant N_{ln} resulting from the residuum in (3.124).

3.4. The Superintegrable Potential V_4 on $D_{\rm II}$. We consider the potential V_4 in the respective coordinate systems

$$\begin{aligned} V_4(u,v) &= \frac{u^2}{bu^2-a}\frac{\hbar^2}{2m}v_0^2 = && (3.126)\\ &= \left(\frac{b\,e^{2\tau_2}}{\cosh^2\tau_1}-a\right)^{-1}\frac{\hbar^2 v_0^2}{2m}\frac{e^{2\tau_2}}{\cosh^2\tau_1} = && (3.127)\\ &= \left(\frac{b\xi^2\eta^2-a}{\xi^2\eta^2}\right)^{-1}\frac{1}{\xi^2+\eta^2}\frac{\hbar^2 v_0^2}{2m}(\xi^2+\eta^2) = && (3.128)\\ &= \left(\frac{bd^2\cosh^2\omega\cos^2\varphi-a}{\cosh^2\omega\cos^2\varphi}\right)^{-1}\times \\ &\times\ \frac{1}{\cosh^2\omega-\cos^2\varphi}\frac{\hbar^2 v_0^2}{2m}(\cosh^2\omega-\cos^2\varphi). && (3.129)\end{aligned}$$

We have displayed the potential in a somewhat more complicated way to demonstrate the effect of the separation procedures. The quantity v_0 enters the formulas in a way that only the respective quantum numbers are altered. We will not go into details, and consider the potential V_4 only in the (u,v) system. For the remaining systems we refer to [15]. Let us note that the separability of V_4 in all the four coordinate systems on $D_{\rm II}$ shows that a quantum system of a three-dimensional analogue of $D_{\rm II}$ is also separable in three-dimensional «cylindrical» versions of the (u,v) system, spherical, parabolic, and elliptic coordinates [16]. The additional quantum number associated with the third coordinate can be identified with v_0.

Inserting V_4 into the path integral in the (u,v) systems yields

$$K^{(V_4)}(u'',u',v'',v';T) = \int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)\frac{bu^2-a}{u^2}\times$$

$$\times \exp\left\{\frac{i}{\hbar}\int_0^T\left[\frac{m}{2}\frac{bu^2-a}{u^2}(\dot{u}^2+\dot{v}^2)-\frac{u^2}{bu^2-a}\frac{\hbar^2 v_0^2}{2m}\right]dt\right\}=$$

$$=\int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int_0^\infty ds''\times$$

$$\times\int_{u(0)=u'}^{u(s'')=u''}\mathcal{D}u(s)\int_{v(0)=v'}^{v(s'')=v''}\mathcal{D}v(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}(\dot{u}^2+\dot{v}^2)-\frac{aE}{u^2}\right]ds+\right.$$

$$\left.+\frac{i}{\hbar}s''\left(bE-\frac{\hbar^2 v_0^2}{2m}\right)\right\}=$$

$$=\int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int_0^\infty\frac{ds''}{2\pi}\int_{-\infty}^{\infty}dk\;\mathrm{e}^{ik(v''-v')}\exp\left(\frac{i}{\hbar}bEs''-\frac{i}{\hbar}\frac{\hbar^2}{2m}(k^2+v_0^2)s''\right)\times$$

$$\times\int_{u(0)=u'}^{u(s'')=u''}\mathcal{D}u(s)\exp\left[\frac{i}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{u}^2-\hbar^2\frac{\lambda^2-\frac{1}{4}}{2mu^2}\right)ds\right]=$$

$$=\int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int_0^\infty ds''\int_{-\infty}^{\infty}dk\;\mathrm{e}^{ik(v''-v')}\,\frac{m\sqrt{u'u''}}{i\hbar s''}\times$$

$$\times\exp\left[\frac{i}{\hbar}\left(bE-\frac{\hbar^2}{2m}(k^2+v_0^2)\right)s''+\frac{i}{\hbar}\frac{m}{2s''}(u'^2+u''^2)\right]I_\lambda\left(\frac{mu'u''}{i\hbar s''}\right) \tag{3.130}$$

