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Three methods to describe the collective motion — Random Phase Approximation (RPA), Wigner Function Moments (WFM) and Green's Function (GF) method are compared in detail, and their physical content is analyzed with an example of a simple model — the harmonic oscillator with quadrupole–quadrupole residual interaction. It is shown that they give identical formulae for eigenfrequencies and transition probabilities of all collective excitations of the model. The exact relation between the RPA and WFM variables and the respective dynamical equations is established. The transformation of the RPA spectrum into the one of WFM is explained. The very close connection of the WFM method with the GF one is demonstrated. A differential equation describing the current lines of RPA modes is established and the current lines of the scissors mode are analyzed as a superposition of rotational and irrotational components. The orthogonality of the spurious state to all physical states is proved rigorously.

Сравниваются три подхода к описанию коллективного движения: приближение случайных фаз (ПСФ), метод моментов функции Вигнера (МФВ) и метод функций Грина (ФГ). Их физическое содержание анализируется на примере простой модели — гармонического осциллятора с квадруполь-квадрупольным остаточным взаимодействием. Показано, что они дают одинаковые формулы для собственных частот и вероятностей переходов всех коллективных возбуждений модели. Установлено точное соотношение между переменными ПСФ и метода МФВ и соответствующими динамическими уравнениями. Объяснено преобразование спектра приближения случайных фаз в спектр метода моментов функции Вигнера. Продемонстрировано близкое родство МФВ- и ФГ-методов. Выведено дифференциальное уравнение, описывающее линии токов в ПСФ и методе функций Грина. Линии токов ножничной моды проанализированы как суперпозиция ротационной и ирротационной компонент. Дано строгое доказательство ортогональности духового состояния всем физическим состояниям модели.

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INTRODUCTION

The aim of the present paper is the systematic comparison of three methods to describe the collective motion. As an example, their competition in the description of the nuclear scissors mode will be considered. This very curious excitation was predicted thirty years ago [1, 2]. Its experimental discovery [3] has initiated a cascade of theoretical studies. An excellent review of their twenty years development was given by D. Zawischa [4]. Very briefly the situation can be described in the following way. All microscopic calculations with effective forces reproduce experimental data with respect to the position and the strength

of the scissors mode, some of them [5] giving also reasonable fragmentation of its strength. However, the situation is more obscure in regard to simple phenomenological models whose aim is to explain the physics of the phenomenon and to interpret it in the most simple and transparent terms. A noticeable discord of the opinions of various authors must be observed here [4]. So, it will be interesting to compare the possibilities, advantages, and disadvantages of various methods in the description of all subtleties of this mode.

The full analysis of the scissors mode in the framework of a solvable model (harmonic oscillator with quadrupole–quadrupole residual interaction (HO + QQ)) was given in [6]. Several points in the understanding of the nature of this mode were clarified: for example, its coexistence with the isovector giant quadrupole resonance (IVGQR), the decisive role of the Fermi surface deformation, and several things more. The Wigner Function Moments (WFM) method was applied to derive analytical expressions for currents of both coexisting modes, their excitation energies, magnetic and electric transition probabilities. Our formulae for energies turned out to be identical with those derived by Hamamoto and Nazarewicz [7] in the framework of the RPA. In [8], we investigated the relation between formulas for transition probabilities derived by two methods. It was shown there that also these formulas are identical. This coincidence motivated us to undertake a detailed comparison of the two approaches and understand the connection and differences between them. One of the important subjects of this comparison is the current distributions. The WFM method, a priori, cannot give the exact results, because it deals only with integrals over the whole phase space. It would therefore be very interesting to evaluate the accuracy of this approximation by comparing the results with the currents obtained from the RPA. Unfortunately, even for this simple model (HO + QQ) it is impossible to derive in the RPA the closed analytical expressions for currents of the scissors mode and IVGQR. That is why we consider in addition Green's Function (GF) method which allows one to find explicit expressions for currents directly.

The HO + QQ model is a very convenient ground for this kind of investigation, because most of the results can be obtained analytically. The basis of all three methods is the same: Time Dependent Hartree–Fock (TDHF) theory in its small amplitude approximation. Strictly speaking, the small amplitude approximation is not compulsory in the WFM method — it allows one to study the large amplitude motion, too. There is no need to describe the merits of the RPA or of the GF method — they are very well known [9]. It is necessary, however, to say a few words about the WFM. Its idea is based on the virial theorems of Chandrasekhar and Lebovitz. These theorems were derived by the authors in fifties in a series of papers, the results of which were summarized in the book [10]. The old astrophysical problems were considered: figures of equilibrium of rotating self-gravitating masses (planets and stars) and their vibration eigenfrequencies. In the classical mechanics the dynamics of such objects is described with the help

of the well-known equations of hydrodynamics, the Euler equation and the continuity equation, which usually leads to very complicated mathematical problems. Chandrasekhar and Lebovitz have shown that the solution of these problems can be found in an essentially simpler and elegant way if one works with moments of the Euler equation (virial theorems). In such a way they reproduced all already known results (obtained by the efforts of many famous mathematicians) and have found the solutions of several old long-standing problems of astrophysics.

In the light of these successes it is natural to expect that the method of moments will also be useful in the nuclear theory, for example, to solve TDHF equations. Really, it is known that the Wigner transform of the TDHF equation for the density matrix is similar to the dynamical equation for the distribution function of the classical kinetic theory (Vlasov equation). In particular, first two moments (in momentum space) of this equation are just the continuity equation and the Euler equation. So, at a glance there is no problem in employing the virial theorems of Chandrasekhar and Lebovitz in the nuclear theory. However, the real situation turns out to be a little bit more complicated, because all moments (in momentum space) of the Vlasov equation are coupled in an infinite set of dynamical equations and, consequently, the problem of their decoupling arises. The way to the solution of this problem with the help of virial theorems was proposed in [11] and described in detail in [12]. Instead of writing the equations of motion for microscopic amplitudes of particle hole excitations (RPA), one writes the dynamical equations for various multipole phase space moments of a nucleus. This allows one to achieve a more direct physical interpretation of the studied phenomenon without going into its detailed microscopic structure and, what is even more important, solves the problem of decoupling. The obtained equations are nonlinear, as it should be, when one deals with the Hartree–Fock theory. In the case of a sufficiently simple interaction they can be solved without a standard linearization procedure, i.e., for large amplitudes, which has been done in [13], where the multiphonon giant resonances were studied. So in this sense the WFM method is more general than the RPA one. In the approximation of small amplitudes, the WFM method was successfully applied to study isoscalar and isovector giant multipole resonances and low-lying collective modes of rotating and nonrotating nuclei with various realistic forces [14–17]. The results of WFM were always very close to similar results obtained with the help of the RPA, which allowed one to suspect the intimate relationship between both the methods. The detailed analysis of the interplay of the two methods turns out to be useful also from a «practical» point of view: first, and most importantly, it allows one to obtain additional insight into the nature of the scissors mode; second, we find new exact mathematical results for the considered model.

The paper is organized as follows. In Sec. 1, we recall the principal points of the WFM formalism and give a summary of the key results of [6] obtained by applying this method to the $HO + QQ$ model. The same model is considered in

Sec. 2 in the frame of the RPA: the formulae for eigenfrequencies, electric and magnetic transition probabilities of the scissors mode are derived, the equations of motion for Transition Matrix Elements (TME) are constructed and the «synthetic» scissors and spurious state are analyzed. The exact interrelation between the RPA and WFM methods and between their variables is established in Sec. 3, where TME equations are derived by the WFM method. Section 4 is devoted to the GF method. The three methods are applied to derive analytical formulae for lines of currents in Sec. 5. The mutual interplay of the three methods is discussed in Conclusion. Various mathematical details are given in Appendices.

1. THE WFM METHOD

The basis of the method is the TDHF equation for the one-body density matrix $\rho^\tau(\mathbf{r}_1, \mathbf{r}_2, t) = \langle \mathbf{r}_1 | \hat{\rho}^\tau(t) | \mathbf{r}_2 \rangle$:

$$i\hbar \frac{\partial \hat{\rho}^\tau}{\partial t} = \left[\hat{H}^\tau, \hat{\rho}^\tau \right], \tag{1}$$

where $\hat{H}^\tau$ is the one-body self-consistent mean field Hamiltonian depending implicitly on the density matrix and τ is an isotopic spin index. It is convenient to modify equation (1) introducing the Wigner transform of the density matrix

$$f^\tau(\mathbf{r}, \mathbf{p}, t) = \int d^3s \, \exp(-i\mathbf{p} \cdot \mathbf{s}/\hbar) \rho^\tau \left(\mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{r} - \frac{\mathbf{s}}{2}, t \right) \tag{2}$$

and of the Hamiltonian

$$H_W^\tau(\mathbf{r}, \mathbf{p}) = \int d^3s \, \exp\left(-i\mathbf{p} \cdot \mathbf{s}/\hbar\right) \left(\mathbf{r} + \frac{\mathbf{s}}{2} \middle| \hat{H}^\tau \middle| \mathbf{r} - \frac{\mathbf{s}}{2} \right). \tag{3}$$

Using (2), (3) one arrives [9] at

$$\frac{\partial f^\tau}{\partial t} = \frac{2}{\hbar} \sin \left\{ \frac{\hbar}{2} \left[(\nabla)^H \cdot (\nabla^p)^f - (\nabla^p)^H \cdot (\nabla)^f \right] \right\} H_W^\tau f^\tau, \tag{4}$$

where the upper index on the bracket stands for the function on which the operator in these brackets acts. It is shown in [13,17] that by integrating equation (4) over the phase space $\{\mathbf{p}, \mathbf{r}\}$ with the weights $x_{i_1} x_{i_2} \cdots x_{i_k} p_{i_{k+1}} \cdots p_{i_{n-1}} p_{i_n}$, where k runs from 0 to n, one can obtain a closed finite set of dynamical equations for Cartesian tensors of the rank n. Taking linear combinations of these equations one is able to represent them through irreducible tensors, which play the role of collective variables of the problem. However, it is more convenient to derive the dynamical equations directly for irreducible tensors using the technique of tensor

products [18]. For this it is necessary to rewrite the Wigner function equation (4) in terms of cyclic variables

$$\frac{\partial f^\tau}{\partial t} = \\ = \frac{2}{\hbar}\sin\left\{\frac{\hbar}{2}\sum_{\alpha=-1}^{1}(-1)^\alpha\left[(\nabla_{-\alpha})^H\cdot(\nabla^p_{\alpha})^f-(\nabla^p_{-\alpha})^H\cdot(\nabla_{\alpha})^f\right]\right\}H^\tau_W f^\tau, \quad (5)$$

with

$$\nabla_{+1}=-\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right), \quad \nabla_0=\frac{\partial}{\partial x_3}, \quad \nabla_{-1}=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\right),$$

$$r_{+1}=-\frac{1}{\sqrt{2}}(x_1+ix_2), \quad r_0=x_3, \quad r_{-1}=\frac{1}{\sqrt{2}}(x_1-ix_2)$$

and the analogous definitions for ∇^p_{+1}, ∇^p_0, ∇^p_{-1}, and p_{+1}, p_0, p_{-1}. The required equations are obtained by integrating (5) with different tensor products of r_α and p_α. Here we consider the case $n=2$.

1.1. Model Hamiltonian, Equations of Motion. The microscopic Hamiltonian of the model, harmonic oscillator plus separable quadrupole–quadrupole residual interaction is given by

$$H=\sum_{i=1}^{A}\left(\frac{\hat{\mathbf{p}}_i^2}{2m}+\frac{1}{2}m\omega^2\mathbf{r}_i^2\right)+\bar{\kappa}\sum_{\mu=-2}^{2}(-1)^\mu\sum_{i}^{Z}\sum_{j}^{N}q_{2-\mu}(\mathbf{r}_i)q_{2\mu}(\mathbf{r}_j)+ \\ +\frac{1}{2}\kappa\sum_{\mu=-2}^{2}(-1)^\mu\left\{\sum_{i\neq j}^{Z}q_{2-\mu}(\mathbf{r}_i)q_{2\mu}(\mathbf{r}_j)+\sum_{i\neq j}^{N}q_{2-\mu}(\mathbf{r}_i)q_{2\mu}(\mathbf{r}_j)\right\}, \quad (6)$$

where the quadrupole operator $q_{2\mu}=\sqrt{16\pi/5}\,r^2Y_{2\mu}$ and N, Z are the numbers of neutrons and protons, respectively. The mean field potential for protons (or neutrons) is

$$V^\tau(\mathbf{r},t)=\frac{1}{2}m\,\omega^2r^2+\sum_{\mu=-2}^{2}(-1)^\mu\tilde{Z}^\tau_{2-\mu}(t)q_{2\mu}(\mathbf{r}), \quad (7)$$

where $\tilde{Z}^n_{2\mu}=\kappa Q^n_{2\mu}+\bar{\kappa}Q^p_{2\mu}$, $\tilde{Z}^p_{2\mu}=\kappa Q^p_{2\mu}+\bar{\kappa}Q^n_{2\mu}$ and the quadrupole moments $Q^\tau_{2\mu}(t)$ are defined as

$$Q^\tau_{2\mu}(t)=\int d\{\mathbf{p},\mathbf{r}\}q_{2\mu}(\mathbf{r})f^\tau(\mathbf{r},\mathbf{p},t) \quad (8)$$

with $\int d\{\mathbf{p},\mathbf{r}\} \equiv 2(2\pi\hbar)^{-3}\int d^3p \int d^3r$, where the factor 2 appears due to summation over spin degrees of freedom. To simplify notation, we omit spin indices, because we consider spin saturated system without the spin-orbit interaction.

Substituting spherical functions by tensor products $r^2Y_{2\mu} = \sqrt{\frac{15}{8\pi}}r^2_{2\mu}$, where

$$r^2_{\lambda\mu} \equiv \{r\otimes r\}_{\lambda\mu} = \sum_{\sigma,\nu} C^{\lambda\mu}_{1\sigma,1\nu} r_\sigma r_\nu$$

and $C^{\lambda\mu}_{1\sigma,1\nu}$ is the Clebsch–Gordan coefficient, one has

$$V^\tau = \frac{1}{2}m\,\omega^2 r^2 + \sum_\mu (-1)^\mu Z^\tau_{2-\mu} r^2_{2\mu}. \tag{9}$$

Here

$$\begin{gathered} Z^n_{2\mu} = \chi R^n_{2\mu} + \bar\chi R^p_{2\mu}, \quad Z^p_{2\mu} = \chi R^p_{2\mu} + \bar\chi R^n_{2\mu}, \quad \chi = 6\kappa, \quad \bar\chi = 6\bar\kappa, \\ R^\tau_{\lambda\mu}(t) = \int d\{\mathbf{p},\mathbf{r}\} r^2_{\lambda\mu} f^\tau(\mathbf{r},\mathbf{p},t). \end{gathered} \tag{10}$$

Integration of equation (5) with the weights $r^2_{\lambda\mu}$, $(rp)_{\lambda\mu} \equiv \{r\otimes p\}_{\lambda\mu}$, and $p^2_{\lambda\mu}$ yields the following set of equations [6]:

$$\begin{aligned} &\frac{d}{dt}R^\tau_{\lambda\mu} - \frac{2}{m}L^\tau_{\lambda\mu} = 0, \quad \lambda = 0,2, \\ &\frac{d}{dt}L^\tau_{\lambda\mu} - \frac{1}{m}P^\tau_{\lambda\mu} + m\,\omega^2 R^\tau_{\lambda\mu} - \\ &\qquad - 2\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\left\{{2\ 1\ 1\atop \lambda\ 1\ j}\right\}(Z^\tau_2 R^\tau_j)_{\lambda\mu} = 0, \quad \lambda = 0,1,2, \\ &\frac{d}{dt}P^\tau_{\lambda\mu} + 2m\,\omega^2 L^\tau_{\lambda\mu} - 4\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\left\{{2\ 1\ 1\atop \lambda\ 1\ j}\right\}(Z^\tau_2 L^\tau_j)_{\lambda\mu} = 0, \quad \lambda = 0,2, \end{aligned} \tag{11}$$

where $\left\{{2\ 1\ 1\atop \lambda\ 1\ j}\right\}$ is the Wigner $6j$-symbol. For the sake of simplicity, the time dependence of tensors is not written out. Further the following notation is introduced:

$$P^\tau_{\lambda\mu}(t) = \int d\{\mathbf{p},\mathbf{r}\} p^2_{\lambda\mu} f^\tau(\mathbf{r},\mathbf{p},t), \quad L^\tau_{\lambda\mu}(t) = \int d\{\mathbf{p},\mathbf{r}\}(rp)_{\lambda\mu} f^\tau(\mathbf{r},\mathbf{p},t). \tag{12}$$

It is necessary to say some words about the physical meaning of the collective variables introduced above. By definition $R^\tau_{2\mu} = Q^\tau_{2\mu}/\sqrt{6}$ and $Q^\tau_{2\mu}$ is the quadrupole moment of the system of particles and $R^\tau_{00} = -Q^\tau_{00}/\sqrt{3}$ with $Q^\tau_{00} = N^\tau\langle r^2\rangle$

being the mean square radius of the same system. By analogy with these variables, defined in the coordinate space, we can say that the variables $P^\tau_{2\mu}$ and P^τ_{00} describe the quadrupole moment and the mean square radius of the same system in a momentum space. The variables $L^\tau_{\lambda\mu}$ describe the coupling of momentum and coordinate spaces. To understand their nature it is useful to recall the definitions [13, 17] of nuclear density and mean velocity:

$$\begin{aligned} n^\tau(\mathbf{r},t) &= \int \frac{2d^3p}{(2\pi\hbar)^3}\, f^\tau(\mathbf{r},\mathbf{p},t), \\ mn^\tau(\mathbf{r},t)u^\tau_i(\mathbf{r},t) &= \int \frac{2d^3p}{(2\pi\hbar)^3}\, p_i f^\tau(\mathbf{r},\mathbf{p},t). \end{aligned} \tag{13}$$

They enter into the definitions (10), (12) of irreducible tensors

$$\begin{aligned} R^\tau_{\lambda\mu}(t) &= \int d^3r \int \frac{2d^3p}{(2\pi\hbar)^3}\, r^2_{\lambda\mu} f^\tau(\mathbf{r},\mathbf{p},t) = \int d^3r\, r^2_{\lambda\mu} n^\tau(\mathbf{r},t), \\ L^\tau_{\lambda\mu}(t) &= \int d^3r \int \frac{2d^3p}{(2\pi\hbar)^3}\, (rp)_{\lambda\mu} f^\tau(\mathbf{r},\mathbf{p},t) = m\int d^3r\, (ru^\tau)_{\lambda\mu} n^\tau(\mathbf{r},t). \end{aligned} \tag{14}$$

The last expression for $L^\tau_{\lambda\mu}$ demonstrates in an obvious way the physical meaning of these variables: being the first order moments of mean velocities they give information about the distribution of these velocities in the nucleus («first» means that velocities are weighted with the coordinate $\mathbf{r}$). Sometimes, if the motion is comparatively simple, this information turns out sufficient to completely determine the velocity field (see Conclusion). In the case of more intricate motions higher order moments are required for a complete description of velocities [17]. In any case the moments of velocities are a very convenient tool to describe the collective motion. For example, the zero order moment of velocity is nothing more than the linear momentum describing the nucleus' center-of-mass motion. One of the first order moments corresponds to the very well known angular momentum of a nucleus. It is connected with the variable $L^\tau_{1\mu}$ by the following relations:

$$L^\tau_{10} = \frac{i}{\sqrt{2}} I^\tau_3, \quad L^\tau_{1\pm1} = \frac{1}{2}(I^\tau_2 \mp iI^\tau_1).$$

It is convenient to rewrite Eqs. (11) in terms of the isoscalar and isovector variables

$$\begin{aligned} R_{\lambda\mu} &= R^n_{\lambda\mu} + R^p_{\lambda\mu}, \quad P_{\lambda\mu} = P^n_{\lambda\mu} + P^p_{\lambda\mu}, \quad L_{\lambda\mu} = L^n_{\lambda\mu} + L^p_{\lambda\mu}, \\ \bar{R}_{\lambda\mu} &= R^n_{\lambda\mu} - R^p_{\lambda\mu}, \quad \bar{P}_{\lambda\mu} = P^n_{\lambda\mu} - P^p_{\lambda\mu}, \quad \bar{L}_{\lambda\mu} = L^n_{\lambda\mu} - L^p_{\lambda\mu}. \end{aligned}$$

So the equations for the neutron and proton systems are transformed into isoscalar and isovector ones. The equations for the isoscalar system are given by

$$\begin{gathered}
\dot{R}_{00} - 2L_{00}/m = 0, \\
\dot{L}_{00} - P_{00}/m + m\,\omega^2 R_{00} - 2\sqrt{5/3}[\chi_0(R_2R_2)_{00} + \chi_1(\bar{R}_2\bar{R}_2)_{00}] = 0, \\
\dot{P}_{00} + 2m\omega^2 L_{00} - 4\sqrt{5/3}[\chi_0(R_2L_2)_{00} + \chi_1(\bar{R}_2\bar{L}_2)_{00}] = 0, \\
\dot{R}_{2\mu} - 2L_{2\mu}/m = 0, \\
\dot{L}_{2\mu} - P_{2\mu}/m + m\,\omega^2 R_{2\mu} - 2\sqrt{1/3}[\chi_0(R_2R_0)_{2\mu} + \chi_1(\bar{R}_2\bar{R}_0)_{2\mu}] - \\
- \sqrt{7/3}[\chi_0(R_2R_2)_{2\mu} + \chi_1(\bar{R}_2\bar{R}_2)_{2\mu}] = 0, \\
\dot{P}_{2\mu} + 2m\omega^2 L_{2\mu} - 4\sqrt{1/3}[\chi_0(R_2L_0)_{2\mu} + \chi_1(\bar{R}_2\bar{L}_0)_{2\mu}] - \\
-2\sqrt{7/3}[\chi_0(R_2L_2)_{2\mu} + \chi_1(\bar{R}_2\bar{L}_2)_{2\mu}] + 2\sqrt{3}[\chi_0(R_2L_1)_{2\mu} + \chi_1(\bar{R}_2\bar{L}_1)_{2\mu}] = 0, \\
\dot{L}_{1\nu} = 0
\end{gathered} \tag{15}$$

and the ones for the isovector system read:

$$\begin{gathered}
\dot{\bar{R}}_{00} - 2\bar{L}_{00}/m = 0, \\
\dot{\bar{L}}_{00} - \bar{P}_{00}/m + m\,\omega^2 \bar{R}_{00} - 2\sqrt{5/3}\chi(R_2\bar{R}_2)_{00} = 0, \\
\dot{\bar{P}}_{00} + 2m\omega^2 \bar{L}_{00} - 4\sqrt{5/3}[\chi_0(R_2\bar{L}_2)_{00} + \chi_1(\bar{R}_2L_2)_{00}] = 0, \\
\dot{\bar{R}}_{2\mu} - 2\bar{L}_{2\mu}/m = 0, \\
\dot{\bar{L}}_{2\mu} - \bar{P}_{2\mu}/m + m\,\omega^2 \bar{R}_{2\mu} - 2\sqrt{1/3}[\chi_0(R_2\bar{R}_0)_{2\mu} + \chi_1(\bar{R}_2R_0)_{2\mu}] - \\
- \sqrt{7/3}\chi(R_2\bar{R}_2)_{2\mu} = 0, \\
\dot{\bar{P}}_{2\mu} + 2m\omega^2 \bar{L}_{2\mu} - 4\sqrt{1/3}[\chi_0(R_2\bar{L}_0)_{2\mu} + \chi_1(\bar{R}_2L_0)_{2\mu}] - \\
-2\sqrt{7/3}[\chi_0(R_2\bar{L}_2)_{2\mu} + \chi_1(\bar{R}_2L_2)_{2\mu}] + 2\sqrt{3}[\chi_0(R_2\bar{L}_1)_{2\mu} + \chi_1(\bar{R}_2L_1)_{2\mu}] = 0, \\
\dot{\bar{L}}_{1\nu} + \sqrt{5}\bar{\chi}(R_2\bar{R}_2)_{1\nu} = 0.
\end{gathered} \tag{16}$$

Here

$$\chi_0 = (\chi + \bar{\chi})/2$$

is an isoscalar strength constant and

$$\chi_1 = (\chi - \bar{\chi})/2$$

is the corresponding isovector one. The last equation of (15) demonstrates the conservation of the isoscalar angular momentum $L_{1\nu}$. The dynamical equation for the isovector angular momentum $\bar{L}_{1\nu}$ (the last equation of (16)) describes the relative (out of phase) motion of the neutron and proton angular momenta; hence it must be responsible for the scissors mode.

Writing out in detail the tensor products one can write out the whole set of 42 coupled equations (including integrals of motion) for the whole set of isoscalar and isovector variables. There is no problem to solve these equations numerically. However, we want to simplify the situation as much as possible to get the results in analytical form giving us a maximum of insight into the nature of the modes.

1) We consider the problem in small-amplitude approximation. Writing all variables as a sum of their equilibrium value plus a small deviation

$$R_{\lambda\mu}(t) = R^{\rm eq}_{\lambda\mu} + \mathcal{R}_{\lambda\mu}(t), \quad P_{\lambda\mu}(t) = P^{\rm eq}_{\lambda\mu} + \mathcal{P}_{\lambda\mu}(t), \quad L_{\lambda\mu}(t) = L^{\rm eq}_{\lambda\mu} + \mathcal{L}_{\lambda\mu}(t),$$
$$\bar{R}_{\lambda\mu}(t) = \bar{R}^{\rm eq}_{\lambda\mu} + \bar{\mathcal{R}}_{\lambda\mu}(t), \quad \bar{P}_{\lambda\mu}(t) = \bar{P}^{\rm eq}_{\lambda\mu} + \bar{\mathcal{P}}_{\lambda\mu}(t), \quad \bar{L}_{\lambda\mu}(t) = \bar{L}^{\rm eq}_{\lambda\mu} + \bar{\mathcal{L}}_{\lambda\mu}(t),$$

we linearize the equations of motion in $\mathcal{R}_{\lambda\mu}$, $\mathcal{P}_{\lambda\mu}$, $\mathcal{L}_{\lambda\mu}$ and $\bar{\mathcal{R}}_{\lambda\mu}$, $\bar{\mathcal{P}}_{\lambda\mu}$, $\bar{\mathcal{L}}_{\lambda\mu}$.

2) We study nonrotating nuclei, i.e., nuclei with $L^{\rm eq}_{1\nu} = \bar{L}^{\rm eq}_{1\nu} = 0$.

3) Only axially symmetric nuclei with $R^{\rm eq}_{2\pm2} = R^{\rm eq}_{2\pm1} = \bar{R}^{\rm eq}_{2\pm2} = \bar{R}^{\rm eq}_{2\pm1} = 0$ are considered.

4) Finally, we suppose that equilibrium deformation and mean square radius of neutrons are equal to that of protons:

$$\bar{R}^{\rm eq}_{20} = \bar{R}^{\rm eq}_{00} = 0. \tag{17}$$

Due to the approximation (17) the isoscalar and isovector sets of equations are decoupled. The isoscalar set is

$$\dot{\mathcal{R}}_{00} - 2\mathcal{L}_{00}/m = 0,$$
$$\dot{\mathcal{L}}_{00} - \mathcal{P}_{00}/m + m\omega^2\mathcal{R}_{00} - 4\sqrt{1/3}\chi_0 R^{\rm eq}_{20}\mathcal{R}_{20} = 0,$$
$$\dot{\mathcal{P}}_{00} + 2m\omega^2\mathcal{L}_{00} - 4\sqrt{1/3}\chi_0 R^{\rm eq}_{20}\mathcal{L}_{20} = 0,$$
$$\dot{\mathcal{R}}_{2\mu} - 2\mathcal{L}_{2\mu}/m = 0,$$
$$\dot{\mathcal{L}}_{2\pm2} - \mathcal{P}_{2\pm2}/m + \left[m\omega^2 - \sqrt{4/3}\chi_0(R^{\rm eq}_{00} + \sqrt{2}R^{\rm eq}_{20})\right]\mathcal{R}_{2\pm2} = 0,$$
$$\dot{\mathcal{L}}_{2\pm1} - \mathcal{P}_{2\pm1}/m + \left[m\omega^2 - \sqrt{4/3}\chi_0(R^{\rm eq}_{00} - R^{\rm eq}_{20}/\sqrt{2})\right]\mathcal{R}_{2\pm1} = 0, \tag{18}$$
$$\dot{\mathcal{L}}_{20} - \mathcal{P}_{20}/m + \left[m\omega^2 - \sqrt{4/3}\chi_0(R^{\rm eq}_{00} - \sqrt{2}R^{\rm eq}_{20})\right]\mathcal{R}_{20} - \sqrt{4/3}\chi_0 R^{\rm eq}_{20}\mathcal{R}_{00} = 0,$$
$$\dot{\mathcal{P}}_{2\pm2} + 2[m\omega^2 - \sqrt{2/3}\chi_0 R^{\rm eq}_{20}]\mathcal{L}_{2\pm2} = 0,$$
$$\dot{\mathcal{P}}_{2\pm1} + 2[m\omega^2 + \sqrt{1/6}\chi_0 R^{\rm eq}_{20}]\mathcal{L}_{2\pm1} \mp \sqrt{6}\chi_0 R^{\rm eq}_{20}]\mathcal{L}_{1\pm1} = 0,$$
$$\dot{\mathcal{P}}_{20} + 2[m\omega^2 + \sqrt{2/3}\chi_0 R^{\rm eq}_{20}]\mathcal{L}_{20} - 4\sqrt{1/3}\chi_0 R^{\rm eq}_{20}\mathcal{L}_{00} = 0,$$
$$\dot{\mathcal{L}}_{1\nu} = 0.$$

The isovector set of equations reads:

$$\begin{gathered}
\dot{\bar{\mathcal{R}}}_{00} - 2\bar{\mathcal{L}}_{00}/m = 0, \\
\dot{\bar{\mathcal{L}}}_{00} - \bar{\mathcal{P}}_{00}/m + m\omega^2\bar{\mathcal{R}}_{00} - \sqrt{4/3}\chi R_{20}^{\mathrm{eq}}\,\bar{\mathcal{R}}_{20} = 0, \\
\dot{\bar{\mathcal{P}}}_{00} + 2m\omega^2\bar{\mathcal{L}}_{00} - 4\sqrt{1/3}\chi_0 R_{20}^{\mathrm{eq}}\,\bar{\mathcal{L}}_{20} = 0, \\
\dot{\bar{\mathcal{R}}}_{2\mu} - 2\bar{\mathcal{L}}_{2\mu}/m = 0, \\
\dot{\bar{\mathcal{L}}}_{2\pm2} - \bar{\mathcal{P}}_{2\pm2}/m + \left[m\omega^2 - \sqrt{2/3}\chi R_{20}^{\mathrm{eq}} - \sqrt{4/3}\chi_1 R_{00}^{\mathrm{eq}}\right]\bar{\mathcal{R}}_{2\pm2} = 0, \\
\dot{\bar{\mathcal{L}}}_{2\pm1} - \bar{\mathcal{P}}_{2\pm1}/m + \left[m\omega^2 + \sqrt{1/6}\chi R_{20}^{\mathrm{eq}} - \sqrt{4/3}\chi_1 R_{00}^{\mathrm{eq}}\right]\bar{\mathcal{R}}_{2\pm1} = 0, \qquad (19) \\
\dot{\bar{\mathcal{L}}}_{20} - \bar{\mathcal{P}}_{20}/m + \left[m\omega^2 + \sqrt{2/3}\chi R_{20}^{\mathrm{eq}} - \sqrt{4/3}\chi_1 R_{00}^{\mathrm{eq}}\right]\bar{\mathcal{R}}_{20} - \sqrt{4/3}\chi_0\mathcal{R}_{20}^{\mathrm{eq}}\,\bar{\mathcal{R}}_{00} = 0, \\
\dot{\bar{\mathcal{P}}}_{2\pm2} + 2[m\omega^2 - \sqrt{2/3}\chi_0 R_{20}^{\mathrm{eq}}]\bar{\mathcal{L}}_{2\pm2} = 0, \\
\dot{\bar{\mathcal{P}}}_{2\pm1} + 2[m\omega^2 + \sqrt{1/6}\chi_0 R_{20}^{\mathrm{eq}}]\bar{\mathcal{L}}_{2\pm1} \mp \sqrt{6}\chi_0 R_{20}^{\mathrm{eq}}\,\bar{\mathcal{L}}_{1\pm1} = 0, \\
\dot{\bar{\mathcal{P}}}_{20} + 2[m\omega^2 + \sqrt{2/3}\chi_0 R_{20}^{\mathrm{eq}}]\bar{\mathcal{L}}_{20} - \sqrt{4/3}\chi_0 R_{20}^{\mathrm{eq}}\,\bar{\mathcal{L}}_{00} = 0, \\
\dot{\bar{\mathcal{L}}}_{1\pm1} \pm \sqrt{3/2}\bar{\chi} R_{20}^{\mathrm{eq}}\bar{\mathcal{R}}_{2\pm1} = 0, \\
\dot{\bar{\mathcal{L}}}_{10} = 0.
\end{gathered}$$

Due to the axial symmetry, the angular momentum projection is a good quantum number. As a result, every set of equations splits into five independent subsets with $\mu = 0, \pm1, \pm2$. It is known [19], that equations with $\mu = 0$ and $\mu = \pm2$ describe the β and γ modes, respectively. The equations with $\mu = \pm1$ describe the coupled dynamics of the transvers shear mode [10] and the rotational motion — they are the subject of the especial interest in this paper.

1.2. Isoscalar Eigenfrequencies. The dynamics of the isoscalar angular momentum is trivial — no vibrations, this variable is conserved. However, it is necessary to treat this mode carefully because, being the nonvibrational mode with zero eigenfrequency, it gives, nevertheless, a nonzero contribution to the sum rule (see below). Let us analyze the isoscalar set of equations with $\mu = \nu = 1$ in more detail

$$\begin{gathered}
\dot{\mathcal{R}}_{21} - 2\mathcal{L}_{21}/m = 0, \\
\dot{\mathcal{L}}_{21} - \mathcal{P}_{21}/m + \left[m\omega^2 + \sqrt{4/3}\chi_0(R_{20}^{\mathrm{eq}}/\sqrt{2} - R_{00}^{\mathrm{eq}})\right]\mathcal{R}_{21} = 0, \\
\dot{\mathcal{P}}_{21} + 2[m\omega^2 + \sqrt{1/6}\chi_0 R_{20}^{\mathrm{eq}}]\mathcal{L}_{21} - \sqrt{6}\chi_0 R_{20}^{\mathrm{eq}}\,\mathcal{L}_{11} = 0, \\
\dot{\mathcal{L}}_{11} = 0.
\end{gathered} \qquad (20)$$

Using the self-consistent value of the strength constant $\kappa_0 = -\dfrac{m\bar{\omega}^2}{4Q_{00}}$ (see Appendix A), the relations between $Q_{\lambda\mu}$ and $R_{\lambda\mu}$ and the standard definition of the

deformation parameter $Q_{20} = Q_{00}\dfrac{4}{3}\,\delta$ we reduce (20) to

$$\begin{gathered}
\dot{\mathcal{R}}_{21} - 2\mathcal{L}_{21}/m = 0, \\
\dot{\mathcal{L}}_{21} - \mathcal{P}_{21}/m = 0, \\
\dot{\mathcal{P}}_{21} + 2m\bar{\omega}^2\left[\left(1+\frac{\delta}{3}\right)\mathcal{L}_{21} + \delta\mathcal{L}_{11}\right] = 0, \\
\dot{\mathcal{L}}_{11} = 0.
\end{gathered} \tag{21}$$

Imposing the time evolution via $\mathrm{e}^{-i\Omega t}$ for all variables one transforms (21) into a set of algebraic equations. The eigenfrequencies are found from its characteristic equation which reads

$$\Omega^2\left[\Omega^2 - 2\bar{\omega}^2\left(1+\frac{\delta}{3}\right)\right] = 0. \tag{22}$$

The nontrivial solution of this equation gives the frequency of the $\mu = 1$ branch of the isoscalar GQR

$$\Omega^2 = \Omega_{\mathrm{is}}^2 = 2\bar{\omega}^2\left(1+\frac{\delta}{3}\right). \tag{23}$$

Taking into account the relation (A.7) we find that this result coincides with that of [20]. The trivial solution $\Omega = \Omega_0 = 0$ is characteristic of nonvibrational mode corresponding to the obvious integral of motion $\mathcal{L}_{11} = \mathrm{const}$ responsible for the rotational degree of freedom. Having in mind that in the case of harmonic oscillations $\mathcal{L}_{11} = 0$, we can find another, not so obvious, integral. The simple combination of the third and first equations of (21) gives

$$\mathcal{P}_{21} + m^2\bar{\omega}^2\left(1+\frac{\delta}{3}\right)\mathcal{R}_{21} = \mathrm{const}.$$

Assuming here $\delta = 0$, we reproduce our result from [13] for spherical nuclei, saying that the nuclear density and the Fermi surface oscillate out of phase.

1.3. Isovector Eigenfrequencies. The information about the scissors mode is contained in the subset of isovector equations with $\mu = 1$. Let us analyze it in detail:

$$\begin{gathered}
\dot{\bar{\mathcal{R}}}_{21} - 2\bar{\mathcal{L}}_{21}/m = 0, \\
\dot{\bar{\mathcal{L}}}_{21} - \bar{\mathcal{P}}_{21}/m + \left[m\omega^2 + \sqrt{1/6}\chi R_{20}^{\mathrm{eq}} - \sqrt{4/3}\chi_1 R_{00}^{\mathrm{eq}}\right]\bar{\mathcal{R}}_{21} = 0, \\
\dot{\bar{\mathcal{P}}}_{21} + 2[m\omega^2 + \sqrt{1/6}\chi_0 R_{20}^{\mathrm{eq}}]\bar{\mathcal{L}}_{21} - \sqrt{6}\chi_0 R_{20}^{\mathrm{eq}}\,\bar{\mathcal{L}}_{11} = 0, \\
\dot{\bar{\mathcal{L}}}_{11} + \sqrt{3/2}\bar{\chi} R_{20}^{\mathrm{eq}}\bar{\mathcal{R}}_{21} = 0.
\end{gathered} \tag{24}$$

Supposing, as usual, the isovector constant κ_1 to be proportional to the isoscalar one, $\kappa_1 = \alpha\kappa_0$ and using the same definitions as in the isoscalar case we find

$$\begin{gathered}
\dot{\bar{\mathcal{R}}}_{21} - 2\bar{\mathcal{L}}_{21}/m = 0, \\
\dot{\bar{\mathcal{L}}}_{21} - \bar{\mathcal{P}}_{21}/m + m\bar{\omega}^2(1-\alpha)\left(1+\frac{\delta}{3}\right)\bar{\mathcal{R}}_{21} = 0, \\
\dot{\bar{\mathcal{P}}}_{21} + 2m\bar{\omega}^2\left[\left(1+\frac{\delta}{3}\right)\bar{\mathcal{L}}_{21} + \delta\bar{\mathcal{L}}_{11}\right] = 0, \\
\dot{\bar{\mathcal{L}}}_{11} - m\bar{\omega}^2\delta(1-\alpha)\bar{\mathcal{R}}_{21} = 0.
\end{gathered} \tag{25}$$

Imposing the time evolution via $\mathrm{e}^{-i\Omega t}$ one transforms (25) into a set of algebraic equations with the characteristic equation

$$\Omega^4 - 2\Omega^2\bar{\omega}^2(2-\alpha)\left(1+\frac{\delta}{3}\right) + 4\bar{\omega}^4(1-\alpha)\delta^2 = 0. \tag{26}$$

Its solutions are

$$\Omega_{\pm}^2 = \bar{\omega}^2(2-\alpha)\left(1+\frac{\delta}{3}\right) \pm \sqrt{\bar{\omega}^4(2-\alpha)^2\left(1+\frac{\delta}{3}\right)^2 - 4\bar{\omega}^4(1-\alpha)\delta^2}. \tag{27}$$

The high-lying solution Ω_+ gives the frequency Ω_{iv} of the $\mu = 1$ branch of the isovector GQR. The low-lying solution Ω_- gives the frequency Ω_{sc} of the scissors mode.

It is worth noticing that in the case $\bar{\mathcal{L}}_{11} = 0$ the set of equations (25) becomes quite similar to (21). Its characteristic equation reduces to the equation

$$\Omega^3 - 2\Omega\bar{\omega}^2(2-\alpha)\left(1+\frac{\delta}{3}\right) = 0, \tag{28}$$

implying that there exists an integral of motion analogous to the isoscalar one:

$$\bar{\mathcal{P}}_{21} + m^2\bar{\omega}^2\left(1+\frac{\delta}{3}\right)\bar{\mathcal{R}}_{21} = \mathrm{const}.$$

The nontrivial solution of (28) gives the IVGQR frequency for the case, when rotational degrees of freedom are neglected:

$$\Omega^2 = 2\bar{\omega}^2(2-\alpha)\left(1+\frac{\delta}{3}\right). \tag{29}$$

Now let us fix the value of the coefficient α. The experimental fact is: the energy of an isovector GQR is practically two times higher than that of an isoscalar one. Assuming $\delta = 0$, we have

$$\Omega_+^2 = \Omega_{\mathrm{iv}}^2 = 2\omega^2(2-\alpha).$$

The simple comparison of this expression with (23) shows that the experimental observation is satisfied by $\alpha = -2$. Then Eq. (27) gives the following formulae for both energies:

$$\begin{aligned} \Omega_{\rm iv}^2 &= 4\bar{\omega}^2 \left(1 + \frac{\delta}{3} + \sqrt{\left(1 + \frac{\delta}{3}\right)^2 - \frac{3}{4}\delta^2} \right), \\ \Omega_{\rm sc}^2 &= 4\bar{\omega}^2 \left(1 + \frac{\delta}{3} - \sqrt{\left(1 + \frac{\delta}{3}\right)^2 - \frac{3}{4}\delta^2} \right). \end{aligned} \tag{30}$$

In the limit of small deformations one can write for IVGQR energy

$$E_{\rm iv}^2 \simeq 8(\hbar\bar{\omega})^2 \left(1 + \frac{\delta}{3}\right) \left(1 - \frac{3}{16}\delta^2\right). \tag{31}$$

For $\alpha = -2$, formula (29) gives: $E_{\rm iv}^2 = 8(\hbar\bar{\omega})^2(1 + \delta/3)$. Comparing it with (31) one sees that the influence of rotational degrees of freedom on the IVGQR energy is very small.

The scissors mode energy in the limit of small deformation is

$$E_{\rm sc} \approx \sqrt{\frac{3}{2}}\,\hbar\omega_0\delta, \tag{32}$$

which is quite close to the result of Hilton [21]: $E_{\rm sc} \approx \sqrt{1 + 0.66}\,\hbar\omega_0\delta$.

It is interesting to study the role of the Fermi Surface Deformation (FSD) for the formation of IVGQR and the scissors mode. Neglecting in (25) the variable $\bar{\mathcal{P}}_{21}(t)$, which is responsible for FSD, we find that the frequency of IVGQR (being determined mainly by the neutron–proton interaction) is changed not very much:

$$\Omega_{\rm iv}^2 = 2\bar{\omega}^2(1 - \alpha)\left(1 + \frac{\delta}{3}\right).$$

Comparing this formula (for $\alpha = -2$) with (30), one sees that in the limit of small deformation one obtains $\Omega_{\rm iv}^2 \simeq 6\omega_0^2$ instead of $\Omega_{\rm iv}^2 \simeq 8\omega_0^2$. One should recall that also for the Isovector Giant Dipole Resonance the distortion of Fermi sphere plays only a minor role.

It is also easy to see that omitting $\bar{\mathcal{P}}_{21}(t)$ in (25), one obtains zero energy for the scissors mode independent of the strength of the residual interaction. Thus, the nuclear elasticity discovered by G. F. Bertsch [22] is the single origin for the restoring force of the scissors mode. So one can conclude that this mode is in its essence a pure quantum mechanical phenomenon. This agrees with the conclusion of the papers [23, 24]: classically (i.e., without Fermi surface deformation) the scissors mode is a zero energy mode.