($\lambda^2-1/4=2maE/\hbar^2$). We observe that the principal effect of the introduction of V_4 consists in a change in the quantum number k which can be formulated as $\tilde{k}^2=k^2+v_0^2$. We can therefore write down the solution by referring to [15] and get

$$G^{(V_4)}(u'',u',v'',v';E)=\frac{2m\sqrt{u'u''}}{i\hbar}\int_{-\infty}^{\infty}dk\;\mathrm{e}^{ik(v''-v')}\,I_\lambda\times$$

$$\times\left(\sqrt{\tilde{k}^2-\frac{2mbE}{\hbar^2}}\,u_<\right)K_\lambda\left(\sqrt{\tilde{k}^2-\frac{2mbE}{\hbar^2}}\,u_>\right)= \tag{3.131}$$

$$= \frac{\hbar}{\pi^2} \int\limits_{-\infty}^{\infty} dk \, \frac{e^{ik(v''-v')}}{2\pi} \times$$

$$\times \int\limits_0^{\infty} \frac{2p \sinh \pi p dp}{\dfrac{\hbar^2}{2m|a|}\left(p^2 + \dfrac{1}{4}\right) - E} K_{ip}\left(\sqrt{\tilde{k}^2 - \frac{2mbE}{\hbar^2}}\, u'\right) K_{ip}\left(\sqrt{\tilde{k}^2 - \frac{2mbE}{\hbar^2}}\, u''\right), \tag{3.132}$$

with

$$\lambda = \sqrt{\frac{1}{4} - \frac{2m|a|E}{\hbar^2}} \equiv ip. \tag{3.133}$$

The wave functions and the energy spectrum are read off:

$$\Psi_{pk}^{(V_4)}(u, v) = \frac{e^{ikv}}{\sqrt{2\pi}} \frac{\sqrt{2p \sinh \pi p}}{\pi} K_{ip}\left(\sqrt{\tilde{k}^2 - \frac{2mbE}{\hbar^2}}\, u\right), \tag{3.134}$$

$$E = \frac{\hbar^2}{2m|a|}\left(p^2 + \frac{1}{4}\right). \tag{3.135}$$

4. SUMMARY AND DISCUSSION

In this paper we have discussed superintegrable potentials on spaces of non-constant curvature. The results are very satisfactory. According to [4, 27, 28], there are three potentials on $D_{\rm I}$, four potentials on $D_{\rm II}$, five potentials on $D_{\rm III}$, and four potentials on $D_{\rm IV}$, respectively. We could solve many of the emerging quantum mechanical problems. To give an overview, we summarize our results in Table 5. We list for each space the corresponding potentials including the general form of the solution (if explicitly possible). We omit the trivial potentials here, because they are separable in all corresponding coordinate systems.

We were able to solve the various path-integral representations, because we have now to our disposal not only the basic path integrals for the harmonic oscillator, the linear oscillator, the radial harmonic oscillator, and the Pöschl–Teller potential, but also path-integral identities derived from path integration on harmonic spaces like the elliptic and spheroidal path-integral representations with their more complicated special functions [13, 17, 22]. This includes also numerous transformation techniques to find a particular solution based on one of the basic solutions. Various analysis techniques can be applied to find not only an expression for the Green function but also for the wave functions and the energy spectrum.