1.4. Linear Response and Transition Probabilities. A direct way of calculating the reduced transition probabilities is provided by the theory of the linear response of a system to a weak external field

$$\hat{F}(t) = \hat{F}\exp(-i\Omega t) + \hat{F}^{\dagger}\exp(i\Omega t), \tag{33}$$

where $\hat{F} = \sum_{s=1}^{A} \hat{f}_s$ is a one-body operator. A convenient form of the response theory is, e.g., given by Lane [25] (see also Sec. 4). The matrix elements of the operator $\hat{F}$ obey the relation

$$|\langle\nu|\hat{F}|0\rangle|^2 = \hbar \lim_{\Omega\to\Omega_\nu} (\Omega - \Omega_\nu)\overline{\langle\psi|\hat{F}|\psi\rangle \exp(-i\Omega t)}, \tag{34}$$

where $|0\rangle$ and $|\nu\rangle$ are the stationary wave functions of unperturbed ground and excited states; ψ is the wave function of the perturbed ground state, $\Omega_\nu = (E_\nu - E_0)/\hbar$ are the normal frequencies, the bar means averaging over a time interval much larger than $1/\Omega$, Ω being the frequency of the external field $\hat{F}(t)$. To use formula (34) in the frame of WFM method, one must solve two problems [17]:

(1) to express the matrix element $\langle\psi|\hat{F}|\psi\rangle$ in terms of collective variables of the system,

(2) to find the solution of the dynamic equations for these variables in the presence of the external field.

The first problem is solved with the help of the formula for the Wigner transformation of a product of two operators [9]

$$\langle\psi|\hat{F}|\psi\rangle^{\tau} = \int d^3r \int d^3r' \rho^{\tau}(\mathbf{r},\mathbf{r}',t)\hat{F}(\mathbf{r}',\mathbf{r}) = \int d^3r \int \frac{2d^3p}{(2\pi\hbar)^3} \times$$
$$\times \exp\left(\frac{\hbar}{2i}(\nabla^F_{\mathbf{r}}\cdot\nabla^f_{\mathbf{p}} - \nabla^F_{\mathbf{p}}\cdot\nabla^f_{\mathbf{r}})\right) F_W(\mathbf{r},\mathbf{p}) f^{\tau}(\mathbf{r},\mathbf{p},t). \tag{35}$$

To deal with the second problem we add the field (33) to the mean field potential (9). The equation for the Wigner function (4) is then modified by the term

$$F^{\tau}_{\mathrm{ext}} = \frac{2}{\hbar}\sin\left(\frac{\hbar}{2}(\nabla^F_{\mathbf{r}}\cdot\nabla^f_{\mathbf{p}} - \nabla^F_{\mathbf{p}}\cdot\nabla^f_{\mathbf{r}})\right) \times$$
$$\times \left(F_W \exp(-i\Omega t) + F^*_W \exp(i\Omega t)\right) f^{\tau}. \tag{36}$$

Proceeding in the same way as before one obtains equations for all collective variables needed to calculate $\langle\psi|\hat{F}|\psi\rangle^{\tau}$. The only new element now is the presence of the term F^{τ}_{ext} that makes the equations for the moments inhomogeneous.

1.5. $B(M1)$ **Factors.** To calculate the magnetic transition probability, it is necessary to excite the system with the following external field:

$$\hat{F} = \hat{F}^p_{\lambda\mu'} = \sum_{s=1}^{Z} \hat{f}_{\lambda\mu'}(s),$$
$$\hat{f}_{\lambda\mu'} = -i\frac{2}{\lambda+1}\nabla(r^\lambda Y_{\lambda\mu'})[\mathbf{r}\times\nabla]\mu_N, \quad \mu_N = \frac{e\hbar}{2mc}. \tag{37}$$

We are interested in the dipole operator ($\lambda = 1$). In the cyclic coordinates it looks like

$$\hat{f}_{1\mu'} = -\mu_N\sqrt{\frac{3}{2\pi}}\sum_{\nu,\sigma} C^{1\mu'}_{1\nu,1\sigma} r_\nu \nabla_\sigma, \quad \hat{f}^\dagger_{1\mu'} = -\hat{f}^*_{1\mu'} = (-1)^{\mu'}\hat{f}_{1-\mu'}. \tag{38}$$

Its Wigner transformation is

$$(\hat{f}_{1\mu'})_W = \gamma\sum_{\nu,\sigma} C^{1\mu'}_{1\nu,1\sigma} r_\nu p_\sigma = \gamma(rp)_{1\mu'},$$

where $\gamma = -\dfrac{i}{\hbar}\sqrt{\dfrac{3}{2\pi}}\mu_N$. For its matrix element we have

$$\langle\psi|\hat{F}^p_{1\mu}|\psi\rangle = \gamma L^p_{1\mu'} = \frac{\gamma}{2}(L_{1\mu'} - \bar{L}_{1\mu'}) = \frac{\gamma}{2}(\mathcal{L}_{1\mu'} - \bar{\mathcal{L}}_{1\mu'}). \tag{39}$$

Here we have taken into account that $L^{\mathrm{eq}}_{\lambda\mu'} = \bar{L}^{\mathrm{eq}}_{\lambda\mu'} = 0$. The contribution of $\hat{F}_{1\mu'}(t)$ to the equation for the Wigner function is

$$F_{\mathrm{ext}} = \gamma\left(F_{\mu'}\,\mathrm{e}^{-i\Omega t} + (-1)^{\mu'}F_{-\mu'}\,\mathrm{e}^{i\Omega t}\right)$$

with

$$F_{\mu'} = \sum_{\nu\sigma} C^{1\mu'}_{1\nu,1\sigma}[p_\sigma\nabla^p_\nu - r_\nu\nabla^r_\sigma]f^p.$$

Integration of $F_{\mu'}$ with the weights $r^2_{\lambda\mu}$, $(rp)_{\lambda\mu}$ and $p^2_{\lambda\mu}$ yields

$$\int d\{\mathbf{p},\mathbf{r}\}r^2_{\lambda\mu}F_{\mu'} = 2\sqrt{3(2\lambda+1)}\sum_{k,\pi} C^{k\pi}_{\lambda\mu,1\mu'}\{^{11\lambda}_{k11}\}R^p_{k\pi}(\mathrm{eq}),$$
$$\int d\{\mathbf{p},\mathbf{r}\}(rp)_{\lambda\mu}F_{\mu'} = \sqrt{3(2\lambda+1)}\sum_{k,\pi}[(-1)^\lambda + (-1)^k]C^{k\pi}_{\lambda\mu,1\mu'}\{^{11\lambda}_{k11}\}L^p_{k\pi}(\mathrm{eq}),$$
$$\int d\{\mathbf{p},\mathbf{r}\}p^2_{\lambda\mu}F_{\mu'} = 2\sqrt{3(2\lambda+1)}\sum_{k,\pi} C^{k\pi}_{\lambda\mu,1\mu'}\{^{11\lambda}_{k11}\}P^p_{k\pi}(\mathrm{eq}).$$

A simple analysis of these expressions shows that the external field modifies only the proton part of the set of equations (11) with $\lambda = 2$:

$$\begin{aligned}
&\frac{d}{dt}R^p_{2\mu} - \ldots = \\
&\qquad = -\gamma\sqrt{3}\left[C^{2\mu+\mu'}_{2\mu,1\mu'}R^p_{2\mu+\mu'}(\mathrm{eq})\,\mathrm{e}^{-i\Omega t} + (-1)^{\mu'}C^{2\mu-\mu'}_{2\mu,1-\mu'}R^p_{2\mu-\mu'}(\mathrm{eq})\,\mathrm{e}^{i\Omega t}\right], \\
&\frac{d}{dt}L^p_{2\mu} - \ldots = 0, \\
&\frac{d}{dt}P^p_{2\mu} + \ldots = \\
&\qquad = -\gamma\sqrt{3}\left[C^{2\mu+\mu'}_{2\mu,1\mu'}P^p_{2\mu+\mu'}(\mathrm{eq})\,\mathrm{e}^{-i\Omega t} + (-1)^{\mu'}C^{2\mu-\mu'}_{2\mu,1-\mu'}P^p_{2\mu-\mu'}(\mathrm{eq})\,\mathrm{e}^{i\Omega t}\right].
\end{aligned} \tag{40}$$

The modifications of the respective isoscalar and isovector equations are obvious.

The $\mu' = 0$ component of the external field does not disturb a nucleus due to its axial symmetry. Let us consider the case of $\mu' = 1$. According to formula (39) we have to find the tensors $\bar{\mathcal{L}}_{11}$ and $\mathcal{L}_{11}$. The tensor $\bar{\mathcal{L}}_{11}$ is found by solving the modified (as in (40)) set of equations (24):

$$\begin{gathered}
\dot{\bar{\mathcal{R}}}_{21} - 2\bar{\mathcal{L}}_{21}/m = -\gamma\sqrt{3/8}R^{\mathrm{eq}}_{20}\,\mathrm{e}^{i\Omega t}, \\
\dot{\bar{\mathcal{L}}}_{21} - \bar{\mathcal{P}}_{21}/m + \left[m\omega^2 + \sqrt{1/6}\chi R^{\mathrm{eq}}_{20} - \sqrt{4/3}\chi_1 R^{\mathrm{eq}}_{00}\right]\bar{\mathcal{R}}_{21} = 0, \\
\dot{\bar{\mathcal{P}}}_{21} + 2[m\omega^2 + \sqrt{1/6}\chi_0 R^{\mathrm{eq}}_{20}]\bar{\mathcal{L}}_{21} - \sqrt{6}\chi_0 R^{\mathrm{eq}}_{20}\,\bar{\mathcal{L}}_{11} = -\gamma\sqrt{3/8}P^{\mathrm{eq}}_{20}\,\mathrm{e}^{i\Omega t}, \\
\dot{\bar{\mathcal{L}}}_{11} + \sqrt{3/2}\bar{\chi}R^{\mathrm{eq}}_{20}\bar{\mathcal{R}}_{21} = 0.
\end{gathered} \tag{41}$$

It is clear that the time dependence of all variables must be $\mathrm{e}^{i\Omega t}$. The required variable is determined by the ratio of two determinants

$$\bar{\mathcal{L}}_{11} = \frac{\Delta_{\bar{\mathcal{L}}}}{\Delta_{\mathrm{iv}}}\,\mathrm{e}^{i\Omega t},$$

where Δ_{iv} is the determinant of (24) and

$$\Delta_{\bar{\mathcal{L}}} = \frac{3}{4}\gamma\bar{\chi}R^{\mathrm{eq}}_{20}\left[R^{\mathrm{eq}}_{20}\left(2\omega^2 + \sqrt{\frac{2}{3}}\frac{\chi_0}{m}R^{\mathrm{eq}}_{20} - \Omega^2\right) + \frac{2}{m^2}P^{\mathrm{eq}}_{20}\right].$$

At equilibrium the set of dynamic equations (11) considerably simplify turning into the set of equations of equilibrium. Taking into account one of them we find

$$\frac{1}{m}P^{\mathrm{eq}}_{20} = m\omega^2 R^{\mathrm{eq}}_{20} - \frac{2}{\sqrt{3}}\chi_0 R^{\mathrm{eq}}_{20}R^{\mathrm{eq}}_{00} + \frac{2}{\sqrt{6}}\chi_0(R^{\mathrm{eq}}_{20})^2.$$

It turns out, that for the self-consistent value of χ_0 this expression is equal to zero, i.e., $P_{20}^{\rm eq}=0$. This means that deformed nuclei have spherical Fermi surface. So we have

$$\Delta_{\bar{\mathcal{L}}}=\frac{3}{4}\gamma\bar{\kappa}Q_{20}^2\left[2\bar{\omega}^2\left(1+\frac{\delta}{3}\right)-\Omega^2\right].$$

Looking at the isoscalar counterpart of the set of equations (41) one easily finds that the isoscalar tensor $\mathcal{L}_{11}=0$.

Writing now the determinant $\Delta_{\rm iv}$ as

$$\Delta_{\rm iv}=(\Omega^2-\Omega_{\rm iv}^2)(\Omega^2-\Omega_{\rm sc}^2), \tag{42}$$

we easily can find the limit (34). For the case, where $|\nu\rangle=|{\rm sc}\rangle$, we have

$$|\langle{\rm sc}|\hat{F}_{11}^p|0\rangle|^2=\frac{-\gamma\hbar\Delta_{\bar{\mathcal{L}}}(\Omega_{\rm sc})}{[(\Omega_{\rm sc}^2-\Omega_{\rm iv}^2)4\Omega_{\rm sc}]}.$$

The matrix element for $|\nu\rangle=|{\rm iv}\rangle$ is obtained simply by changing indices ${\rm sc}\leftrightarrow{\rm iv}$. Applying the standard values of parameters

$$\kappa_1=\alpha\kappa_0,\quad 4\kappa_0Q_{00}=-m\bar{\omega}^2,\quad \kappa_0Q_{20}=-\frac{\delta}{3}m\bar{\omega}^2$$

we arrive at the following expressions for transition probabilities:

$$B(M1)_{\rm sc}=2|\langle{\rm sc}|\hat{F}_{11}^p|0\rangle|^2=\frac{1-\alpha}{4\pi}\frac{m\bar{\omega}^2}{\hbar}Q_{00}\delta^2\frac{\Omega_{\rm sc}^2-2(1+\delta/3)\bar{\omega}^2}{\Omega_{\rm sc}(\Omega_{\rm sc}^2-\Omega_{\rm iv}^2)}\,\mu_N^2, \tag{43}$$

$$B(M1)_{\rm iv}=2|\langle{\rm iv}|\hat{F}_{11}^p|0\rangle|^2=\frac{1-\alpha}{4\pi}\frac{m\bar{\omega}^2}{\hbar}Q_{00}\delta^2\frac{\Omega_{\rm iv}^2-2(1+\delta/3)\bar{\omega}^2}{\Omega_{\rm iv}(\Omega_{\rm iv}^2-\Omega_{\rm sc}^2)}\,\mu_N^2. \tag{44}$$

These two formulae can be joined into one expression by a simple transformation of the denominators. Really, we have from (27)

$$\pm(\Omega_{\rm iv}^2-\Omega_{\rm sc}^2)=\pm(\Omega_+^2-\Omega_-^2)=\pm2\sqrt{\bar{\omega}^4(2-\alpha)^2\left(1+\frac{\delta}{3}\right)^2-4\bar{\omega}^4(1-\alpha)\delta^2}=$$

$$=2\Omega_\pm^2-2\bar{\omega}^2(2-\alpha)\left(1+\frac{\delta}{3}\right)=2\Omega_\pm^2-(2-\alpha)(\omega_x^2+\omega_z^2). \tag{45}$$

Using these relations in formulae (43) and (44), we obtain the expression for the $B(M1)$ values valid for both excitations

$$B(M1)_\nu=2|\langle\nu|\hat{F}_{11}^p|0\rangle|^2=$$

$$=\frac{1-\alpha}{8\pi}\frac{m\bar{\omega}^2}{\hbar}Q_{00}\delta^2\frac{\Omega_\nu^2-2(1+\delta/3)\bar{\omega}^2}{\Omega_\nu[\Omega_\nu^2-\bar{\omega}^2(2-\alpha)(1+\delta/3)]}\mu_N^2. \tag{46}$$

Taking into account the relation $Q_{00}^0 \dfrac{m\omega_0}{\hbar} \simeq \dfrac{1}{2}\left(\dfrac{3}{2}A\right)^{4/3}$, which is usually [19] used to fix the value of the harmonic oscillator frequency ω_0, we obtain the following estimate for the transition probability of the scissors mode:

$$B(M1)\uparrow = 2\,|\langle \mathrm{sc}|\hat{F}_{11}^p|0\rangle|^2 = \frac{(3/2)^{11/6}}{16\pi} A^{4/3}\delta\,\mu_N^2 = 0.042A^{4/3}\delta\,\mu_N^2,$$

which practically coincides with the result of [26]: $B(M1)\uparrow = 0.043A^{4/3}\delta\,\mu_N^2$, obtained with the help of the microscopic approach based on the evaluation of the sum rules.

1.6. $B(E2)$ **Factors.** To calculate the $B(E2)$ factor it is necessary to excite the system with the external field operator

$$\hat{F} = \hat{F}_{2\mu'}^p = er^2 Y_{2\mu'} = \beta r_{2\mu'}^2, \quad \hat{F}_{2\mu'}^\dagger = \hat{F}_{2\mu'}^* = (-1)^{\mu'}\hat{F}_{2-\mu'}, \tag{47}$$

where $\beta = e\sqrt{\dfrac{15}{8\pi}}$. Its Wigner transform is identical to (47): $(\hat{F}_{2\mu'}^p)_W = \beta r_{2\mu'}^2$. The matrix element is given by

$$\langle\psi|\hat{F}_{2\mu'}^p|\psi\rangle = \beta R_{2\mu'}^p = \frac{1}{2}\beta(R_{2\mu'} - \bar{R}_{2\mu'}). \tag{48}$$

The contribution of $\hat{F}_{2\mu'}(t)$ to the equation for the Wigner function is

$$F_{\mathrm{ext}} = 2\beta\left(F_{\mu'}\,\mathrm{e}^{-i\Omega t} + (-1)^{\mu'}F_{-\mu'}\,\mathrm{e}^{i\Omega t}\right)$$

with

$$F_{\mu'} = \sum_{\nu,\sigma} C_{1\nu,1\sigma}^{2\mu'} r_\nu \nabla_\sigma^p f^p.$$

Integration of $F_{\mu'}$ with the weights $r_{\lambda\mu}^2$, $(rp)_{\lambda\mu}$ and $p_{\lambda\mu}^2$ yields

$$\int d\{\mathbf{p},\mathbf{r}\} r_{\lambda\mu}^2 F_{\mu'} = 0,$$

$$\int d\{\mathbf{p},\mathbf{r}\} (rp)_{\lambda\mu} F_{\mu'} = \sqrt{5(2\lambda+1)}\sum_{k,\pi} C_{\lambda\mu,2\mu'}^{k\pi}\{{}^{11\lambda}_{k21}\} R_{k\pi}^p(\mathrm{eq}),$$

$$\int d\{\mathbf{p},\mathbf{r}\} p_{\lambda\mu}^2 F_{\mu'} = [1+(-1)^\lambda]\sqrt{5(2\lambda+1)}\sum_{k\pi} C_{\lambda\mu,2\mu'}^{k\pi}\{{}^{11\lambda}_{k21}\} L_{k\pi}^p(\mathrm{eq}).$$

The external field modifies the set of equations (11) in the following way:

$$\frac{d}{dt}L^p_{1\mu}+\ldots = \\ = -\beta\sqrt{3}\left[C^{2\mu+\mu'}_{1\mu,2\mu'}R^p_{2\mu+\mu'}(\mathrm{eq})\,\mathrm{e}^{-i\Omega t}+(-1)^{\mu'}C^{2\mu-\mu'}_{1\mu,2-\mu'}R^p_{2\mu-\mu'}(\mathrm{eq})\,\mathrm{e}^{i\Omega t}\right], \tag{49}$$

$$\frac{d}{dt}L^p_{2\mu}-\ldots = \frac{\beta}{\sqrt{3}}\left[\left(2\sqrt{5}C^{00}_{2\mu,2\mu'}R^p_{00}(\mathrm{eq})+\sqrt{7}C^{2\mu+\mu'}_{2\mu,2\mu'}R^p_{2\mu+\mu'}(\mathrm{eq})\right)\,\mathrm{e}^{-i\Omega t}\,=\right. \\ \left.=\,+(-1)^{\mu'}\left(2\sqrt{5}C^{00}_{2\mu,2-\mu'}R^p_{00}(\mathrm{eq})+\sqrt{7}C^{2\mu-\mu'}_{2\mu,2-\mu'}R^p_{2\mu-\mu'}(\mathrm{eq})\right)\,\mathrm{e}^{i\Omega t}\right].$$

Let us consider the case of $\mu'=1$ ($\mu'=-1$ gives the same result). According to formula (48), we have to find the tensors $\bar{R}_{21}$ and R_{21}. The value of $\bar{R}_{21}=\bar{\mathcal{R}}_{21}$ is found by solving the modified (as in (49)) set of equations (25)

$$\begin{gathered}\dot{\bar{\mathcal{R}}}_{21}-2\bar{\mathcal{L}}_{21}/m=0,\\ \dot{\bar{\mathcal{L}}}_{21}-\bar{\mathcal{P}}_{21}/m+m\bar{\omega}^2(1-\alpha)\left(1+\frac{\delta}{3}\right)\bar{\mathcal{R}}_{21}=\frac{e}{3}\sqrt{\frac{15}{8\pi}}\left(1+\frac{\delta}{3}\right)Q_{00}\,\mathrm{e}^{i\Omega t},\\ \dot{\bar{\mathcal{P}}}_{21}+2m\bar{\omega}^2\left[\left(1+\frac{\delta}{3}\right)\bar{\mathcal{L}}_{21}+\delta\bar{\mathcal{L}}_{11}\right]=0,\\ \dot{\bar{\mathcal{L}}}_{11}-m\bar{\omega}^2\delta(1-\alpha)\bar{\mathcal{R}}_{21}=-e\sqrt{\frac{15}{8\pi}}\frac{\delta}{3}Q_{00}\,\mathrm{e}^{i\Omega t}.\end{gathered} \tag{50}$$

It is obvious that the time dependence of all variables must be $\mathrm{e}^{i\Omega t}$. The required variable is determined by the ratio of two determinants

$$\bar{\mathcal{R}}_{21}=\frac{\Delta_{\bar{\mathcal{R}}}}{\Delta_{\mathrm{iv}}}\,\mathrm{e}^{i\Omega t},$$

where Δ_{iv} is the determinant of (25) and

$$\Delta_{\bar{\mathcal{R}}}=-\frac{\beta}{m}\left[\frac{2}{3}\Omega^2\left(Q^{\mathrm{eq}}_{00}+\frac{1}{4}Q^{\mathrm{eq}}_{20}\right)+\frac{1}{m}Q^{\mathrm{eq}}_{20}\sqrt{\frac{3}{2}}\chi_0R^{\mathrm{eq}}_{20}\right]= \\ =-\frac{2}{3}\frac{\beta}{m}Q^{\mathrm{eq}}_{00}\left[\Omega^2\left(1+\frac{\delta}{3}\right)-2\bar{\omega}^2\delta^2\right].$$

Analogously variable $\mathcal{R}_{21}$ is found from the modified set of equations (21), which have exactly the same right-hand sides as (50) but with the opposite signs.

The limit (34) is calculated with the help of expression (42) for Δ_{iv} and the analogous expression for Δ_{is} (the determinant of (21)):

$$\Delta_{\mathrm{is}}=(\Omega^2-\Omega_0^2)(\Omega^2-\Omega_{\mathrm{is}}^2).$$

In the case $|\nu\rangle = |\mathrm{sc}\rangle$ we find

$$|\langle \mathrm{sc}|\hat{F}^p_{21}|0\rangle|^2 = -\beta\frac{\hbar}{2}\Delta_{\bar{\mathcal{R}}}\frac{(\Omega_{\mathrm{sc}})}{(\Omega^2_{\mathrm{sc}}-\Omega^2_{\mathrm{iv}})2\Omega_{\mathrm{sc}}} = \\ = \frac{e^2\hbar}{m}\frac{5}{8\pi}Q_{00}\frac{(1+\delta/3)\Omega^2_{\mathrm{sc}}-2(\bar{\omega}\delta)^2}{\Omega_{\mathrm{sc}}(\Omega^2_{\mathrm{sc}}-\Omega^2_{\mathrm{iv}})}. \quad (51)$$

In the case $|\nu\rangle = |\mathrm{iv}\rangle$ formula (34) gives

$$|\langle \mathrm{iv}|\hat{F}^p_{21}|0\rangle|^2 = -\beta\frac{\hbar}{2}\Delta_{\bar{\mathcal{R}}}\frac{(\Omega_{\mathrm{iv}})}{2\Omega_{\mathrm{iv}}(\Omega^2_{\mathrm{iv}}-\Omega^2_{\mathrm{sc}})} = \\ = \frac{e^2\hbar}{m}\frac{5}{16\pi}Q_{00}\frac{(1+\delta/3)\Omega^2_{\mathrm{iv}}-2(\bar{\omega}\delta)^2}{\Omega_{\mathrm{iv}}(\Omega^2_{\mathrm{iv}}-\Omega^2_{\mathrm{sc}})}. \quad (52)$$

In the case $|\nu\rangle = |\mathrm{is}\rangle$ formula (34) gives

$$|\langle \mathrm{is}|\hat{F}^p_{21}|0\rangle|^2 = -\beta\frac{\hbar}{2}\Delta_{\mathcal{R}}\frac{(\Omega_{\mathrm{is}})}{2\Omega_{\mathrm{is}}(\Omega^2_{\mathrm{is}}-\Omega^2_0)} = \\ = \frac{e^2\hbar}{m}\frac{5}{16\pi}Q_{00}\frac{(1+\delta/3)\Omega^2_{\mathrm{is}}-2(\bar{\omega}\delta)^2}{[\Omega_{\mathrm{is}}]^3}. \quad (53)$$

Formula (34) allows one to calculate the matrix element $|\langle\nu|\hat{F}^p_{21}|0\rangle|^2$ also in the case when $|\nu\rangle = |\Omega_0\rangle$, i.e., for the rotational state corresponding to the trivial solution of (22):

$$|\langle\Omega_0|\hat{F}^p_{21}|0\rangle|^2 = -\beta\frac{\hbar}{2}\Delta_{\mathcal{R}}\frac{(\Omega_0)}{2\Omega_0(\Omega^2_0-\Omega^2_{\mathrm{is}})} = \frac{e^2\hbar}{m}\frac{5}{8\pi}Q_{00}\frac{\delta^2}{2\Omega_0(1+\delta/3)}. \quad (54)$$

The value of this matrix element is obviously infinite due to the zero value of Ω_0. However, below this expression will be useful to calculate the energy weighted sum rule.

Using relations (45) in formulae (51) and (52), we obtain the expression for the $B(E2)$ values valid for all four excitations

$$B(E2)_\nu = 2|\langle\nu|\hat{F}^p_{21}|0\rangle|^2 = \frac{e^2\hbar}{m}\frac{5}{16\pi}Q_{00}\frac{(1+\delta/3)\Omega^2_\nu-2(\bar{\omega}\delta)^2}{\Omega_\nu[\Omega^2_\nu-\bar{\omega}^2(2-\alpha)(1+\delta/3)]}. \quad (55)$$

The isoscalar values (53), (54) are obtained by assuming $\alpha = 1$.

1.7. Sum Rules. *1.7.1. Magnetic Case.* The magnetic dipole operator (37) is not Hermitian. By definition it is a linear combination of Hermitian operators (components of the angular momentum)

$$\hat{F}_{11} = -\frac{i}{2}\gamma(\hat{I}_x + i\hat{I}_y), \quad \hat{F}_{1-1} = \frac{i}{2}\gamma(\hat{I}_x - i\hat{I}_y).$$

This fact allows one to derive several useful relations:

$$[\hat{F}_{11},[H,\hat{F}_{1-1}]]=\frac{\gamma^2}{4}([\hat{I}_x,[H,\hat{I}_x]]+[\hat{I}_y,[H,\hat{I}_y]]),$$

$$\langle 0|\hat{F}_{11}|\nu\rangle\langle\nu|\hat{F}_{1-1}|0\rangle=\frac{\gamma^2}{2}(|\langle\nu|\hat{I}_x|0\rangle|^2+|\langle\nu|\hat{I}_y|0\rangle|^2)=$$
$$=-(|\langle\nu|\hat{F}_{11}|0\rangle|^2+|\langle\nu|\hat{F}_{1-1}|0\rangle|^2).$$

Using these formulae and the standard sum rule for a Hermitian operator [37]

$$\sum_\nu(E_\nu-E_0)|\langle\nu|\hat{I}_i|0\rangle|^2=\frac{1}{2}\langle 0|[\hat{I}_i,[H,\hat{I}_i]]|0\rangle,$$

one immediately obtains the sum rule for $\hat{F}_{1\pm1}$:

$$\sum_\nu(E_\nu-E_0)(|\langle\nu|\hat{F}_{11}|0\rangle|^2+|\langle\nu|\hat{F}_{1-1}|0\rangle|^2)=-\langle 0|[\hat{F}_{11},[H,\hat{F}_{1-1}]]|0\rangle. \quad (56)$$

It can also be calculated in a more direct way:

$$\langle 0|[\hat{F}_{11},[H,\hat{F}_{1-1}]]|0\rangle=$$
$$=\sum_\nu(E_\nu-E_0)(\langle 0|\hat{F}_{11}|\nu\rangle\langle\nu|\hat{F}_{1-1}|0\rangle+\langle 0|\hat{F}_{1-1}|\nu\rangle\langle\nu|\hat{F}_{11}|0\rangle)=$$
$$=\sum_\nu(E_\nu-E_0)(\langle 0|\hat{F}_{11}|\nu\rangle\langle 0|\hat{F}^\dagger_{1-1}|\nu\rangle^*+\langle 0|\hat{F}_{1-1}|\nu\rangle\langle 0|\hat{F}^\dagger_{11}|\nu\rangle^*). \quad (57)$$

Using here the Hermitian conjugation properties (38) of the operator $\hat{F}_{1\mu}$, one reproduces formula (56).

The double commutator is calculated with the help of (6) and (38):

$$[\hat{F}_{1\phi},[H,\hat{F}_{1\phi'}]]=\frac{15}{2\pi}\bar{\chi}\sum_i^Z\sum_j^N\sum_{\nu,\sigma,\epsilon}(-1)^\nu C^{1\phi}_{2\nu,2\sigma}C^{1\phi'}_{2-\nu,2\epsilon}r^2_{2\epsilon}(i)r^2_{2\sigma}(j)\mu_N^2. \quad (58)$$

Taking into account axial symmetry, one finds the ground state matrix element of (58) (in the Hartree–Fock approximation)

$$\frac{\langle 0|[\hat{F}_{1\phi},[H,\hat{F}_{1\phi'}]]|0\rangle}{\mu_N^2}=\frac{15}{2\pi}\bar{\chi}\sum_\nu(-1)^\nu C^{1\phi}_{2\nu,20}C^{1\phi'}_{2-\nu,20}R^p_{20}R^n_{20}=$$
$$=\frac{15}{8\pi}\delta_{\phi',-\phi}\bar{\chi}(C^{1\phi}_{2\phi,20}R^{\rm eq}_{20})^2.$$

It is obvious that this expression is different from zero only for $\phi = \pm 1$. Hence, the final expression for the right-hand side of (56) is

$$\langle 0|[\hat{F}_{11},[H,\hat{F}_{1-1}]]|0\rangle = \frac{9}{16\pi}\bar{\chi}(R_{20}^{\mathrm{eq}})^2\mu_N^2 = \\ = -\frac{1-\alpha}{4\pi}Q_{00}m\bar{\omega}^2\delta^2\mu_N^2 \equiv -(1-\alpha)\Sigma_0, \quad (59)$$

where, for the sake of convenience, the notation $\Sigma_0 = \dfrac{m\bar{\omega}^2}{4\pi}Q_{00}\delta^2\mu_N^2$ is introduced. The left-hand side of (56) is calculated trivially by multiplying the right-hand side of (43) by E_{sc} and adding it to the right-hand side of (44) multiplied by E_{iv}:

$$\Sigma_{\mathrm{tot}} = \sum_\nu (E_\nu - E_0)\left(|\langle\nu|\hat{F}_{11}|0\rangle|^2 + |\langle\nu|\hat{F}_{1-1}|0\rangle|^2\right) = \\ = 2\left(E_{\mathrm{sc}}|\langle \mathrm{sc}|\hat{F}_{11}|0\rangle|^2 + E_{\mathrm{iv}}|\langle \mathrm{iv}|\hat{F}_{11}|0\rangle|^2\right) = \\ = \Sigma_{\mathrm{sc}} + \Sigma_{\mathrm{iv}} = (1-\alpha)\Sigma_0, \quad (60)$$

where

$$\Sigma_{\mathrm{sc}} = \frac{[E_{\mathrm{sc}}^2 - 2(1+\delta/3)(\hbar\bar{\omega})^2]}{(E_{\mathrm{sc}}^2 - E_{\mathrm{iv}}^2)}(1-\alpha)\Sigma_0 \quad (61)$$

and

$$\Sigma_{\mathrm{iv}} = \frac{[E_{\mathrm{iv}}^2 - 2(1+\delta/3)(\hbar\bar{\omega})^2]}{(E_{\mathrm{iv}}^2 - E_{\mathrm{sc}}^2)}(1-\alpha)\Sigma_0. \quad (62)$$

So, one sees that the sum rule (56) is fulfilled.

1.7.2. Electric Case. The sum rule for $\hat{F}_{2\pm1}$ can easily be obtained by replacing in formula (57) the operators $\hat{F}_{1\pm1}$ by the operators $\hat{F}_{2\pm1}$ and using the Hermitian conjugation properties (47) of the operator $\hat{F}_{2\mu}$:

$$\sum_\nu (E_\nu - E_0)(|\langle\nu|\hat{F}_{21}|0\rangle|^2 + |\langle\nu|\hat{F}_{2-1}|0\rangle|^2) = -\langle 0|[\hat{F}_{21},[H,\hat{F}_{2-1}]]|0\rangle. \quad (63)$$

The double commutator is calculated with the help of (6) and (47):

$$[\hat{F}_{2\phi},[H,\hat{F}_{2\phi'}]] = -20\beta^2\frac{\hbar^2}{m}\sum_i^Z\sum_{\lambda,\sigma} C_{2\phi,2\phi'}^{\lambda\sigma}\{{}^{112}_{\lambda 21}\}r_{\lambda\sigma}^2(i). \quad (64)$$

Taking into account axial symmetry, one finds the ground state matrix element of (64):

$$\langle 0|[\hat{F}_{2\phi},[H,\hat{F}_{2\phi'}]]|0\rangle = -20\beta^2\frac{\hbar^2}{m}\delta_{\phi,-\phi'}\sum_{\lambda=0,2} C_{2\phi,2-\phi}^{\lambda 0}\{{}^{112}_{\lambda 21}\}R_{\lambda 0}^p = \\ = -2\beta^2\frac{\hbar^2}{m}\delta_{\phi,-\phi'}\left((-1)^\phi\frac{2}{\sqrt{3}}R_{00}^p + \frac{1}{\sqrt{6}}R_{20}^p\right). \quad (65)$$

Taking here $\phi = 1$ we obtain the final expression for the right-hand side of (63)

$$\langle 0|[\hat{F}_{21},[H,\hat{F}_{2-1}]]|0\rangle = -2\beta^2\frac{\hbar^2}{m}\left(\frac{2}{3}Q_{00}^p + \frac{1}{6}Q_{20}^p\right) = -e^2\frac{\hbar^2}{m}\frac{5}{4\pi}Q_{00}\left(1+\frac{\delta}{3}\right).$$

The left-hand side of (63) is calculated by summing expressions (51), (52), (53), and (54) multiplied by the respective energies. It is convenient to calculate the isovector and isoscalar contributions separately. The contribution of the isovector modes is

$$2\left(E_{\rm sc}|\langle {\rm sc}|\hat{F}_{21}|0\rangle|^2 + E_{\rm iv}|\langle {\rm iv}|\hat{F}_{21}|0\rangle|^2\right) = \frac{\beta^2\hbar^2}{3m}\left(Q_{00}+\frac{1}{4}Q_{20}\right) = \\ = e^2\frac{\hbar^2}{m}\frac{5}{8\pi}Q_{00}\left(1+\frac{\delta}{3}\right). \quad (66)$$

Exactly the same result is obtained for isoscalar modes:

$$2\left(\hbar\Omega_0|\langle\Omega_0|\hat{F}_{21}|0\rangle|^2 + E_{\rm is}|\langle {\rm is}|\hat{F}_{21}|0\rangle|^2\right) = e^2\frac{\hbar^2}{m}\frac{5}{8\pi}Q_{00}\left(1+\frac{\delta}{3}\right). \quad (67)$$

Hence the sum rule (63) is fulfilled.

It is interesting to compare the contributions of the scissors mode and the rotational mode. The scissors mode (for small δ) yields:

$$2E_{\rm sc}|\langle {\rm sc}|\hat{F}_{21}|0\rangle|^2 \simeq \frac{5}{128\pi}e^2\frac{\hbar^2}{m}Q_{00}\delta^2. \quad (68)$$

The rotational mode yields:

$$2\hbar\Omega_0|\langle\Omega_0|\hat{F}_{21}|0\rangle|^2 = \frac{5}{8\pi}e^2\frac{\hbar^2}{m}Q_{00}\frac{\delta^2}{1+\delta/3}. \quad (69)$$

It is seen that the contribution of the rotational mode is approximately 16 times larger than the one of the scissors mode. This is a very significant number demonstrating the importance of excluding the spurious state from the theoretical results. Indeed, to describe correctly such a subtle phenomenon as the scissors mode, it is compulsory to eliminate the errors from spurious motion whose value can be by an order of magnitude larger than the phenomenon under consideration.

2. RANDOM PHASE APPROXIMATION (RPA)

In this section we now want to derive the analogous equations for energies and transition probabilities from standard RPA theory. RPA equations in the notation of [9] are

$$\sum_{n,j}\left\{\left[\delta_{ij}\delta_{mn}(\epsilon_m-\epsilon_i)+\bar{v}_{mjin}\right]X_{nj}+\bar{v}_{mnij}Y_{nj}\right\} = \hbar\Omega X_{mi},$$
$$\sum_{n,j}\left\{\bar{v}_{ijmn}X_{nj}+\left[\delta_{ij}\delta_{mn}(\epsilon_m-\epsilon_i)+\bar{v}_{inmj}\right]Y_{nj}\right\} = -\hbar\Omega Y_{mi}. \quad (70)$$

According to the definition of the schematic model in [9], the matrix elements of the residual interaction corresponding to the Hamiltonian (6) are written as

$$\bar{v}_{mjin} = \kappa_{\tau\tau'} \mathcal{Q}^{\tau *}_{im} \mathcal{Q}^{\tau'}_{jn}$$

with $\mathcal{Q}_{im} \equiv \langle i|q_{21}|m\rangle$ and $\kappa_{nn} = \kappa_{pp} = \kappa, \quad \kappa_{np} = \bar{\kappa}$. This interaction distinguishes between protons and neutrons, so we have to introduce the isospin indices τ, τ' into the set of RPA equations (70):

$$\begin{gathered}
(\epsilon^{\tau}_{m} - \epsilon^{\tau}_{i}) X^{\tau}_{mi} + \sum_{n,j,\tau'} \kappa_{\tau\tau'} \mathcal{Q}^{\tau *}_{im} \mathcal{Q}^{\tau'}_{jn} X^{\tau'}_{nj} + \sum_{n,j,\tau'} \kappa_{\tau\tau'} \mathcal{Q}^{\tau *}_{im} \mathcal{Q}^{\tau'}_{nj} Y^{\tau'}_{nj} = \hbar\Omega X^{\tau}_{mi}, \\
\sum_{n,j,\tau'} \kappa_{\tau\tau'} \mathcal{Q}^{\tau *}_{mi} \mathcal{Q}^{\tau'}_{jn} X^{\tau'}_{nj} + (\epsilon^{\tau}_{m} - \epsilon^{\tau}_{i}) Y^{\tau}_{mi} + \sum_{n,j,\tau'} \kappa_{\tau\tau'} \mathcal{Q}^{\tau *}_{mi} \mathcal{Q}^{\tau'}_{nj} Y^{\tau'}_{nj} = -\hbar\Omega Y^{\tau}_{mi}.
\end{gathered} \tag{71}$$

The solution of these equations is

$$X^{\tau}_{mi} = \frac{\mathcal{Q}^{\tau *}_{im}}{E - \epsilon^{\tau}_{mi}} K^{\tau}, \quad Y^{\tau}_{mi} = -\frac{\mathcal{Q}^{\tau *}_{mi}}{E + \epsilon^{\tau}_{mi}} K^{\tau} \tag{72}$$

with $E = \hbar\Omega$, $\epsilon^{\tau}_{mi} = \epsilon^{\tau}_{m} - \epsilon^{\tau}_{i}$ and $K^{\tau} = \sum_{\tau'} \kappa_{\tau\tau'} C^{\tau'}$.

The constant C^{τ} is defined as $C^{\tau} = \sum_{n,j} (\mathcal{Q}^{\tau}_{jn} X^{\tau}_{nj} + \mathcal{Q}^{\tau}_{nj} Y^{\tau}_{nj})$. Using here the expressions for X^{τ}_{nj} and Y^{τ}_{nj} given above, one derives the useful relation

$$C^{\tau} = 2S^{\tau} K^{\tau} = 2S^{\tau} \sum_{\tau'} \kappa_{\tau\tau'} C^{\tau'}, \tag{73}$$

where the following notation is introduced:

$$S^{\tau} = \sum_{mi} |\mathcal{Q}^{\tau}_{mi}|^2 \frac{\epsilon^{\tau}_{mi}}{E^2 - (\epsilon^{\tau}_{mi})^2}. \tag{74}$$

Let us write out the relation (73) in detail

$$\begin{aligned}
C^{n} - 2S^{n}(\kappa C^{n} + \bar{\kappa} C^{p}) = 0, \\
C^{p} - 2S^{p}(\bar{\kappa} C^{n} + \kappa C^{p}) = 0.
\end{aligned} \tag{75}$$

The condition for existence of a nontrivial solution of this set of equations gives the secular equation

$$(1 - 2S^{n}\kappa)(1 - 2S^{p}\kappa) - 4S^{n} S^{p} \bar{\kappa}^2 = 0. \tag{76}$$

Making linear combinations of the two equations in (75), we write them in terms of isoscalar and isovector constants $C = C^n + C^p$, $\bar{C} = C^n - C^p$

$$\begin{aligned} C - 2(S^n + S^p)\kappa_0 C - 2(S^n - S^p)\kappa_1 \bar{C} = 0, \\ \bar{C} - 2(S^n - S^p)\kappa_0 C - 2(S^n + S^p)\kappa_1 \bar{C} = 0. \end{aligned} \tag{77}$$

Approximation (17) allows us to decouple the equations for isoscalar and isovector constants. Really, in this case $S^n = S^p \equiv S/2$; hence, we obtain two secular equations

$$1 - 2S\kappa_0 = 0, \quad \text{or} \quad 1 - S\kappa = S\bar{\kappa} \tag{78}$$

in the isoscalar case and

$$1 - 2S\kappa_1 = 0, \quad \text{or} \quad 1 - S\kappa = -S\bar{\kappa} \tag{79}$$

in the isovector one, the difference of both lies in the strength constants only. Having in mind the relation $\kappa_1 = \alpha\kappa_0$, we come to the conclusion that it is sufficient to analyze the isovector case only — the results for isoscalar one are obtained by assuming $\alpha = 1$.

2.1. Eigenfrequencies. The detailed expression for the isovector secular equation is

$$\frac{1}{2\kappa_1} = \sum_{mi} |\mathcal{Q}_{mi}|^2 \frac{\epsilon_{mi}}{E^2 - \epsilon_{mi}^2}. \tag{80}$$

The operator $\mathcal{Q} = q_{21}$ has only two types of nonzero matrix elements $\mathcal{Q}_{mi}$ in the deformed oscillator basis. Matrix elements of the first type couple states of the same major shell. All corresponding transition energies are degenerate: $\epsilon_m - \epsilon_i = \hbar(\omega_x - \omega_z) \equiv \epsilon_0$. Matrix elements of the second type couple states of the different major shells with $\Delta N = 2$. All corresponding transition energies are degenerate, too: $\epsilon_m - \epsilon_i = \hbar(\omega_x + \omega_z) \equiv \epsilon_2$. Therefore, the secular equation can be rewritten as

$$\frac{1}{2\kappa_1} = \frac{\epsilon_0 \mathcal{Q}_0}{E^2 - \epsilon_0^2} + \frac{\epsilon_2 \mathcal{Q}_2}{E^2 - \epsilon_2^2}. \tag{81}$$

The sums $\mathcal{Q}_0 = \sum_{mi(\Delta N=0)} |\mathcal{Q}_{mi}|^2$ and $\mathcal{Q}_2 = \sum_{mi(\Delta N=2)} |\mathcal{Q}_{mi}|^2$ can be calculated analytically (see Appendix B):

$$\mathcal{Q}_0 = \frac{Q_{00}}{m\bar{\omega}^2}\epsilon_0, \quad \mathcal{Q}_2 = \frac{Q_{00}}{m\bar{\omega}^2}\epsilon_2. \tag{82}$$

Let us transform the secular equation (81) in the polynomial form

$$E^4 - E^2[(\epsilon_0^2 + \epsilon_2^2) + 2\kappa_1(\epsilon_0\mathcal{Q}_0 + \epsilon_2\mathcal{Q}_2)] + [\epsilon_0^2\epsilon_2^2 + 2\kappa_1\epsilon_0\epsilon_2(\epsilon_0\mathcal{Q}_2 + \epsilon_2\mathcal{Q}_0)] = 0.$$

Using here the expressions (82) for $\mathcal{Q}_0$, $\mathcal{Q}_2$ and the self-consistent value of the strength constant (A.3), we find

$$E^4 - E^2(1-\alpha/2)(\epsilon_0^2+\epsilon_2^2) + (1-\alpha)\epsilon_0^2\epsilon_2^2 = 0,$$

or

$$\Omega^4 - \Omega^2(2-\alpha)\omega_+^2 + (1-\alpha)\omega_-^4 = 0, \tag{83}$$

with the notation $\omega_+^2 = \omega_x^2 + \omega_z^2$ and $\omega_-^4 = (\omega_x^2 - \omega_z^2)^2$. This result coincides with that of [7]. By a trivial rearrangement of the terms in (83) one obtains the useful relation

$$\Omega^2(\Omega^2 - \omega_+^2) = (1-\alpha)(\Omega^2\omega_+^2 - \omega_-^4). \tag{84}$$

Inserting expressions (A.3) for ω_x^2,ω_z^2 into (83), we find $\omega_+^2 = 2\bar{\omega}^2(1+\delta/3)$, $\omega_-^4 = 4\delta^2\bar{\omega}^4$ and reproduce formula (26) for the isovector case. Taking $\alpha = 1$, we reproduce also formula (22) for the isoscalar case.