We also observe a new feature of superintegrable potentials. We learned from our investigation of potential problems on $D_{\rm I}$ that degeneracy for superintegrable

Table 5. **Solutions of the path integration for superintegrable potentials in Darboux spaces**

Space and potential	Solution in terms of the wave functions
$D_{\rm I}$	
V_1: (u,v)	Hermite polynomials × Parabolic cylinder functions
Parabolic	No explicit solution
V_2: (u,v)	Hermite polynomials × Parabolic cylinder functions
(r,q)	Hermite polynomials × Parabolic cylinder functions
V_3: (u,v)	Product of Airy functions
(r,q)	Product of Airy functions
Parabolic	Product of Airy functions
$D_{\rm II}$	
V_1: (u,v)	Hermite polynomial × Whittaker functions*
Parabolic	No explicit solution
V_2: (u,v)	Laguerre polynomial × Whittaker functions*
Polar	Gegenbauer polynomial × Whittaker functions*
Elliptic	No explicit solution
V_3: Polar	Gegenbauer polynomial × Whittaker functions*
Parabolic	Gegenbauer polynomial × Whittaker functions*
Displaced Elliptic	No explicit solution
V_4: (u,v)	Product of Bessel functions
Polar	Bessel functions × Legendre functions
Parabolic	Product of Whittaker functions*
Elliptic	Spheroidal wave functions

* The notion Whittaker functions means in all cases for a discrete spectrum Laguerre polynomials and for a continuous spectrum Whittaker functions $W_{\mu,\nu}(z)$, respectively $M_{\mu,\nu}(z)$.

potentials does not follow automatically. In fact, our (counter-)examples show that the usually accepted opinion that superintegrability and degeneracy of a quantum system are equivalent statements is not true in general. It would be interesting to formulate the precise additional mathematical requirements that these statements are actually true in general. In our case the nonequivalence of these two notions comes from the boundary conditions which had to be imposed on $D_{\rm I}$ in order to guarantee a well-defined Hilbert space.

We found in all cases a discrete and a continuous spectrum for the superintegrable potentials. We also could compare some limiting cases, e.g., for the Darboux space $D_{\rm II}$, where we could recover the corresponding solutions for the two-dimensional Euclidean space and the two-dimensional hyperboloid. On $D_{\rm I}$ the energy spectra are only determined by a transcendental equation due to the boundary condition for the coordinate u. On $D_{\rm II}$ we found analogues of the singular oscillator, the Holt potential and the Coulomb potential in two-dimensional Euclidean space. We could recover these limiting cases in the equations for the energy spectra. The equations for the energy spectra were on $D_{\rm II}$ algebraic equations in the second and fourth orders in the energy. This allows several solutions depending on the specific values of the parameters a and b and possible further boundary conditions. Also semibound states may be possible.

In the forthcoming publications we will treat the two remaining Darboux spaces $D_{\rm III}$ and $D_{\rm IV}$, respectively. In particular, on $D_{\rm III}$ there is already a discrete spectrum possible for the free motion, which has the form

$$E_{nl} = -\frac{\hbar^2}{2m}\frac{b}{a^2}(2n+2l+1)^2 \tag{4.1}$$

yielding for $b > 0$ an infinite number of bound states. This is similar to the motion on the $SU(1,1)$ hyperboloid, where continuous and discrete spectra exist [2]. On $D_{\rm III}$ there are five superintegrable potentials and on $D_{\rm IV}$ there are four superintegrable potentials.

Let us finally discuss the following issue: Let us consider a three-dimensional generalization of the Darboux space $D_{\rm II}$ with a line element

$$ds^2 = \frac{bu^2 - a}{u^2}(du^2 + dv^2 + dw^2), \tag{4.2}$$

and w is the new variable. $D_{\rm II}$ has the property that for $a = 0$, $b = 1$, we recover the two-dimensional Euclidean plane, and all four coordinate systems on the two-dimensional Euclidean plane are also separable coordinate systems on $D_{\rm II}$ for the Schrödinger, respectively the Helmholtz equation. However, in order to set up a well-defined quantum theory, a curvature term $(\hbar^2/2m)(R/8)$ must be introduced in the quantum Hamiltonian [16, 26]. In the present case of (4.2), which we might call three-dimensional Darboux space II, for short $D_{3d-{\rm II}}$, it is easy to check that all eleven systems for the three-dimensional Euclidean plane which separate the Schrödinger, respectively the Helmholtz equation, also separate the Schrödinger, respectively the Helmholtz equation on $D_{3d-{\rm II}}$. As has been shown in [16], the corresponding quantum motion can be explicitly evaluated by means of the path integral with the energy spectrum

$$E = \frac{\hbar^2}{2m|a|}(p^2 + 1). \tag{4.3}$$

As is well known, there are four minimally superintegrable potentials in three-dimensional Euclidean space and five maximally superintegrable potentials, and it is obvious how to construct maximally superintegrable potentials on $D_{3d-{\rm II}}$. In the forthcoming publication the details will be worked out.