2.2. $B(E2)$ **Factors.** According to [9], the transition probability for the one-body operator $\hat{F} = \sum_{s=1}^{A} \hat{f}_s$ is calculated by means of the formulae

$$\begin{aligned} \langle 0|\hat{F}^\tau|\nu\rangle &= \sum_{mi}(f_{im}^\tau X_{mi}^{\tau\nu} + f_{mi}^\tau Y_{mi}^{\tau\nu}), \\ \langle \nu|\hat{F}^\tau|0\rangle &= \sum_{mi}(f_{mi}^\tau X_{mi}^{\tau\nu} + f_{im}^\tau Y_{mi}^{\tau\nu}). \end{aligned} \tag{85}$$

Quadrupole excitations are described by the operator (47) with $\hat{f}_{2\mu} = er^2Y_{2\mu} = \tilde{e}\mathcal{Q}$, where $\tilde{e} = e\sqrt{\dfrac{5}{16\pi}}$. Using the expressions (72) for X_{mi}^τ, Y_{mi}^τ, we get

$$\langle 0|\hat{F}_{21}^p|\nu\rangle = 2\tilde{e}K_\nu^p \sum_{mi} |\mathcal{Q}_{mi}^p|^2 \frac{\epsilon_{mi}^p}{E_\nu^2 - (\epsilon_{mi}^p)^2} = 2\tilde{e}K_\nu^p S_\nu^p = \tilde{e}C_\nu^p. \tag{86}$$

The constant C_ν^p is determined by the normalization condition

$$\delta_{\nu,\nu'} = \sum_{mi,\tau}(X_{mi}^{\tau\nu*}X_{mi}^{\tau\nu'} - Y_{mi}^{\tau\nu*}Y_{mi}^{\tau\nu'}),$$

that gives

$$\begin{aligned} &\frac{1}{(C_\nu^p)^2} = \\ &= E_\nu \sum_{mi}\left[\frac{|\mathcal{Q}_{mi}^p|^2}{(S_\nu^p)^2}\frac{\epsilon_{mi}^p}{[E_\nu^2 - (\epsilon_{mi}^p)^2]^2} + \frac{(C_\nu^n)^2}{(C_\nu^p)^2}\frac{|\mathcal{Q}_{mi}^n|^2}{(S_\nu^n)^2}\frac{\epsilon_{mi}^n}{[E_\nu^2 - (\epsilon_{mi}^n)^2]^2}\right]. \end{aligned} \tag{87}$$

The ratio C^n/C^p is determined by any of the equations (75):

$$\frac{C^n}{C^p} = \frac{1-2S^p\kappa}{2S^p\bar{\kappa}} = \frac{2S^n\bar{\kappa}}{1-2S^n\kappa}. \tag{88}$$

Formula (87) is considerably simplified by the approximation (17), when $S^p = S^n \equiv S/2$, $\epsilon^p_{mi} = \epsilon^n_{mi}$, $\mathcal{Q}^p_{mi} = \mathcal{Q}^n_{mi}$. Applying the second forms of formulae (78), (79) it is easy to find that in this case $C^n/C^p = \pm 1$. As a result, the final expression for $B(E2)$ value is

$$B(E2)_\nu = 2|\langle 0|\hat{F}^p_{21}|\nu\rangle|^2 = 2\tilde{e}^2\left(16E_\nu\kappa_1^2\sum_{mi}|\mathcal{Q}_{mi}|^2\frac{\epsilon_{mi}}{(E^2_\nu-\epsilon^2_{mi})^2}\right)^{-1}. \tag{89}$$

With the help of formulae (82) this expression can be transformed into

$$B(E2)_\nu = \frac{5}{8\pi}\frac{e^2Q_{00}}{m\bar{\omega}^2\alpha^2E_\nu}\left[\frac{\epsilon_0^2}{(E^2_\nu-\epsilon^2_0)^2}+\frac{\epsilon_2^2}{(E^2_\nu-\epsilon^2_2)^2}\right]^{-1} = \\ = \frac{5}{16\pi}\frac{e^2\hbar Q_{00}}{m\bar{\omega}^2\Omega_\nu}\frac{(\Omega^2_\nu\omega^2_+-\omega^4_-)^2}{\Omega^4_\nu\omega^2_+-2\Omega^2_\nu\omega^4_-+\omega^2_+\omega^4_-}. \tag{90}$$

At first sight, this expression has nothing in common with (55). Nevertheless, it can be shown that they are identical. To this end, we analyze carefully the denominator of the last expression in (90). Summing it with the secular equation (83) (multiplied by ω^2_+), which obviously does not change its value, we find after elementary combinations

$$\begin{aligned}\mathrm{Denom} &= \Omega^4_\nu\omega^2_+-2\Omega^2_\nu\omega^4_-+\omega^2_+\omega^4_-+\omega^2_+[\Omega^4_\nu-\Omega^2_\nu(2-\alpha)\omega^2_+ +(1-\alpha)\omega^4_-] = \\ &= \omega^2_+\Omega^2_\nu[2\Omega^2_\nu-(2-\alpha)\omega^2_+]-\omega^4_-[2\Omega^2_\nu-(2-\alpha)\omega^2_+] = \\ &= (\Omega^2_\nu\omega^2_+-\omega^4_-)[2\Omega^2_\nu-(2-\alpha)\omega^2_+]. \end{aligned}\tag{91}$$

This result allows us to write the final expression as

$$B(E2)_\nu = \frac{5}{16\pi}\frac{e^2\hbar}{m\bar{\omega}^2}Q_{00}\frac{\Omega^2_\nu\omega^2_+-\omega^4_-}{\Omega_\nu[2\Omega^2_\nu-(2-\alpha)\omega^2_+]}, \tag{92}$$

which coincides with (55). By simple transformations this formula is reduced to the result of Hamamoto and Nazarewicz [7] (taking into account that they published it without the constant factor $\dfrac{5}{32\pi}\dfrac{e^2\hbar}{m\omega_0}Q^0_{00}$).

2.3. $B(M1)$ **Factors.** In accordance with formulae (37), (72), (85) the magnetic transition matrix element is given by

$$\langle 0|\hat{F}^p_{11}|\nu\rangle = K^p_\nu \sum_{mi}\left[\frac{(\hat{f}^p_{11})_{im}\mathcal{Q}^{p*}_{im}}{E_\nu - \epsilon^p_{mi}} - \frac{(\hat{f}^p_{11})_{mi}\mathcal{Q}^{p*}_{mi}}{E_\nu + \epsilon^p_{mi}}\right]. \quad (93)$$

As is shown in Appendix B, the matrix element $(f^p_{11})_{im}$ is proportional to $\mathcal{Q}^p_{im}$ (formula (B.16). So, expression (93) is reduced to

$$\langle 0|\hat{F}^p_{11}|\nu\rangle = -K^p_\nu \frac{\tilde{e}\hbar}{2c\sqrt{5}}(\omega_x^2 - \omega_z^2)^p \sum_{mi}\left[\frac{\mathcal{Q}^p_{im}\mathcal{Q}^{p*}_{im}}{\epsilon^p_{im}(E_\nu - \epsilon^p_{mi})} - \frac{\mathcal{Q}^p_{mi}\mathcal{Q}^{p*}_{mi}}{\epsilon^p_{mi}(E_\nu + \epsilon^p_{mi})}\right] =$$

$$= K^p_\nu \frac{\tilde{e}\hbar}{c\sqrt{5}}(\omega_x^2 - \omega_z^2)^p E_\nu \sum_{mi}\frac{|\mathcal{Q}^p_{mi}|^2}{\epsilon^p_{mi}[E^2_\nu - (\epsilon^p_{mi})^2]}. \quad (94)$$

With the help of approximation (17) and expressions (82) for $\mathcal{Q}_0$, $\mathcal{Q}_2$ we find

$$\langle 0|\hat{F}^p_{11}|\nu\rangle = \frac{C^p_\nu}{2S^p_\nu}\frac{\tilde{e}\hbar}{c\sqrt{5}}(\omega_x^2 - \omega_z^2)\frac{Q_{00}}{2m\bar{\omega}^2}\left(\frac{E_\nu}{E^2_\nu - \epsilon^2_0} + \frac{E_\nu}{E^2_\nu - \epsilon^2_2}\right) =$$

$$= -2\kappa_1 C^p_\nu \frac{\tilde{e}}{c\sqrt{5}}(\omega_x^2 - \omega_z^2)\frac{Q_{00}}{m\bar{\omega}^2}\frac{\Omega_\nu(\Omega^2_\nu - \omega^2_+)}{\alpha(\Omega^2_\nu\omega^2_+ - \omega^4_-)} =$$

$$= \frac{C^p_\nu}{2}\frac{\tilde{e}}{c\sqrt{5}}(\omega_x^2 - \omega_z^2)\frac{1-\alpha}{\Omega_\nu}. \quad (95)$$

Relation (84) and the self-consistent value of the strength constant $\kappa_1 = \alpha\kappa_0$ were used in the last step. For the magnetic transition probability we have

$$B(M1)_\nu = 2|\langle 0|\hat{F}^p_{11}|\nu\rangle|^2 = 2\frac{(C^p_\nu)^2}{4}\frac{\tilde{e}^2}{5c^2}\omega^4_-\frac{(1-\alpha)^2}{\Omega^2_\nu} =$$

$$= \frac{\omega^4_-}{20c^2}\frac{(1-\alpha)^2}{\Omega^2_\nu}B(E2)_\nu. \quad (96)$$

This relation between $B(M1)$ and $B(E2)$ was also found (up to the factor $1/(20c^2)$) by Hamamoto and Nazarewicz [7]. Substituting expression (92) for $B(E2)$ into (96) we reproduce (with the help of relation (84)) formula (46).

2.4. «Synthetic» Scissors and Spurious State. The nature of collective excitations calculated with the method of Wigner function moments is quite easily revealed analyzing the role of collective variables describing the phenomenon. The solution of this problem in the RPA approach is not so obvious. That is why the nature of the low-lying states has often been established by considering

overlaps of these states with the «pure scissors state» [27, 28] or «synthetic state» [7] produced by the action of the scissors operator

$$\hat{S}_x = \mathcal{N}^{-1}(\langle I_x^{n\,2}\rangle \hat{I}_x^p - \langle I_x^{p\,2}\rangle \hat{I}_x^n)$$

on the ground state

$$|\mathrm{Syn}\rangle = \hat{S}_x|0\rangle.$$

In the considered model the overlap of the «synthetic» state with the real scissors mode (and with IVGQR) can be calculated analytically. Surprisingly, it was not done until now. Let us at first modify the definition of the «synthetic» state. Due to axial symmetry one can use the $\hat{I}_y^\tau$ component instead of $\hat{I}_x^\tau$, or any of their linear combinations, for example, the $\mu = 1$ component of the magnetic operator $\hat{F}_{1\mu}^\tau$, which is much more convenient for us. The terms $\langle I_x^{\tau\,2}\rangle$ are introduced to ensure the orthogonality of the synthetic scissors to the spurious state $|\mathrm{Sp}\rangle = (\hat{I}^n + \hat{I}^p)|0\rangle$. However, we do not need these terms because the collective states $|\nu\rangle$ of our model are already orthogonal to $|\mathrm{Sp}\rangle$ (see below); hence, the overlaps $\langle \mathrm{Syn}|\nu\rangle$ will be free from any admixtures of $|\mathrm{Sp}\rangle$. So, we use the following definitions of the synthetic and spurious states:

$$|\mathrm{Syn}\rangle = \mathcal{N}^{-1}(\hat{F}_{11}^p - \hat{F}_{11}^n)|0\rangle, \quad |\mathrm{Sp}\rangle = (\hat{F}_{11}^p + \hat{F}_{11}^n)|0\rangle.$$

Let us demonstrate the orthogonality of the spurious state to all the rest of the states $|\nu\rangle$. As the first step it is necessary to show that the secular equation (76) has the solution $E = 0$. We need the expression for $S^\tau(E = 0) \equiv S^\tau(0)$. In accordance with (74), we have

$$S^\tau(E) = \left[\frac{\epsilon_0 \mathcal{Q}_0}{E^2 - \epsilon_0^2} + \frac{\epsilon_2 \mathcal{Q}_2}{E^2 - \epsilon_2^2}\right]^\tau, \quad S^\tau(0) = -\left[\frac{\mathcal{Q}_0}{\epsilon_0} + \frac{\mathcal{Q}_2}{\epsilon_2}\right]^\tau.$$

The expressions for $\mathcal{Q}_0^\tau$, $\mathcal{Q}_2^\tau$ are easily extracted from formulae (B.10), (B.11):

$$\begin{aligned} \mathcal{Q}_0^\tau &= \frac{\hbar}{m} Q_{00}^\tau \left[\frac{1 + \frac{4}{3}\delta}{\omega_x} - \frac{1 - \frac{2}{3}\delta}{\omega_z}\right]^\tau, \\ \mathcal{Q}_2^\tau &= \frac{\hbar}{m} Q_{00}^\tau \left[\frac{1 + \frac{4}{3}\delta}{\omega_x} + \frac{1 - \frac{2}{3}\delta}{\omega_z}\right]^\tau. \end{aligned} \tag{97}$$

So we find

$$S^\tau(0) = -\frac{\hbar}{m}Q^\tau_{00}\left[\frac{1+\frac{4}{3}\delta}{\omega_x}\left(\frac{1}{\epsilon_2}+\frac{1}{\epsilon_0}\right)+\frac{1-\frac{2}{3}\delta}{\omega_z}\left(\frac{1}{\epsilon_2}-\frac{1}{\epsilon_0}\right)\right]^\tau = \\ = -\frac{\hbar^2}{m}\frac{4\delta^\tau Q^\tau_{00}}{\epsilon^\tau_2\epsilon^\tau_0} = -\frac{1}{m}\frac{3Q^\tau_{20}}{(\omega^2_x-\omega^2_z)^\tau}, \quad (98)$$

where, in accordance with (B.12),

$$(\omega^2_x-\omega^2_z)^p = -\frac{6}{m}(\kappa Q^p_{20}+\bar\kappa Q^n_{20}), \quad (\omega^2_x-\omega^2_z)^n = -\frac{6}{m}(\kappa Q^n_{20}+\bar\kappa Q^p_{20}). \quad (99)$$

Finally, we get

$$2S^p(0) = \frac{Q^p_{20}}{\kappa Q^p_{20}+\bar\kappa Q^n_{20}}, \quad 1-2S^p(0)\kappa = \frac{\bar\kappa Q^n_{20}}{\kappa Q^p_{20}+\bar\kappa Q^n_{20}}, \\ 2S^n(0) = \frac{Q^n_{20}}{\kappa Q^n_{20}+\bar\kappa Q^p_{20}}, \quad 1-2S^n(0)\kappa = \frac{\bar\kappa Q^p_{20}}{\kappa Q^n_{20}+\bar\kappa Q^p_{20}}.$$

It is easy to see that substituting these expressions into (76) we obtain an identity; therefore, the secular equation has a zero energy solution.

For the second step it is necessary to calculate the overlap $\langle \mathrm{Sp}|\nu\rangle$. Summing (94) with an analogous expression for neutrons, we get

$$\langle \mathrm{Sp}|\nu\rangle = \frac{\tilde e\hbar}{c\sqrt{5}}E_\nu\sum_\tau K^\tau_\nu(\omega^2_x-\omega^2_z)^\tau\sum_{mi}\frac{|\mathcal{Q}^\tau_{mi}|^2}{\epsilon^\tau_{mi}(E^2_\nu-\epsilon^2_{mi})^\tau} = \\ = \frac{\tilde e\hbar}{c\sqrt{5}}E_\nu\sum_\tau K^\tau_\nu(\omega^2_x-\omega^2_z)^\tau\sum_{mi}\frac{|\mathcal{Q}^\tau_{mi}|^2\epsilon^\tau_{mi}}{(\epsilon^2_{mi})^\tau(E^2_\nu-\epsilon^2_{mi})^\tau}. \quad (100)$$

Applying the algebraical identity

$$\frac{1}{\epsilon^2(E^2-\epsilon^2)} = \frac{1}{E^2}\left(\frac{1}{\epsilon^2}+\frac{1}{E^2-\epsilon^2}\right)$$

and remembering the definition (74) of S^τ, we rewrite (100) as

$$\langle \mathrm{Sp}|\nu\rangle = \frac{\tilde e\hbar}{c\sqrt{5}E_\nu}\sum_\tau K^\tau_\nu(\omega^2_x-\omega^2_z)^\tau(S^\tau-S^\tau(0)) = \frac{\tilde e\hbar}{c\sqrt{5}}\frac{K^p_\nu}{E_\nu}\times \\ \times\left[(\omega^2_x-\omega^2_z)^p(S^p-S^p(0))+(\omega^2_x-\omega^2_z)^n(S^n-S^n(0))\frac{K^n_\nu}{K^p_\nu}\right]. \quad (101)$$

In accordance with (73) and (88),

$$\frac{K_\nu^n}{K_\nu^p} = \frac{1-2S^p\kappa}{2S^n\bar{\kappa}}. \tag{102}$$

Noting now (see formula (98)) that $(\omega_x^2-\omega_z^2)^\tau S^\tau(0) = -\dfrac{3}{m}Q_{20}^\tau$ and taking into account relations (99), we find

$$\begin{aligned}
\langle \mathrm{Sp}|\nu\rangle &= \beta\left\{[(\kappa Q_2^p+\bar{\kappa}Q_2^n)2S^p - Q_2^p]+[(\kappa Q_2^n+\bar{\kappa}Q_2^p)2S^n - Q_2^n]\frac{1-2S^p\kappa}{2S^n\bar{\kappa}}\right\} = \\
&= \beta\left\{[(2S^p\kappa-1)Q_2^p + 2S^p\bar{\kappa}Q_2^n] + [(2S^n\kappa-1)Q_2^n + 2S^n\bar{\kappa}Q_2^p)]\frac{1-2S^p\kappa}{2S^n\bar{\kappa}}\right\} = \\
&= \beta\left\{2S^p\bar{\kappa}Q_2^n + (2S^n\kappa-1)Q_2^n\frac{1-2S^p\kappa}{2S^n\bar{\kappa}}\right\} = \\
&= \beta\frac{Q_2^n}{2S^n\bar{\kappa}}\left\{2S^n\bar{\kappa}2S^p\bar{\kappa} - (1-2S^n\kappa)(1-2S^p\kappa)\right\} = 0,
\end{aligned} \tag{103}$$

where $\beta = -\frac{3}{m}\dfrac{\tilde{e}\hbar}{c\sqrt{5}}\dfrac{K_\nu^p}{E_\nu}$ and $Q_2 \equiv Q_{20}$. The expression in the last curly brackets coincides obviously with the secular equation (76) that proves the orthogonality of the spurious state to all physical states of the considered model. So we can conclude that, strictly speaking, this is not a spurious state, but one of the exact eigenstates of the model corresponding to the integral of motion $I^n + I^p$. In other words [9]: «In fact these excitations are not really spurious, but they represent a different type of motion which has to be treated separately». The same conclusion was made by N. Lo Iudice [29] who solved this problem approximately with the help of several assumptions (a small deformation limit, for example).

The problem of the «spurious» state being solved, the calculation of the overlaps $\langle \mathrm{Syn}|\nu\rangle$ becomes trivial. Really, we have shown that $\langle 0|\hat{F}_{11}^n + \hat{F}_{11}^p|\nu\rangle = 0$. That means that $\langle 0|\hat{F}_{11}^n|\nu\rangle = -\langle 0|\hat{F}_{11}^p|\nu\rangle$; hence, $\langle \mathrm{Syn}|\nu\rangle = \mathcal{N}^{-1}\langle 0|\hat{F}_{11}^p - \hat{F}_{11}^n|\nu\rangle = 2\mathcal{N}^{-1}\langle 0|\hat{F}_{11}^p|\nu\rangle$ and

$$U^2 \equiv |\langle \mathrm{Syn}|\nu\rangle|^2 = 2\mathcal{N}^{-2}B(M1)_\nu. \tag{104}$$

The nontrivial part of the problem is the calculation of the normalization factor $\mathcal{N}$. It is important not to forget about the time dependence of the synthetic state which should be determined by the external field:

$$|\mathrm{Syn}\,(t)\rangle = \mathcal{N}^{-1}[(\hat{F}_{11}^p - \hat{F}_{11}^n)\,\mathrm{e}^{-i\Omega t} + (\hat{F}_{11}^p - \hat{F}_{11}^n)^\dagger\,\mathrm{e}^{i\Omega t}]|0\rangle.$$

Then we have

$$\mathcal{N}^2 = 2\langle 0|(\hat{F}^p_{11} - \hat{F}^n_{11})^\dagger(\hat{F}^p_{11} - \hat{F}^n_{11})|0\rangle =$$
$$= 2\sum_{\rm ph}\langle 0|(\hat{F}^p_{11} - \hat{F}^n_{11})^\dagger|{\rm ph}\rangle\langle{\rm ph}|(\hat{F}^p_{11} - \hat{F}^n_{11})|0\rangle = 2\sum_{\rm ph}|\langle{\rm ph}|(\hat{F}^p_{11} - \hat{F}^n_{11})|0\rangle|^2 =$$
$$= 2\sum_{\tau,{\rm ph}}|\langle{\rm ph}|\hat{F}^\tau_{11}|0\rangle|^2 = 2\sum_{\tau,{\rm ph}}|(f^\tau_{11})_{\rm ph}|^2. \quad (105)$$

With the help of relation (B.16) we find

$$\mathcal{N}^2 = \frac{2}{5}\left(\frac{e\hbar}{2c}\right)^2\sum_{\tau,{\rm ph}}\left(\omega_-^4\frac{|\langle{\rm ph}|r^2Y_{21}|0\rangle|^2}{\epsilon^2_{\rm ph}}\right)^\tau =$$
$$= \frac{1}{8\pi}\left(\frac{e\hbar}{2c}\right)^2\sum_\tau(\omega_-^4)^\tau\left(\frac{\mathcal{Q}_0}{\epsilon_0^2} + \frac{\mathcal{Q}_2}{\epsilon_2^2}\right)^\tau. \quad (106)$$

Expressions for $\mathcal{Q}^\tau_0$, $\mathcal{Q}^\tau_2$, ω^τ_x, ω^τ_z are given by formulae (97), (B.12). To get a definite number, it is necessary to make some assumption concerning the relation between neutron and proton equilibrium characteristics. As usual, we apply the approximation (17), i.e., suppose $Q^n_{00} = Q^p_{00}$, $Q^n_{20} = Q^p_{20}$. It is easy to check that in this case formulae for $\omega^\tau_{x,z}$ are reduced to the ones for the isoscalar case, namely (A.3), and $\mathcal{Q}^\tau_0 = \mathcal{Q}_0/2$, $\mathcal{Q}^\tau_2 = \mathcal{Q}_2/2$, where $\mathcal{Q}_0$ and $\mathcal{Q}_2$ are given by (82). So we get

$$\mathcal{N}^2 = \frac{\omega_-^4}{8\pi}\left(\frac{e\hbar}{2c}\right)^2\frac{Q_{00}}{m\bar{\omega}^2}\left(\frac{1}{\epsilon_0} + \frac{1}{\epsilon_2}\right) = \frac{\delta}{2\pi}\frac{m\omega_x}{\hbar}Q_{00}\mu_N^2. \quad (107)$$

The estimation of the overlap for ^{156}Gd with $\delta = 0.27$ gives $\mathcal{N}^2 = 34.72\mu_N^2$ and $U^2 = 0.53$ (see Eq. (104)), that is two times larger than the result of [27] obtained in QRPA calculations with the Skyrme forces. The disagreement can naturally be attributed to the difference in forces and especially to the lack of pair correlations in our approach. In a small deformation limit $U^2 = \dfrac{1}{2}\sqrt{\dfrac{3}{2}} \approx 0.6$.

This is the maximum possible overlap of the «pure» (or «synthetic») scissors with the real scissors. The increasing of δ and /or taking into account pairing correlations decreases its value, which is confirmed by numerous microscopic calculations with various forces [4]. Such small overlap leads inevitably to the conclusion that the original model of counter rotating rigid rotors [30] has not very much in common with the real scissors mode, the correct description of which requires the proper treatment of the Fermi surface deformation and the coupling with IVGQR.

2.4.1. Superdeformation. The pair correlations are not considered in this paper. Nevertheless, our formulae (30), (46) can be successfully used for the description of superdeformed nuclei where the pairing is very weak [7, 30]. For example, applying them to the superdeformed nucleus ^{152}Dy ($\delta \simeq 0.6$, $\hbar\omega_0 = 41/A^{1/3}$ MeV), we get

$$E_{\rm iv} = 20.8 \text{ MeV}, \qquad B(M1)_{\rm iv} = 15.9\mu_N^2$$

for the isovector GQR and

$$E_{\rm sc} = 4.7 \text{ MeV}, \qquad B(M1)_{\rm sc} = 20.0\mu_N^2$$

for the scissors mode. There are not so many results of other calculations to compare with. As a matter of fact, there are only two papers considering this problem.

The phenomenological TRM model [30] predicts

$$E_{\rm iv} \simeq 26 \text{ MeV}, \quad B(M1)_{\rm iv} \simeq 26\mu_N^2, \quad E_{\rm sc} \simeq 6.1 \text{ MeV}, \quad B(M1)_{\rm sc} \simeq 22\mu_N^2.$$

The only existing microscopic calculation [7] in the framework of QRPA with separable forces gives

$$E_{\rm iv} \simeq 28 \text{ MeV}, \quad B(M1)_{\rm iv} \simeq 37\mu_N^2, \quad E_{\rm sc} \simeq 5-6 \text{ MeV}, \quad B(M1)_{1^+} \simeq 23\mu_N^2.$$

Here $B(M1)_{1^+}$ denotes the total $M1$ orbital strength carried by the calculated $K^\pi = 1^+$ QRPA excitations modes in the energy region below 20 MeV.

It is easy to see that in the case of IVGQR one can speak, at least, about qualitative agreement. Our results for $E_{\rm sc}$ and $B(M1)_{\rm sc}$ are in good agreement with that of phenomenological model and with $E_{\rm sc}$ and $B(M1)_{1^+}$ of Hamamoto and Nazarewicz.

It is possible to extract from the histogram of [7] the value of the overlap of calculated low-lying 1^+ excitations with the synthetic scissors state: $|\langle \mathrm{Syn}|1^+\rangle|^2 \approx 0.4$. The result of our calculation $U^2 = 0.43$ agrees with it very well. So, the comparison of our calculations with that of QRPA shows, that we have excellent agreement in superdeformed nuclei and rather large disagreement in moderately deformed nuclei. On the other hand, it is known [7] that pairing is very weak at the superdeformation and becomes important at moderate deformations. Therefore, as a consequence, the correct treatment of pair correlations is important for an accurate description of the scissors mode.

2.5. Equations of Motion. Let us look on WFM equations of motion from the RPA point of view. Is it possible to construct something similar in the RPA approach? Equations (11) are written for average values of operators and are valid for the description of the arbitrary amplitude motion. One should compare with RPA their linearized version (18), (19). The variables of these equations are the

variations of the above-mentioned average values. It is natural to suppose some correspondence between the variation of the average value of the operator $\hat{F}$ and the matrix element of the type $\langle 0|\hat{F}|\nu\rangle$ used to calculate transition probabilities. To check this idea we have to derive dynamical equations for matrix elements of the operators $r^2_{\lambda\mu}$, $\hat{p}^2_{\lambda\mu}$, and $(r\hat{p})_{\lambda\mu}$ and to compare them with the linearized equations (11). To this end, we combine the RPA equations (71) in accordance with the definition (85) of matrix elements:

$$\hbar\Omega_\nu \sum_{mi}(f^\tau_{im}X^{\tau\nu}_{mi} + f^\tau_{mi}Y^{\tau\nu}_{mi}) = \sum_{mi}\epsilon_{mi}(f^\tau_{im}X^{\tau\nu}_{mi} - f^\tau_{mi}Y^{\tau\nu}_{mi}) + \\ + K^\tau_\nu \sum_{mi}(f^\tau_{im}\mathcal{Q}^{\tau*}_{im} - f^\tau_{mi}\mathcal{Q}^{\tau*}_{mi}). \quad (108)$$

Taking into account the relations

$$\epsilon_{mi}f_{im} = [\hat{f}, H_0]_{im}, \quad \epsilon_{mi}f_{mi} = -[\hat{f}, H_0]_{mi},$$

one rewrites Eq. (108) as

$$\hbar\Omega_\nu\langle 0|\hat{F}^\tau|\nu\rangle = \sum_{mi}\{[\hat{f}^\tau, H^\tau_0]_{im}X^{\tau\nu}_{mi} + [\hat{f}^\tau, H^\tau_0]_{mi}Y^{\tau\nu}_{mi} + \\ + K^\tau_\nu(f^\tau_{im}\mathcal{Q}^{\tau*}_{im} - f^\tau_{mi}\mathcal{Q}^{\tau*}_{mi})\}. \quad (109)$$

The Hamiltonian of the axially deformed harmonic oscillator corresponding to the mean field (9) is

$$H^\tau_0(\mathbf{r}) = \sum_{s=1}^{N_\tau}\left\{\frac{\hat{\mathbf{p}}^2_s}{2m} + \frac{1}{2}m\omega^2\mathbf{r}^2_s + Z^\tau_{20}(\mathrm{eq})r^2_{20}(s)\right\}. \quad (110)$$

Let us consider the operator $\hat{f} = \sqrt{6}\, r^2_{21} = q_{21} = \mathcal{Q}$. Calculating the commutator

$$[r^2_{21}, H_0] = i\hbar\frac{2}{m}(r\hat{p})_{21}$$

we find from (109) the following equation:

$$\hbar\Omega_\nu\left\langle 0\left|\sum_{s=1}^{N_\tau}\mathcal{Q}^\tau_s\right|\nu\right\rangle = i\hbar\sqrt{6}\frac{2}{m}\sum_{mi}\{((r\hat{p})_{21})^\tau_{im}X^{\tau\nu}_{mi} + ((r\hat{p})_{21})^\tau_{mi}Y^{\tau\nu}_{mi}\} + \\ + K^\tau_\nu\sum_{mi}(\mathcal{Q}^\tau_{im}\mathcal{Q}^{\tau*}_{im} - \mathcal{Q}^\tau_{mi}\mathcal{Q}^{\tau*}_{mi}). \quad (111)$$

Taking into account relations $(\mathcal{Q}^*)_{im} = (\mathcal{Q})^*_{mi}$ and $|\mathcal{Q}_{mi}|^2 = |\mathcal{Q}_{im}|^2$, we find that the last sum in (111) is equal to zero. Applying again formula (85) and

introducing the notation $\hat{R}_{\lambda\mu} = \sum_{s=1}^{A}(r_s^2)_{\lambda\mu}$, $\hat{L}_{\lambda\mu} = \sum_{s=1}^{A}(r_s\hat{p}_s)_{\lambda\mu}$, we write (111) as

$$-i\Omega_\nu\langle 0|\hat{R}^\tau_{21}|\nu\rangle = \frac{2}{m}\langle 0|\hat{L}^\tau_{21}|\nu\rangle. \tag{112}$$

Identifying the matrix elements $\langle 0|\hat{R}^\tau_{21}|\nu\rangle$ and $\langle 0|\hat{L}^\tau_{21}|\nu\rangle$ with variables $\mathcal{R}^\tau_{21}$ and $\mathcal{L}^\tau_{21}$, respectively, we reproduce the variation of the first equation in (11), or first equations of (18), (19) (having in mind the time dependence via $\mathrm{e}^{-i\Omega t}$).

Let us consider the operator $\hat{f} = (r\hat{p})_{21}$. The required commutator is evaluated to be

$$[(r\hat{p})_{21}, H_0] = i\frac{\hbar}{m}\hat{p}^2_{21} - i\hbar m\omega^2 r^2_{21} - i\frac{\hbar}{\sqrt{6}}Z_{20}(\mathrm{eq})r^2_{21}.$$

With this result equation (109) looks as

$$\begin{aligned}\hbar\Omega_\nu\langle 0|\hat{L}^\tau_{21}|\nu\rangle &= i\frac{\hbar}{m}\langle 0|\hat{P}^\tau_{21}|\nu\rangle - i\hbar m\omega^2\langle 0|\hat{R}^\tau_{21}|\nu\rangle - \\ &- i\frac{\hbar}{\sqrt{6}}Z^\tau_{20}(\mathrm{eq})\langle 0|\hat{R}^\tau_{21}|\nu\rangle + K^\tau_\nu\sum_{mi}[((r\hat{p})^\tau_{21})_{im}\mathcal{Q}^{\tau *}_{im} - ((r\hat{p})^\tau_{21})_{mi}\mathcal{Q}^{\tau *}_{mi}],\end{aligned} \tag{113}$$

where the notation $\hat{P}_{\lambda\mu} = \sum_{s=1}^{A}(\hat{p}_s^2)_{\lambda\mu}$ has been introduced. The last sum in (113) is calculated with the help of formula (B.17). Using the fact that $\epsilon_{im} = -\epsilon_{mi}$, one gets

$$\begin{aligned}\sum_{mi}[((r\hat{p})_{21})^\tau_{im}\mathcal{Q}^{\tau *}_{im} - ((r\hat{p})_{21})^\tau_{mi}\mathcal{Q}^{\tau *}_{mi}] &= \\ = -i\frac{m}{2\hbar}\sum_{mi}\epsilon^\tau_{mi}[(r^2_{21})^\tau_{im}\mathcal{Q}^{\tau *}_{im} + (r^2_{21})^\tau_{mi}\mathcal{Q}^{\tau *}_{mi}] &= \\ = -i\frac{m}{\hbar\sqrt{6}}\sum_{mi}\epsilon^\tau_{mi}|\mathcal{Q}^\tau_{mi}|^2 = -i\frac{m}{\hbar\sqrt{6}}(\epsilon^\tau_0\mathcal{Q}^\tau_0 + \epsilon^\tau_2\mathcal{Q}^\tau_2) &= -i\hbar\frac{4}{\sqrt{6}}\left(1+\frac{\delta}{3}\right)Q^\tau_{00}.\end{aligned}$$

According to the definitions (see formulae (73), (85)) we have

$$K^n_\nu = \sum_\tau \kappa_{n\tau}C^\tau_\nu = \frac{\chi\langle 0|\hat{R}^n_{21}|\nu\rangle + \bar{\chi}\langle 0|\hat{R}^p_{21}|\nu\rangle}{\sqrt{6}},$$

$$K^p_\nu = \sum_\tau \kappa_{p\tau}C^\tau_\nu = \frac{\chi\langle 0|\hat{R}^p_{21}|\nu\rangle + \bar{\chi}\langle 0|\hat{R}^n_{21}|\nu\rangle}{\sqrt{6}}.$$

So, equation (113) (let us say, for neutrons) is transformed into

$$-i\Omega_\nu\langle 0|\hat{L}^n_{21}|\nu\rangle = +\frac{1}{m}\langle 0|\hat{P}^n_{21}|\nu\rangle - m\omega^2\langle 0|\hat{R}^n_{21}|\nu\rangle - \frac{1}{\sqrt{6}}Z^n_{20}(\mathrm{eq})\langle 0|\hat{R}^n_{21}|\nu\rangle - \\ - \frac{2}{3}\left(1+\frac{\delta}{3}\right)Q^n_{00}(\chi\langle 0|\hat{R}^n_{21}|\nu\rangle + \bar{\chi}\langle 0|\hat{R}^p_{21}|\nu\rangle). \quad (114)$$

The equation for protons is obtained by interchanging indices n and p. One has to compare this equation with the variation of the second equation in (11) with $\lambda = 2$, $\mu = 1$. Let us write this variation in detail:

$$\frac{d}{dt}\mathcal{L}^\tau_{21} - \frac{1}{m}\mathcal{P}^\tau_{21} + m\omega^2\mathcal{R}^\tau_{21} - 2\sqrt{5}\sum_{j=0,2}\sqrt{2j+1}\{^{11j}_{221}\}\times \\ \times\sum_{\sigma,\nu}C^{21}_{2\sigma,j\nu}[Z^\tau_{2\sigma}(\mathrm{eq})\mathcal{R}^\tau_{j\nu} + \delta Z^\tau_{2\sigma}R^\tau_{j\nu}(\mathrm{eq})] = 0.$$

We recall that only $R^\tau_{00}(\mathrm{eq})$ and $R^\tau_{20}(\mathrm{eq})$ have nonzero values, so this equation is reduced to

$$\frac{d}{dt}\mathcal{L}^\tau_{21} - \frac{1}{m}\mathcal{P}^\tau_{21} + m\omega^2\mathcal{R}^\tau_{21} - 10\{^{112}_{221}\}C^{21}_{20,21}Z^\tau_{20}(\mathrm{eq})\mathcal{R}^\tau_{21} - \\ - 2\sqrt{5}\delta Z^\tau_{21}[\{^{110}_{221}\}C^{21}_{21,00}R^\tau_{00}(\mathrm{eq}) + \sqrt{5}\{^{112}_{221}\}C^{21}_{21,20}R^\tau_{20}(\mathrm{eq})] = 0.$$

In agreement with definition (9) of $Z^\tau_{\lambda\mu}$ its variation is

$$\delta Z^n_{2\mu} = \chi\mathcal{R}^n_{2\mu} + \bar{\chi}\mathcal{R}^p_{2\mu}, \quad \delta Z^p_{2\mu} = \chi\mathcal{R}^p_{2\mu} + \bar{\chi}\mathcal{R}^n_{2\mu}.$$

Substituting $6j$ symbols and Clebsch–Gordan coefficients by their numerical values, we obtain finally (e.g., for neutrons)

$$\frac{d}{dt}\mathcal{L}^n_{21} - \frac{1}{m}\mathcal{P}^n_{21} + m\omega^2\mathcal{R}^n_{21} + \frac{1}{\sqrt{6}}Z^n_{20}(\mathrm{eq})\mathcal{R}^n_{21} + \\ + \frac{2}{3}\left(1+\frac{\delta}{3}\right)Q^n_{00}(\chi\mathcal{R}^n_{21} + \bar{\chi}\mathcal{R}^p_{21}) = 0. \quad (115)$$

This equation coincides obviously with (114) if we assume the time dependence via $\mathrm{e}^{-i\Omega t}$ and identify the matrix elements $\langle 0|\hat{R}^\tau_{21}|\nu\rangle$, $\langle 0|\hat{L}^\tau_{21}|\nu\rangle$, and $\langle 0|\hat{P}^\tau_{21}|\nu\rangle$ with the variables $\mathcal{R}^\tau_{21}$, $\mathcal{L}^\tau_{21}$, and $\mathcal{P}^\tau_{21}$, respectively.

Let us consider finally the operator $\hat{f} = (\hat{p}^2)_{21}$. The required commutator is

$$[(\hat{p}^2)_{21}, H_0] = -2i\hbar m\omega^2(r\hat{p})_{21} + 4\sqrt{5}i\hbar\sum_{j=1}^{2}\sqrt{2j+1}\{^{11j}_{221}\}C^{21}_{20,j1}Z_{20}(\mathrm{eq})(r\hat{p})_{j1},$$

and one obtains from (109) the following equation:

$$\hbar\Omega_\nu\langle 0|\hat{P}^\tau_{21}|\nu\rangle = -2i\hbar m\omega^2\langle 0|\hat{L}^\tau_{21}|\nu\rangle + \\ + 4\sqrt{5}i\hbar\sum_{j=1}^{2}\sqrt{2j+1}\{^{11j}_{221}\}C^{21}_{20,j1}Z^\tau_{20}(\mathrm{eq})\langle 0|\hat{L}^\tau_{j1}|\nu\rangle + \\ + K^\tau_\nu\sum_{mi}[((\hat{p}^2)^\tau_{21})_{im}\mathcal{Q}^{\tau *}_{im} - ((\hat{p}^2)^\tau_{21})_{mi}\mathcal{Q}^{\tau *}_{mi}]. \quad (116)$$

It is easy to show (with the help of formula (B.18)) that the last sum is equal to zero. This equation must be compared with the variation of the last equation in (11) with $\lambda = 2$, $\mu = 1$. Let us write it in detail. Taking into account that $L^\tau_{\lambda\mu}(\mathrm{eq}) = 0$, we find the equation

$$\frac{d}{dt}\mathcal{P}^\tau_{21} + 2m\omega^2\mathcal{L}^\tau_{21} - 4\sqrt{5}\sum_{j=1}^{2}\sqrt{2j+1}\{^{11j}_{221}\}C^{21}_{20,j1}Z^\tau_{20}(\mathrm{eq})\mathcal{L}^\tau_{j1} = 0, \quad (117)$$

which obviously coincides with (116), if we assume the $\mathrm{e}^{-i\Omega t}$ time dependence and identify the proper RPA matrix elements with the respective WFM variables.

We will show in the next section that Eqs. (112), 114), (116) can be derived from the proper equations of (11) exactly, without the primitive procedure of identifying RPA matrix elements with WFM variables.

3. WFM VERSUS RPA

The exact relation between RPA matrix elements and the respective WFM variables can be established with the help of the linear response theory. Let us first recall, following Appendix D of [9], the necessary definitions concerning the density and the density matrix.

The density operator is defined as

$$\hat{\rho}(\mathbf{r}) = \sum_{s=1}^{A}\delta(\mathbf{r} - \hat{\mathbf{r}}_s) = \sum_{kq} d_{kq}(\mathbf{r})a^\dagger_k a_q, \quad (118)$$

where $d_{kq}(\mathbf{r}) = \langle k|\delta(\mathbf{r} - \hat{\mathbf{r}})|q\rangle = \sum_{\sigma,\tau}\phi^*_k(\mathbf{r}\sigma\tau)\phi_q(\mathbf{r}\sigma\tau)$ and $\phi_q(\mathbf{r}\sigma\tau)$ are single-particle wave functions. Indices k, q include spin and isospin quantum numbers σ and τ.