Acknowledgements. This work was supported by the Heisenberg–Landau programme.

The authors are grateful to Ernie Kalnins for fruitful and pleasant discussions on superintegrability and separating coordinate systems. C. Grosche would like to thank the organizers of the XII International Conference on Symmetry Methods in Physics, Yerevan, Armenia, July 3–8, for the warm hospitality during the stay in Yerevan.

G. S. Pogosyan acknowledges the support of the Dirección General de Asuntos del Personal Académico, Universidad Nacional Autónoma de México (DGAPA–UNAM) by the grant 102603 *Optica Matemática*, SEP-CONACYT project 44845 and *PROMEP* 103.5/05/1705.

REFERENCES

1. *Pocketbook of Mathematical Functions* / Ed. Abramowitz M., Stegun I. A. Frankfurt/Main: Harry Deutsch, 1984.
2. *Böhm M., Junker G.* Path Integration Over Compact and Noncompact Rotation Groups // J. Math. Phys. 1987. V. 28. P. 1978–1994.
3. *Boyer C. P., Kalnins E. G., Winternitz P.* Completely Integrable Relativistic Hamiltonian Systems and Separation of Variables in Hermitian Hyperbolic Spaces // J. Math. Phys. 1983. V. 24. P. 2022–2034.
4. *Daskaloyannis C., Ypsilantis K.* Unified Treatment and Classification of Superintegrable Systems with Integrals Quadratic in Momenta on a Two-Dimensional Manifold // J. Math. Phys. 2006. V. 45. P. 042904.
5. *Duru I. H.* Path Integrals Over $SU(2)$ Manifold and Related Potentials // Phys. Rev. D. 1984. V. 30. P. 2121–2127.
6. *Evans N. W.* Superintegrability in Classical Mechanics // Phys. Rev. A. 1990. V. 41. P. 5666;
Evans N. W. Group Theory of the Smorodinsky–Winternitz System // J. Math. Phys. V. 32. 1991. P. 3369;
Evans N. W. Superintegrability of the Winternitz System // Phys. Lett. A. 1990. V. 147. P. 483.
7. *Feynman R. P., Hibbs A.* Quantum Mechanics and Path Integrals. N. Y.: McGraw Hill, 1965.
8. *Fischer W., Leschke H., Müller P.* Path Integration in Quantum Physics by Changing the Drift of the Underlying Diffusion Process: Application of Legendre Processes // Ann. Phys. (N. Y.). 1993. V. 227. P. 206–221.
9. *Friš J. et al.* On Higher Symmetries in Quantum Mechanics // Phys. Lett. 1965. V. 16. P. 354;
Friš J. et al. Symmetry Groups in Classical and Quantum Mechanics // Sov. J. Nucl. Phys. 1967. V. 4. P. 444.
10. *Gradshteyn I. S., Ryzhik I. M.* Table of Integrals, Series, and Products. N. Y.: Acad. Press, 1980.
11. *Grosche C.* The Path Integral on the Poincaré Upper Half-Plane with a Magnetic Field and for the Morse Potential // Ann. Phys. (N. Y.). 1988. V. 187. P. 110–134.
12. *Grosche C.* Path Integration via Summation of Perturbation Expansions and Application to Totally Reflecting Boundaries and Potential Steps // Phys. Rev. Lett. 1993. V. 71. P. 1–4.
13. *Grosche C.* Path Integrals, Hyperbolic Spaces, and Selberg Trace Formulae. Singapore: World Sci., 1996.
14. *Grosche C.* Path Integration on Hermitian Hyperbolic Space // J. Phys. A: Math. Gen. 2005. V. 38. P. 3625–3650.
15. *Grosche C.* Path Integration on Darboux Spaces // Part. Nucl. 2006. V. 37. P. 368–389.