The density of particles in the system depends on its state Ψ and is defined as the average value of a density operator over this state:

$$\rho(\mathbf{r}) = \langle\Psi|\hat{\rho}(\mathbf{r})|\Psi\rangle = \sum_{kq} d_{kq}(\mathbf{r})\rho_{qk} =$$

$$= A \sum_{\sigma,\tau,\ldots,\sigma_A,\tau_A} \int d^3r_2 \cdots d^3r_A |\Psi(\mathbf{r}\sigma\tau, \mathbf{r}_2\sigma_2\tau_2, \ldots, \mathbf{r}_A\sigma_A\tau_A)|^2, \quad (119)$$

where $\rho_{qk} = \langle\Psi|a_k^\dagger a_q|\Psi\rangle$. The particle density (119) can be interpreted as the diagonal element (in the coordinate space representation) of *the density matrix* which is defined as

$$\rho(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma'\tau') = \sum_{kq} \phi_k^*(\mathbf{r}'\sigma'\tau')\phi_q(\mathbf{r}\sigma\tau)\langle\Psi|a_k^\dagger a_q|\Psi\rangle =$$

$$= \sum_{kq} d_{kq}(\mathbf{r}'\sigma'\tau', \mathbf{r}\sigma\tau)\rho_{qk} \quad (120)$$

with $d_{kq}(\mathbf{r}'\sigma'\tau', \mathbf{r}\sigma\tau) = \phi_k^*(\mathbf{r}'\sigma'\tau')\phi_q(\mathbf{r}\sigma\tau)$. The average value of the arbitrary one-body operator

$$\hat{F} = \sum_{s=1}^{A} \hat{f}_s = \sum_{kq} f_{kq} a_k^\dagger a_q \quad (121)$$

is written in terms of the density matrix as

$$\langle\Psi|\hat{F}|\Psi\rangle = \sum_{kq} f_{kq}\langle\Psi|a_k^\dagger a_q|\Psi\rangle = \sum_{kq} f_{kq}\rho_{qk} = \mathrm{Tr}\,(f\rho).$$

Let us consider the system to be in the weak external time-dependent field

$$\hat{W}(t) = \hat{W}\exp(-i\Omega t) + \hat{W}^\dagger \exp(i\Omega t), \quad (122)$$

where $\hat{W} = \sum_{kq} w_{kq} a_k^\dagger a_q$ is a one-body operator. The change of the ground state wave function produced by this field is found by using the time-dependent perturbation theory [31]:

$$\Psi(t) = |0\rangle + \sum_\nu |\nu\rangle \left[c_\nu \,\mathrm{e}^{-i\Omega t} - \bar{c}_\nu^* \,\mathrm{e}^{i\Omega t}\right]. \quad (123)$$

Here $|0\rangle$ and $|\nu\rangle$ are stationary eigenstates of the unperturbed system and

$$\begin{aligned} c_\nu &= \frac{\langle\nu|\hat{W}|0\rangle}{\hbar(\Omega-\Omega_\nu)} = \sum_{kq}\frac{\langle\nu|a_k^\dagger a_q|0\rangle}{\hbar(\Omega-\Omega_\nu)}w_{kq}, \\ \bar{c}_\nu &= \frac{\langle 0|\hat{W}|\nu\rangle}{\hbar(\Omega+\Omega_\nu)} = \sum_{kq}\frac{\langle 0|a_k^\dagger a_q|\nu\rangle}{\hbar(\Omega+\Omega_\nu)}w_{kq}. \end{aligned} \tag{124}$$

Inserting this expression into formula (120), we obtain the perturbed density matrix

$$\rho(\mathbf{r}\sigma\tau,\mathbf{r}'\sigma'\tau',t) = \rho_0(\mathbf{r}\sigma\tau,\mathbf{r}'\sigma'\tau') + \delta\rho(\mathbf{r}\sigma\tau,\mathbf{r}'\sigma'\tau',t),$$

where $\rho_0(\mathbf{r}\sigma\tau,\mathbf{r}'\sigma'\tau')$ is the unperturbed (equilibrium) density matrix

$$\rho_0(\mathbf{r}\sigma\tau,\mathbf{r}'\sigma'\tau') = \sum_{kq} d_{qk}(\mathbf{r}'\sigma'\tau',\mathbf{r}\sigma\tau)\langle 0|a_q^\dagger a_k|0\rangle = \sum_{kq} d_{qk}(\mathbf{r}'\sigma'\tau',\mathbf{r}\sigma\tau)\rho_{kq}^{(0)}$$

and $\delta\rho(\mathbf{r}\sigma\tau,\mathbf{r}'\sigma'\tau',t)$ is the change of the density matrix

$$\delta\rho(\mathbf{r}\sigma\tau,\mathbf{r}'\sigma'\tau',t) = \sum_{kq} d_{qk}(\mathbf{r}'\sigma'\tau',\mathbf{r}\sigma\tau)\rho_{kq}^{(1)}(t) \tag{125}$$

with

$$\begin{aligned} \rho_{kq}^{(1)}(t) = \sum_\nu \big[&(\langle 0|a_q^\dagger a_k|\nu\rangle c_\nu - \langle\nu|a_q^\dagger a_k|0\rangle\bar{c}_\nu)\,\mathrm{e}^{-i\Omega t} + \\ &+(\langle\nu|a_q^\dagger a_k|0\rangle c_\nu^* - \langle 0|a_q^\dagger a_k|\nu\rangle\bar{c}_\nu^*)\,\mathrm{e}^{i\Omega t}\big]. \end{aligned} \tag{126}$$

Deriving (125) we neglected the terms proportional to $|\hat{W}|^2$. At this stage it is necessary to remind that we work in a Hartree–Fock approximation. This means that stationary states $|0\rangle$, $|\nu\rangle$ are Slater determinants; matrix $\rho_{kq}^{(0)} = \rho_q\delta_{kq}$ is diagonal with $\rho_q = 1$ for levels below the Fermi level and $\rho_q = 0$ for levels above the Fermi level. The requirement $(\rho_0+\delta\rho)^2 = (\rho_0+\delta\rho)$ leads to the well-known [9] property of the matrix $\rho_{kq}^{(1)}$: it has only particle–hole nonvanishing matrix elements. Looking to formula (126) we see that it is possible for the matrix elements $\langle 0|a_q^\dagger a_k|\nu\rangle$ to be different from zero only for particle–hole combinations of indices q,k. Consequently, the summation over k,q in formula (124) for c_n and $\bar{c}_n$ will also be restricted only to particle–hole pairs. So we can write $\rho_{kq}^{(1)}$ as

$$\rho_{kq}^{(1)}(t) = \sum_{k'q'}\left[R_{kq,k'q'}(\Omega)\,\mathrm{e}^{-i\Omega t} + R^*_{qk,k'q'}(\Omega)\,\mathrm{e}^{i\Omega t}\right]w_{k'q'},$$

where

$$R_{kq,k'q'}(\Omega)=\sum_{\nu}\left(\frac{\langle 0|a_q^{\dagger}a_k|\nu\rangle\langle\nu|a_{k'}^{\dagger}a_{q'}|0\rangle}{\hbar(\Omega-\Omega_{\nu})}-\frac{\langle 0|a_{k'}^{\dagger}a_{q'}|\nu\rangle\langle\nu|a_q^{\dagger}a_k|0\rangle}{\hbar(\Omega+\Omega_{\nu})}\right)$$

is the RPA response function [9], where the index pairs kq and $k'q'$ are restricted to particle–hole pairs. For the change of the arbitrary operator average value we have

$$\delta\langle\Psi|\hat{F}|\Psi\rangle=\sum_{kq}f_{kq}\rho_{qk}^{(1)}. \tag{127}$$

We now are ready to analyze the WFM variables. The first one is

$$R_{\lambda\mu}^{\tau}(t)=2(2\pi\hbar)^{-3}\int d^3p\int d^3r r_{\lambda\mu}^2 f^{\tau}(\mathbf{r},\mathbf{p},t).$$

Using here definitions of the Wigner function and the δ function we find

$$\begin{aligned}R_{\lambda\mu}^{\tau}(t)&=\frac{2}{(2\pi\hbar)^3}\int d^3r\, r_{\lambda\mu}^2\int d^3s\int d^3p\,\exp\left(-i\mathbf{p}\cdot\mathbf{s}/\hbar\right)\rho^{\tau}\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2},t\right)=\\&=2\int d^3r\, r_{\lambda\mu}^2\rho^{\tau}(\mathbf{r},\mathbf{r},t)=\sum_{\sigma}\int d^3r\, r_{\lambda\mu}^2\rho(\mathbf{r}\sigma\tau,\mathbf{r}\sigma\tau,t)=\\&=\sum_{kq}\sum_{\sigma}\int d^3r\, r_{\lambda\mu}^2\phi_k^*(\mathbf{r}\sigma\tau)\phi_q(\mathbf{r}\sigma\tau)\langle\Psi|a_k^{\dagger}a_q|\Psi\rangle=\\&=\sum_{kq}(r_{\lambda\mu}^2)_{kq}^{\tau}\langle\Psi|a_k^{\dagger}a_q|\Psi\rangle=\langle\Psi|\sum_{kq}(r_{\lambda\mu}^2)_{kq}^{\tau}a_k^{\dagger}a_q|\Psi\rangle=\\&=\langle\Psi|\sum_{s=1}^{N_{\tau}}r_{\lambda\mu}^2(s)|\Psi\rangle=\langle\Psi|\hat{R}_{\lambda\mu}^{\tau}|\Psi\rangle,\end{aligned} \tag{128}$$

i.e., this is just the ground state expectation value of the operator $\hat{R}_{\lambda\mu}=\sum_{s=1}^{A}(r_s^2)_{\lambda\mu}$. In accordance with (127) the variation of this variable is

$$\begin{aligned}\delta R_{\lambda\mu}^{\tau}(t)\equiv\mathcal{R}_{\lambda\mu}^{\tau}(t)&=\sum_{kq}(r_{\lambda\mu}^2)_{kq}^{\tau}\rho_{qk}^{(1)}(t)=\\&=\sum_{\nu}(\langle 0|\hat{R}_{\lambda\mu}^{\tau}|\nu\rangle c_{\nu}-\langle\nu|\hat{R}_{\lambda\mu}^{\tau}|0\rangle\bar{c}_{\nu})\,\mathrm{e}^{-i\Omega t}+\\&\quad+\sum_{\nu}(\langle\nu|\hat{R}_{\lambda\mu}^{\tau}|0\rangle c_{\nu}^*-\langle 0|\hat{R}_{\lambda\mu}^{\tau}|\nu\rangle\bar{c}_{\nu}^*)\,\mathrm{e}^{i\Omega t}.\end{aligned} \tag{129}$$

Equation (129) demonstrates in an obvious way the structure of the variable $\delta R_{\lambda\mu}$. It is a linear combination of the transition matrix elements $\langle 0|\hat{R}_{\lambda\mu}|\nu\rangle$ which are, in turn, linear combinations of RPA amplitudes X_{kq}, Y_{kq}. In a similar way we can show that the variables $L^{\tau}_{\lambda\mu}(t)$ and $P^{\tau}_{\lambda\mu}(t)$ defined by (12) are just the ground state expectation values of the operators $\hat{L}_{\lambda\mu} = \sum_{s=1}^{A}(r_s\hat{p}_s)_{\lambda\mu}$ and $\hat{P}_{\lambda\mu} = \sum_{s=1}^{A}(\hat{p}_s^2)_{\lambda\mu}$, respectively:

$$L^{\tau}_{\lambda\mu}(t) = \langle\Psi|\hat{L}^{\tau}_{\lambda\mu}|\Psi\rangle, \quad P^{\tau}_{\lambda\mu}(t) = \langle\Psi|\hat{P}^{\tau}_{\lambda\mu}|\Psi\rangle.$$

Variations of these variables are

$$\begin{aligned}\delta L^{\tau}_{\lambda\mu}(t) \equiv \mathcal{L}^{\tau}_{\lambda\mu}(t) &= \sum_{kq}((r\hat{p})_{\lambda\mu})^{\tau}_{kq}\rho^{(1)}_{qk}(t) = \\ &= \sum_{\nu}(\langle 0|\hat{L}^{\tau}_{\lambda\mu}|\nu\rangle c_\nu - \langle\nu|\hat{L}^{\tau}_{\lambda\mu}|0\rangle\bar{c}_\nu)\,\mathrm{e}^{-i\Omega t} + \\ &\quad + \sum_{\nu}(\langle\nu|\hat{L}^{\tau}_{\lambda\mu}|0\rangle c^*_\nu - \langle 0|\hat{L}^{\tau}_{\lambda\mu}|\nu\rangle\bar{c}^*_\nu)\,\mathrm{e}^{i\Omega t}, \quad (130)\end{aligned}$$

and

$$\begin{aligned}\delta P^{\tau}_{\lambda\mu}(t) \equiv \mathcal{P}^{\tau}_{\lambda\mu}(t) &= \sum_{kq}(p^2_{\lambda\mu})^{\tau}_{kq}\rho^{(1)}_{qk}(t) = \\ &= \sum_{\nu}(\langle 0|\hat{P}^{\tau}_{\lambda\mu}|\nu\rangle c_\nu - \langle\nu|\hat{P}^{\tau}_{\lambda\mu}|0\rangle\bar{c}_\nu)\,\mathrm{e}^{-i\Omega t} + \\ &\quad + \sum_{\nu}(\langle\nu|\hat{P}^{\tau}_{\lambda\mu}|0\rangle c^*_\nu - \langle 0|\hat{P}^{\tau}_{\lambda\mu}|\nu\rangle\bar{c}^*_\nu)\,\mathrm{e}^{i\Omega t}. \quad (131)\end{aligned}$$

Inserting the expressions for $\delta R_{\lambda\mu}$, $\delta L_{\lambda\mu}$ into the variation of the first equation of (11)

$$\frac{d}{dt}\mathcal{R}_{\lambda\mu} - \frac{2}{m}\mathcal{L}_{\lambda\mu} = 0,$$

we find

$$-i\Omega\sum_{\nu}(\langle 0|\hat{R}_{\lambda\mu}|\nu\rangle c_\nu - \langle\nu|\hat{R}_{\lambda\mu}|0\rangle\bar{c}_\nu) = \frac{2}{m}\sum_{\nu}(\langle 0|\hat{L}_{\lambda\mu}|\nu\rangle c_\nu - \langle\nu|\hat{L}_{\lambda\mu}|0\rangle\bar{c}_\nu).$$

It is sufficient to consider only the part with the $\mathrm{e}^{-i\Omega t}$ time dependence. Multiplying this equation by $(\Omega - \Omega_\nu)$ and taking the limit $\Omega \to \Omega_\nu$, we reproduce the RPA equation (112).

Inserting the expressions for $\delta R_{\lambda\mu}$, $\delta L_{\lambda\mu}$, and $\delta P_{\lambda\mu}$ into Eq. (115) and performing the described limiting procedure we obtain Eq. (114). Analogously, inserting formulae (131), (130) into Eq. (117) and performing the same limiting procedure we get the RPA equation (116).

So, there exists one-to-one correspondence between the set of dynamical equations for WFM variables and the set of dynamical equations for Transition Matrix Elements (TME). This correspondence makes obvious the fact that both the sets have the same eigenvalues. On the other hand, the TME equations are just linear combinations of the RPA equations. Therefore we can conclude that RPA and WFM approaches generate identical eigenvalues. In this sense both approaches are equivalent in all aspects. This concerns, for instance, also the transition probabilities. However, for this equivalence to be exact, one needs to work in the full space in both approaches, that is in the complete particle hole space in RPA and taking all phase space moments of all powers in WFM, a task which can hardly be tackled in general.

The difference of the two approaches then shows up if truncations of the dimension of the equations have to be operated. In RPA one usually solves the equations with a restricted number of discrete particle hole pairs, i.e., the dimension of the RPA matrix is finite (in some works the RPA equations for finite nuclei are, however, solved in full space, including continuum states [32, 33]). The result of such a diagonalization usually yields a huge number of discrete eigenvalues approximating more or less the spectrum one would obtain from a solution in the full space. For instance, resonances in the continuum (e.g., giant resonances) will be mocked up by a bunch of discrete states whose envelope may simulate the full solution. Reducing of the dimension of the particle hole space too much may lead to a situation where the full solution is only approximated rather badly and in an uncontrolled manner.

In the WFM method, the dynamical equations for Cartesian tensors of the rank $n = 2$ are coupled (by the interaction terms in (5)) with dynamical equations for tensors of the rank $n = 3$, these equations being coupled with the ones for tensors of the rank $n = 4$ and so on up to $n = \infty$. Here one hopes that the essential part of physics is described by a small number of the lowest rank tensors. The hope is based on the assumption that the higher rank tensors (moments) are responsible for the more refined details and that neglecting them does not appreciably influence the description of the more global physics which is described with the lower rank tensors. This assumption is substantiated in the past applications of the WFM method to realistic situations with the Skyrme forces for the description of collective nuclear modes [14, 15, 17]. In those works it has indeed been demonstrated that even with a very limited number of low-rank phase space moments one can faithfully reproduce the centroid position of the collective states. From these studies it is then permitted to assume that the inclusion of higher and higher rank moments will just give raise to a refinement

of the gross structure obtained with the low rank tensors. A formal convergence study of this type has been performed in the infinite matter case [34] where it was indeed shown that the moment method allows one to approach the full solution in an optimized way.

The net result is, that WFM and RPA approximate the exact infinite spectrum of the quantum mechanical problem by a finite number of eigenfrequencies of the classical problem with, however, different convergence.

An analogous situation occurs with transition probabilities. Let us analyze, for example, the expression (129) for the WFM variable $\delta R^{\tau}_{\lambda\mu}(t) = \mathcal{R}^{\tau}_{\lambda\mu}(t)$. Using the definition (124) of c_ν with the external field operator $\hat{W} = \hat{R}^{\tau\dagger}_{\lambda\mu}$, we find

$$\delta R^{\tau}_{\lambda\mu}(t) = \sum_{\nu=1}^{N_c} \left(\frac{\langle 0|\hat{R}^{\tau}_{\lambda\mu}|\nu\rangle\langle\nu|\hat{R}^{\tau\dagger}_{\lambda\mu}|0\rangle}{\hbar(\Omega-\Omega_\nu)} - \frac{\langle\nu|\hat{R}^{\tau}_{\lambda\mu}|0\rangle\langle 0|\hat{R}^{\tau\dagger}_{\lambda\mu}|\nu\rangle}{\hbar(\Omega+\Omega_\nu)} \right) \mathrm{e}^{-i\Omega t} =$$

$$= \sum_{\nu=1}^{N_c} \left(\frac{|\langle 0|\hat{R}^{\tau}_{\lambda\mu}|\nu\rangle|^2}{\hbar(\Omega-\Omega_\nu)} - \frac{|\langle 0|\hat{R}^{\tau\dagger}_{\lambda\mu}|\nu\rangle|^2}{\hbar(\Omega+\Omega_\nu)} \right) \mathrm{e}^{-i\Omega t}. \quad (132)$$

The summation limit N_c depends on the method of calculation. In the case of the exact solution $N_c = \infty$, for RPA N_c is usually of the order of several *hundreds* or *thousands*, for WFM N_c usually is not more than around *a dozen*. Naturally, the eigenvalues Ω_ν and eigenstates $|\nu\rangle$ are different in each case. So, the strength, which in RPA was distributed over hundreds or thousands levels, in WFM is concentrated only on several levels, i.e., averaging of levels is accompanied by the redistribution of the strength. The variable $\delta R^{\tau}_{\lambda\mu}(t)$ is the quantum mechanical observable, so its value should not depend on the basis $|\nu\rangle$. Hence, the right-hand sides of (132), calculated by two methods, should coincide if both methods are mutually consistent. This statement can, e.g., be checked with the help of sum rules. Generally, in RPA sum rules are well fulfilled for a sufficiently large particle hole space which in realistic cases can become quite significant, whereas in WFM sum rules are generally already well fulfilled even with a small number of low rank moments (see, e.g., [14, 15, 17, 35]).

The essential difference between WFM and RPA methods lies in their practical use. The RPA equations (70) are constructed in such a way that the increase in dimension does not cause any formal problems and finally it is only a question of computer power what dimension can be handled. Quite on the contrary, the increase of dimension in WFM is a nontrivial task. Beyond a certain order of the moments even the reduction of the Cartesian tensors to the irreducible ones becomes a very difficult task. However, the spirit of WFM is rather to reproduce the gross structure of a couple of prominent collective states, a situation which it can handle very efficiently.

In conclusion WFM and RPA are equivalent when the full particle hole configuration space in RPA and the infinite number of moments in WFM are considered. However, under truncation of the spaces both methods have different convergence properties. In the general case for WFM only a few moments are sufficient to get the correct gross structure of the collective part of the spectrum, whereas in RPA one in general must take into account a quite large configuration space to produce reasonable results.

4. GREEN'S FUNCTION METHOD

One of the important subjects of comparing RPA and WFM methods are the current distributions. The WFM method, a priori, cannot give the exact results, because it deals only with integrals over the whole phase space. It would therefore be very interesting to evaluate the accuracy of this approximation by comparing it with the exact result. Unfortunately, even for the simple model HO+QQ it is impossible to derive in RPA closed analytical expressions for currents of the scissors mode and IVGQR. That is why we consider in this section Green's Function (GF) method, which allows one to find explicit expressions for the currents directly.

Following the paper of H. Kohl, P. Schuck, and S. Stringari [36] we will consider at first the isoscalar case. Conserving on the right-hand side of Eq. (4) only the first term of the sin-function expansion leads to the Vlasov equation

$$\frac{\partial f}{\partial t} = \nabla H_W \cdot \nabla^p f - \nabla^p H_W \cdot \nabla f. \tag{133}$$

In our case the Wigner transform H_W coincides with the classical Hamiltonian H_c. Having in mind small amplitude vibrations we have to linearize (133): $f = f_0 + f_1$, $H_c = H_0 + H_1$, with f_0 being the solution of the time independent equation. The linearized version of (133) is

$$\frac{\partial f_1}{\partial t} + \nabla^p H_0 \cdot \nabla f_1 - \nabla H_0 \cdot \nabla^p f_1 = S(\mathbf{r}, \mathbf{p}, t), \tag{134}$$

where $S(\mathbf{r}, \mathbf{p}, t) = \nabla H_1 \cdot \nabla^p f_0$. This equation will be solved with Green's function method. We have

$$\left(\frac{\partial}{\partial t} + \nabla^p H_0 \cdot \nabla - \nabla H_0 \cdot \nabla^p \right) G^{(t-t')}(\mathbf{r}\mathbf{p}, \mathbf{r}'\mathbf{p}') = \\ = \delta(\mathbf{r} - \mathbf{r}')\delta(\mathbf{p} - \mathbf{p}')\delta(t - t') \tag{135}$$

with [36]

$$G^{(t-t')}(\mathbf{r}\mathbf{p}, \mathbf{r}'\mathbf{p}') = \delta[\mathbf{r}_c(\mathbf{r}, \mathbf{p}, t' - t) - \mathbf{r}']\delta[\mathbf{p}_c(\mathbf{r}, \mathbf{p}, t' - t) - \mathbf{p}']\theta(t - t'),$$

where $\mathbf{r}_c(\mathbf{r},\mathbf{p},t'-t)$, $\mathbf{p}_c(\mathbf{r},\mathbf{p},t'-t)$ are solutions of classical equations of motion with initial conditions $\mathbf{r}$, $\mathbf{p}$. The solution of (134) can be written as

$$f_1(\mathbf{r},\mathbf{p},t) = f_1^h + \int_{-\infty}^{\infty} dt' \int d^3r' d^3p' G^{(t-t')}(\mathbf{rp},\mathbf{r'p'}) S(\mathbf{r}',\mathbf{p}',t') = \\ = f_1^h + \int_{-\infty}^{t} dt' S(\mathbf{r}_c,\mathbf{p}_c,t'), \quad (136)$$

where f_1^h is the solution of the homogeneous equation, which does not play any role at resonance and therefore will be omitted in the forthcoming.