16. *Grosche C.* Path Integral Approach for Spaces of Non-Constant Curvature in Three Dimensions. DESY Report, DESY 05–221. 2005; quant-ph/0511135; Proc. of the II Intern. Workshop on Superintegrable Systems in Classical and Quantum Mechanics, Dubna, June 27–July 1, 2005 (in press); Phys. At. Nucl. 2006. V. 69.

17. *Grosche C. et al.* Quantum Motion on the Three-Dimensional Sphere: The Ellipso-Cylindrical Bases // J. Phys. A: Math. Gen. 1997. V. 30. P. 1629–1657.

18. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Discussion for Smorodinsky–Winternitz Potentials. I. Two- and Three-Dimensional Euclidean Space // Fortschr. Phys. 1995. Bd. 43. S. 453–521.

19. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Discussion for Smorodinsky–Winternitz Potentials. II. The Two- and Three-Dimensional Sphere // Fortschr. Phys. 1995. Bd. 43. S. 523–563.

20. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Approach to Superintegrable Potentials on the Two-Dimensional Hyperboloid // Part. Nucl. 1996. V. 27. P. 244–278.

21. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Approach for Superintegrable Potentials on the Three-Dimensional Hyperboloid // Part. Nucl. 1997. V. 28. P. 486–519.

22. *Grosche C., Steiner F.* Handbook of Feynman Path Integrals // Springer Tracts in Mod. Phys. Berlin; Heidelberg, 1998. V. 145.

23. *Izmest'ev A. A. et al.* Contractions of Lie Algebras and Separation of Variables. The n-Dimensional Sphere // J. Math. Phys. 1999. V. 40. P. 1549–1573;
Izmest'ev A. A. et al. Contractions of Lie Algebras and Separation of Variables. Interbases Expansions // J. Phys. A: Math. Gen. 2001. V. 34. P. 521–554.

24. *Kalnins E. G.* Separation of Variables for Riemannian Spaces of Constant Curvature. Essex: Longman Scientific & Technical, 1986.

25. *Kalnins E. G. et al.* Completeness of Superintegrability in Two-Dimensional Constant-Curvature Spaces // J. Phys. A: Math. Gen. 2001. V. 34. P. 4705–4720.

26. *Kalnins E. G., Krees J. R., Miller W. (Jr).* Second Order Superintegrable Systems in Conformally Flat Spaces // Quant. Syst. 2006. V. 2D.

27. *Kalnins E. G. et al.* Superintegrable Systems in Darboux Spaces // J. Math. Phys. 2003. V. 44. P. 5811–5848.

28. *Kalnins E. G., Kress J. M., Winternitz P.* Superintegrability in a Two-Dimensional Space of Nonconstant Curvature // J. Math. Phys. 2002. V. 43. P. 970–983.

29. *Kalnins E. G. et al.* Superintegrability on the Two-Dimensional Hyperboloid II // J. Math. Phys. 1999. V. 40. P. 2291–2306.

30. *Kalnins E. G., Miller W. (Jr)., Pogosyan G. S.* Superintegrability and Associated Polynomial Solutions: Euclidean Space and the Sphere in Two Dimensions // J. Math. Phys. 1996. V. 37. P. 6439–6467.

31. *Kalnins E. G., Miller W. (Jr)., Pogosyan G. S.* Superintegrability on the Two-Dimensional Hyperboloid // J. Math. Phys. 1997. V. 38. P. 5416–5433.

32. *Kalnins E. G., Miller W. (Jr)., Pogosyan G. S.* Contractions of Lie Algebras: Applications to Special Functions and Separation of Variables // J. Phys. A: Math. Gen. 1999. V. 32. P. 4709-4732.

33. *Kalnins E. G., Miller W. (Jr)., Pogosyan G. S.* Completeness of Multiseparable Superintegrability on the Complex 2-Sphere // J. Phys. A: Math. Gen. 2000. V. 33. P. 6791–6806.

34. *Kalnins E. G., Miller W. (Jr)., Pogosyan G. S.* Completeness of Multiseparable Superintegrability in $E_{2,C}$ // Ibid. P. 4105–4120.

35. *Kalnins E. G., Miller W. (Jr)., Pogosyan G. S.* Exact and Quasi-Exact Solvability of Second-Order Superintegrable Quantum Systems. I. Euclidean Space Preliminaries // J. Math. Phys. 2006. V. 47. P. 033502.

36. *Kalnins E. G. et al.* Superintegrability in Three-Dimensional Euclidean Space // J. Math. Phys. 1999. V. 40. P. 708–725.

37. *Khachidze T. T., Khelashvili A. A.* Manifestations of the Hidden Symmetry of the Coulomb Problem in Relativistic Quantum Mechanics — from Pauli to Dirac Electron. quant-ph/0507257.

38. *Kleinert H.* Path Integrals in Quantum Mechanics, Statistics and Polymer Physics. Singapore: World Sci., 1990.

39. *Kleinert H., Mustapic I.* Summing the Spectral Representations of Pöschl–Teller and Rosen–Morse Fixed-Energy Amplitudes // J. Math. Phys. 1992. V. 33. P. 643–662.

40. *Koenigs G.* Sur les géodésiques a intégrales quadratiques // Lecons sur la théorie générale des surface. Chelsea Publishing, 1972. V. 4. P. 368–404.

41. *Létourneau P., Vinet L.* Superintegrable Systems: Polynomial Algebras and Quasi-Exactly Solvable Hamiltonians // Ann. Phys. (N. Y.). 1995. V. 243. P. 144–168.

42. *Mardoyan L. G. et al.* Quantum Systems with Hidden Symmetry. Interbasis Expansions. Fizmatlit, 2006 (in Russian).

43. *Meixner J., Schäfke F. W.* Mathieusche Funktionen und Sphäroidfunktionen. Berlin: Springer, 1954.

44. *Miller W. (Jr.)* Symmetry and Separation of Variables. Providence, Rhode Island: Addison-Wesley, 1977.

45. *Moon F., Spencer D.* Field Theory Handbook. Berlin: Springer, 1961.

46. *Morse P. M., Feshbach H.* Methods of Theoretical Physics. N. Y.: McGraw-Hill, 1953.

47. *Olevskiĭ M. N.* Triorthogonal Systems in Spaces of Constant Curvature in Which the Equation $\Delta_2 u + \lambda u = 0$ Allows the Complete Separation of Variables // Math. Sb. 1950. V. 27. P. 379–426 (in Russian).

48. *Patera J., Pogosyan G., Winternitz P.* Graded Contractions of the Lie Algebra $e(2,1)$ // J. Phys. A: Math. Gen. 1999. V. 32. P. 805–826.

49. *Schulman L. S.* Techniques and Applications of Path Integration. N. Y.: John Wiley & Sons, 1981.

50. *Turbiner A. V.* Quasi-Exactly Solvable Problems and $\mathfrak{gsl}(2)$ Algebra // Commun. Math. Phys. 1988. V. 118. P. 467–474.

51. *Ushveridze A.* Quasi-Exactly Solvable Models in Quantum Mechanics. Bristol: Inst. of Phys. Publ., 1994.

52. *Winternitz P. et al.* Symmetry Groups in Classical and Quantum Mechanics // Sov. J. Nucl. Phys. 1967. V. 4. P. 444–450.

53. *Wojciechowski S.* Superintegrability of the Calogero–Moser System // Phys. Lett. A. 1983. V. 95. P. 279–281.