We consider the axially deformed harmonic oscillator H_0 with the quadrupole–quadrupole residual interaction $V_{\rm res} = H_1$. The derivation of the proper single-particle Hamiltonian from the original microscopic Hamiltonian (6) can be found in Appendix A. We have

$$H_0 = \frac{p^2}{2m} + \frac{m}{2}[\omega_x^2(x^2+y^2) + \omega_z^2 z^2].$$

We are interested in the part of the residual interaction with $|\mu| = 1$. In accordance with formula (A.1) it can be written as

$$V_{\rm res} = -\kappa_0[Q_{21}(t)q_{2-1}(\mathbf{r}) + Q_{2-1}(t)q_{21}(\mathbf{r})] = 12\kappa_0 Q_1(t)[xz+yz]$$

with

$$Q_1(t) = 2\int d\{\mathbf{p},\mathbf{r}\} f(\mathbf{r},\mathbf{p},t)xz = 2\int d\{\mathbf{p},\mathbf{r}\} f(\mathbf{r},\mathbf{p},t)yz = \\ = 2\int d\{\mathbf{p},\mathbf{r}\}[f_0(\mathbf{r},\mathbf{p}) + f_1(\mathbf{r},\mathbf{p},t)]xz = 2\int d\{\mathbf{p},\mathbf{r}\} f_1(\mathbf{r},\mathbf{p},t)xz.$$

With the help of the Thomas–Fermi approximation for the static distribution function

$$f_0 = \theta(\epsilon_F - H_0)$$

the right-hand side of (134) is found to be

$$S(\mathbf{r},\mathbf{p},t) = -12\frac{\kappa_0}{m} Q_1(t)\delta(\epsilon_F - H_0)[p_x z + p_z x + p_y z + p_z y].$$

The classical trajectories are determined by the solution of the Hamilton equations $\dot r_{c,i} = \dfrac{\partial H_0}{\partial p_i}$, $\dot p_{c,i} = -\dfrac{\partial H_0}{\partial r_i}$ with $i = x,y,z$. In our case they are

$$r_{c,i}(t) = r_i \cos\omega_i t + \frac{p_i}{m\omega_i}\sin\omega_i t, \quad p_{c,i}(t) = p_i\cos\omega_i t - m\omega_i r_i \sin\omega_i t.$$

Formula (136) then gives

$$f_1(\mathbf{r},\mathbf{p},t) = -6\frac{\kappa_0}{m}\delta(\epsilon_F - H_0)\int\limits_{-\infty}^{t} dt' Q_1(t')\times$$
$$\times\left\{\frac{1}{\omega_x}(p_x+p_y)z\,[\omega_+\cos\,\omega_+(t'-t)+\omega_-\cos\,\omega_-(t'-t)]+\right.$$
$$+\frac{1}{\omega_z}p_z(x+y)[\omega_+\cos\,\omega_+(t'-t)-\omega_-\cos\,\omega_-(t'-t)]+$$
$$+\frac{1}{m\omega_x\omega_z}(p_x+p_y)p_z[\omega_+\sin\,\omega_+(t'-t)-\omega_-\sin\,\omega_-(t'-t)]-$$
$$\left.-\,m(x+y)z[\omega_+\sin\,\omega_+(t'-t)+\omega_-\sin\,\omega_-(t'-t)]\right\}, \quad (137)$$

where $\omega_\pm = \omega_x \pm \omega_z$.

So, we have derived a complicated integral equation for the perturbed distribution function which may not easily be solved in general. As a matter of fact, the analytic possibilities of Green's function method are, without further consideration, exhausted at this point.

In order to proceed to the evaluation of the eigenfrequencies and transition probabilities we again apply the method of moments. Integrating (137) over the whole phase space with the weights xz, p_xp_z, $zp_x + xp_z$, and $zp_x - xp_z$, we obtain the following set of coupled integral equations:

$$Q_1(t) = \beta\int\limits_{-\infty}^{t} dt' Q_1(t')[\omega_+\sin\omega_+(t'-t)+\omega_-\sin\omega_-(t'-t)],$$
$$P_1(t) = -\beta m^2\omega_x\omega_z\int\limits_{-\infty}^{t} dt' Q_1(t')[\omega_+\sin\omega_+(t'-t)-\omega_-\sin\omega_-(t'-t)],$$
$$L_1(t) = -\beta m\int\limits_{-\infty}^{t} dt' Q_1(t')[\omega_+^2\cos\omega_+(t'-t)+\omega_-^2\cos\omega_-(t'-t)], \quad (138)$$
$$I_y(t) = -\beta m\omega_+\omega_-\int\limits_{-\infty}^{t} dt' Q_1(t')[\cos\omega_+(t'-t)+\cos\omega_-(t'-t)],$$

where

$$\beta = \frac{2\kappa_0\pi^3\epsilon_F^4}{m^2\omega_x^4\omega_z^3}\frac{4}{(2\pi\hbar)^3} =$$

$$= 12\kappa_0\int d\{\mathbf{p},\mathbf{r}\}x^2z^2\delta(\epsilon_F - H_0) = \frac{12\kappa_0}{m^4\omega_x^2\omega_z^2}\int d\{\mathbf{p},\mathbf{r}\}p_x^2p_z^2\delta(\epsilon_F - H_0) =$$

$$= \frac{12\kappa_0}{m^2\omega_z^2}\int d\{\mathbf{p},\mathbf{r}\}x^2p_z^2\delta(\epsilon_F - H_0) = \frac{12\kappa_0}{m^2\omega_x^2}\int d\{\mathbf{p},\mathbf{r}\}z^2p_x^2\delta(\epsilon_F - H_0)$$

and the following notation is introduced

$$P_1(t) = 2\int d\{\mathbf{p},\mathbf{r}\}f_1(\mathbf{r},\mathbf{p},t)p_xp_z,$$

$$L_1(t) = 2\int d\{\mathbf{p},\mathbf{r}\}f_1(\mathbf{r},\mathbf{p},t)(zp_x + xp_z),$$

$$I_y(t) = 2\int d\{\mathbf{p},\mathbf{r}\}f_1(\mathbf{r},\mathbf{p},t)(zp_x - xp_z).$$

By simple means these equations are reduced to a set of differential equations. At first, we perform time derivatives of all equations in (138):

$$\begin{aligned}
\dot{Q}_1(t) &= -\beta\int\limits_{-\infty}^{t} dt'Q_1(t')[\omega_+^2\cos\omega_+(t'-t) + \omega_-^2\cos\omega_-(t'-t)],\\
\dot{P}_1(t) &= \beta m^2\omega_x\omega_z\int\limits_{-\infty}^{t} dt'Q_1(t')[\omega_+^2\cos\omega_+(t'-t) - \omega_-^2\cos\omega_-(t'-t)],\\
\dot{L}_1(t) &= -\beta m\Bigg\{(\omega_+^2+\omega_-^2)Q_1(t)+\\
&\qquad + \int\limits_{-\infty}^{t} dt'Q_1(t')[\omega_+^3\sin\omega_+(t'-t) + \omega_-^3\sin\omega_-(t'-t)]\Bigg\},\\
\dot{I}_y(t) &= -\beta m\omega_+\omega_-\Bigg\{2Q_1(t)+\\
&\qquad + \int\limits_{-\infty}^{t} dt'Q_1(t')[\omega_+\sin\omega_+(t'-t) + \omega_-\sin\omega_-(t'-t)]\Bigg\}.
\end{aligned} \tag{139}$$

Solving (138) with respect of obvious time integrals we can exclude them from (139). We have

$$
\begin{aligned}
2\beta m^2\omega_x\omega_z\omega_- \int\limits_{-\infty}^{t} dt' Q_1(t')\sin\omega_-(t'-t) &= m^2\omega_x\omega_z Q_1(t) + P_1(t),\\
2\beta m^2\omega_x\omega_z\omega_+ \int\limits_{-\infty}^{t} dt' Q_1(t')\sin\omega_+(t'-t) &= m^2\omega_x\omega_z Q_1(t) - P_1(t),\\
4\beta m\omega_x\omega_z \int\limits_{-\infty}^{t} dt' Q_1(t')\cos\omega_-(t'-t) &= L_1(t) + \frac{\omega_+}{\omega_-} I_y(t),\\
-4\beta m\omega_x\omega_z \int\limits_{-\infty}^{t} dt' Q_1(t')\cos\omega_+(t'-t) &= L_1(t) + \frac{\omega_-}{\omega_+} I_y(t).
\end{aligned}
\tag{140}
$$

Substituting the time integrals in (139) by the proper expressions from (140) we find

$$
\begin{aligned}
\dot{Q}_1(t) &= \frac{1}{m} L_1(t),\\
\dot{L}_1(t) &= -m(2\beta+1)(\omega_x^2+\omega_z^2)Q_1(t) + \frac{2}{m}P_1(t),\\
\dot{P}_1(t) &= -\frac{m}{2}[(\omega_x^2+\omega_z^2)L_1(t) - (\omega_x^2-\omega_z^2)I_y(t)],\\
\dot{I}_y(t) &= -m(2\beta+1)\omega_+\omega_- Q_1(t).
\end{aligned}
\tag{141}
$$

Due to the conservation of the angular momentum the right-hand side of the last equation must be equal to zero. So we have the requirement

$$
2\beta + 1 = 0, \quad \text{or} \quad \kappa_0 = -\frac{m^2\omega_x^4\omega_z^3}{4\pi^3\epsilon_F^4}\frac{(2\pi\hbar)^3}{4}. \tag{142}
$$

With the help of the relation

$$
\begin{aligned}
A\langle r^2\rangle = 2\int d\{\mathbf{p},\mathbf{r}\} r^2 f_0 = 2\int d\{\mathbf{p},\mathbf{r}\} r^2\theta(\epsilon_F - H_0) =\\
= \frac{\pi^3\epsilon_F^4(\omega_x^2+2\omega_z^2)}{3m\omega_x^4\omega_z^3}\frac{4}{(2\pi\hbar)^3}
\end{aligned}
\tag{143}
$$

and formulae (A.3) for ω_x, ω_z, the expression for κ_0 is reduced to

$$
\kappa_0 = -\frac{m(\omega_x^2+2\omega_z^2)}{12A\langle r^2\rangle} = -\frac{m\bar{\omega}^2}{4A\langle r^2\rangle}, \tag{144}
$$

which is just the familiar expression for the self-consistent value of the strength constant (see Appendix A). This is a rather interesting result, because the well-known formula is obtained without the usual self-consistency requirement [37]. As is known, in the absence of external fields the angular momentum of any system is conserved. The short range interparticle interactions depending on the module of the interparticle distance $|\mathbf{r}_i - \mathbf{r}_j|$ create the scalar, i.e., rotational invariant, mean field, which exactly repeats the shape of the nucleus. When we imitate the mean field by a rotational invariant function, the angular momentum will be conserved independently of the shape of this function due to a pure mathematical reason: angular momentum operator commutes with a scalar field. If we use the nonrotational invariant function (as in our case), mathematics does not help and the shape of the function becomes important. If the function does not follow exactly the shape of the system, the latter will react on this inconsistency as on the external field, that leads to the nonconservation of an angular momentum. Therefore the requirement of the angular momentum conservation in this case becomes equivalent to the requirement of the self-consistency. This is seen very well in the method of moments. Integrating equation (134) over the phase space with the weight $zp_x - xp_z$, we obtain the dynamical equation for I_y

$$\frac{d}{dt}I_y = m(\omega_z^2 - \omega_x^2)Q_1 + 12\kappa_0(\langle x^2\rangle - \langle z^2\rangle)Q_1. \tag{145}$$

The requirement of the angular momentum conservation gives the following relation:

$$m(\omega_z^2 - \omega_x^2) = 12\kappa_0(\langle z^2\rangle - \langle x^2\rangle). \tag{146}$$

Obviously, it is the requirement of the consistency between the shapes of the potential and the nucleus. In principle, this relation is less restrictive than the standard self-consistency requirement [37]. However, the latter satisfies equation (146) which can be easily checked with the help of Appendix A.

So, finally the set of Eqs. (141) is reduced to

$$\begin{aligned}
\dot{Q}_1(t) &= \frac{1}{m}L_1(t),\\
\dot{L}_1(t) &= \frac{2}{m}P_1(t),\\
\dot{P}_1(t) &= -m\bar{\omega}^2\left[\left(1+\frac{1}{3}\delta\right)L_1(t) - \delta I_y(t)\right],\\
\dot{I}_y(t) &= 0.
\end{aligned} \tag{147}$$

Taking into account the relations between the definitions of variables in (21) and (147)

$$Q_1 = -\mathrm{Re}\,\mathcal{R}_{21}, \quad P_1 = -\mathrm{Re}\,\mathcal{P}_{21}, \quad L_1 = -2\mathrm{Re}\,\mathcal{L}_{21}$$

(which follow from formulae $r_{21}^2 = -z(x+iy)$ and $(rp)_{21} = -\frac{1}{2}[zp_x + xp_z + i(zp_y + yp_z)]$) and $I_y = 2\,\mathrm{Re}\,\mathcal{L}_{11}$, it is easy to see that the last set of equations is identical to (21).

With the help of relations (140) the Wigner function (137) can be written in terms of the Wigner function moments

$$f_1(\mathbf{r},\mathbf{p},t) = \frac{3\kappa_0}{\beta m^2}\delta(\epsilon_F - H_0)\Big\{[L_1(t) + I_y(t)]\frac{1}{\omega_x^2}(p_x+p_y)z + \\ + [L_1(t) - I_y(t)]\frac{1}{\omega_z^2}p_z(x+y) + P_1(t)\left(\frac{2}{m^2\omega_x^2\omega_z^2}\right)(p_x+p_y)p_z + \\ + Q_1(t)2m^2z(x+y)\Big\}. \quad (148)$$

Taking into account equations of motion (147) and the time dependence of variables via $\mathrm{e}^{-i\Omega t}$ (which leads to the equality $I_y = 0$) one finds

$$f_1(\mathbf{r},\mathbf{p},t) = \frac{3\kappa_0}{\beta m^2}\delta(\epsilon_F - H_0)\left\{-i\Omega m\left[\frac{1}{\omega_x^2}(p_x+p_y)z + \frac{1}{\omega_z^2}p_z(x+y)\right] - \right. \\ \left. - \left(\frac{1}{\omega_x^2} + \frac{1}{\omega_z^2}\right)(p_x+p_y)p_z + 2m^2z(x+y)\right\}Q_1(t).$$

In the case of $\delta = 0$ it reproduces the result of [36].

Having the Wigner function one can calculate transition probabilities in the same way as in WFM method.

Let us consider now the problem with two sorts of particles: neutrons and protons. All variables and parameters acquire isotopic index τ. The part of the residual interaction with $|\mu| = 1$, in accordance with formula (9) becomes $V_1^\tau = Z_1^\tau(t)[xz + yz]$ with $Z_1^n(t) = 12(\kappa Q_1^n + \bar{\kappa}Q_1^p)$, $Z_1^p(t) = 12(\kappa Q_1^p + \bar{\kappa}Q_1^n)$ and $Q_1^\tau = \int d\{\mathbf{p},\mathbf{r}\}f_1^\tau(\mathbf{r},\mathbf{p},t)xz$. The expression for the Wigner function is obtained from formula (137) by changing the factor $6\kappa Q_1(t')$ by $\frac{1}{2}Z_1^\tau(t')$. The dynamical equations for isovector variables $\bar{Q}_1 = Q_1^n - Q_1^p$, $\bar{P}_1 = P_1^n - P_1^p$, $\bar{L}_1 = L_1^n - L_1^p$, and $\bar{I}_y = I_y^n - I_y^p$ can be derived (in approximation (17)) exactly in the same way as the equations for isoscalar ones. As is expected, they coincide with (25).

As we see, in the considered simple model all results of WFM method are identical to that of Green's Function (GF) method. Having in mind also that both methods generate the same set of dynamical equations for collective variables (Wigner function moments), one could suspect their identity. In general, this is not quite true. The principal difference between the two methods is more or less

obvious. In the GF method, one finds first the formal solution of Eq. (5) and only afterwards one takes the phase space moments of the found Wigner function to obtain the final solution of the physical problem. In the WFM method one takes from the beginning the phase space moments of Eq. (5) without any attempts to find the «natural» expression for the Wigner function. It is worth noting also, that the initial conditions in the two methods are quite different. It is the static, or equilibrium, distribution function f_0 in GF method, which is not always known exactly, and one is forced to use some approximations. In the WFM method the initial conditions are given by the equilibrium values of the natural nucleus characteristics (such as the mean square radius, the quadrupole moment, etc.) which can be taken from the experiment.

The reason of coincidence of all results is quite simple. For the harmonic oscillator with multipole–multipole residual interaction of arbitrary rank (multipolarity) the equations of both methods can be derived without any approximations — the interaction of the multipolarity n generates the set of dynamical equations for tensors (moments) of the rank n. For the GF method this is easily seen from formula (136). In the case of the WFM method it is seen very well from the structure of Eq. (134). When one takes the moments of rank n, neither the left-hand side nor the right-hand side of this equation, can generate moments of rank higher than n. The coincidence of results in the case of $n = 3$ was demonstrated in [38].

The power and simplicity of the GF method are restricted by the potentials for which the analytical solutions for classical trajectories are known. In the case of realistic forces the GF method loses its simplicity and transparency, whereas the WFM method does not meet any difficulties and continues to be a convenient and powerful tool for the description of the collective motion what was demonstrated by calculations with Skyrme forces [17]. For an illustration of this property of the WFM method, currents are a good example, because the procedure of their construction with WFM is general enough to be used for any type of force (see Subsec. 5.1 below and [6]).

5. FLOWS

We are interested in the trajectories of infinitesimal displacements of neutrons and protons during their vibrational motion, i.e., in the lines of currents. The infinitesimal displacements are determined by the magnitudes and directions of the nucleon velocities $\mathbf{u}(\mathbf{r}, t)$, given by

$$mn(\mathbf{r}, t)\mathbf{u}(\mathbf{r}, t) = \int \frac{4d^3p}{(2\pi\hbar)^3}\, \mathbf{p} f(\mathbf{r}, \mathbf{p}, t) =$$
$$= \frac{4}{(2\pi\hbar)^3} \int d^3s \int d^3p\, \mathbf{p} \exp\left(-i\mathbf{p}\cdot\mathbf{s}/\hbar\right) \rho\left(\mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{r} - \frac{\mathbf{s}}{2}, t\right) =$$

$$= -2i\hbar\{(\nabla - \nabla')\rho(\mathbf{r}, \mathbf{r}', t)\}_{r=r'} = -\frac{i\hbar}{2}\sum_{\sigma,\tau}\{(\nabla - \nabla')\rho(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma\tau, t)\}_{r=r'} =$$

$$= -\frac{i\hbar}{2}\sum_{pq}\sum_{\sigma,\tau}\{\phi_p^*(\mathbf{r}\sigma\tau)\nabla\phi_q(\mathbf{r}\sigma\tau) - \phi_q(\mathbf{r}\sigma\tau)\nabla\phi_p^*(\mathbf{r}\sigma\tau)\}\langle\Psi|a_p^\dagger a_q|\Psi\rangle =$$

$$= m\sum_{pq} j_{pq}(\mathbf{r})\rho_{qp}(t) = m\langle\Psi|\sum_{pq} j_{pq}(\mathbf{r})a_p^\dagger a_q|\Psi\rangle = m\langle\Psi|\hat{J}(\mathbf{r})|\Psi\rangle. \quad (149)$$

The current density operator $\hat{J}(\mathbf{r})$ has the standard quantum mechanical definition [9]:

$$\hat{J}(\mathbf{r}) = \sum_{s=1}^{A}\hat{j}_s(\mathbf{r}) = -\frac{i\hbar}{2m}\sum_{s=1}^{A}[\delta(\mathbf{r} - \hat{\mathbf{r}}_s)\nabla_s + \nabla_s\delta(\mathbf{r} - \hat{\mathbf{r}}_s)] = \sum_{pq} j_{pq}(\mathbf{r})a_p^\dagger a_q,$$

$$j_{pq}(\mathbf{r}) = -\frac{i\hbar}{2m}\langle p|[\delta(\mathbf{r} - \hat{\mathbf{r}})\nabla + \nabla\delta(\mathbf{r} - \hat{\mathbf{r}})]|q\rangle =$$

$$= \frac{i\hbar}{2m}\sum_{\sigma,\tau}[\phi_q(\mathbf{r}\sigma\tau)\nabla\phi_p^*(\mathbf{r}\sigma\tau) - \phi_p^*(\mathbf{r}\sigma\tau)\nabla\phi_q(\mathbf{r}\sigma\tau)] =$$

$$= 4\frac{i\hbar}{2m}[\phi_q(\mathbf{r})\nabla\phi_p^*(\mathbf{r}) - \phi_p^*(\mathbf{r})\nabla\phi_q(\mathbf{r})].$$

The variation of $\mathbf{u}$ generated by the external field (122) is

$$n^{\mathrm{eq}}(\mathbf{r})\delta\mathbf{u}(\mathbf{r}, t) = \sum_{pq} j_{pq}(\mathbf{r})\rho_{qp}^{(1)}(t) =$$

$$= \sum_{\nu}[\langle 0|\hat{J}(\mathbf{r})|\nu\rangle c_\nu - \langle\nu|\hat{J}(\mathbf{r})|0\rangle\bar{c}_\nu]\,\mathrm{e}^{-i\Omega t} +$$

$$+ \sum_{\nu}[\langle\nu|\hat{J}(\mathbf{r})|0\rangle c_\nu^* - \langle 0|\hat{J}(\mathbf{r})|\nu\rangle\bar{c}_\nu^*]\,\mathrm{e}^{i\Omega t}. \quad (150)$$

To proceed further, three options are possible.

5.1. WFM Method. The first way was developed within the WFM approach [17]. It allows one to derive an approximate analytical expression for $\delta\mathbf{u}(\mathbf{r}, t)$. The main idea lies in the parameterization of infinitesimal displacements $\boldsymbol{\xi}_i(\mathbf{r}, t) \equiv dx_i$, which are represented by the expansion

$$\xi_i(\mathbf{r}, t) = G_i(t) + \sum_{j=1}^{3} G_{i,j}(t)x_j + \sum_{j,k=1}^{3} G_{i,jk}(t)x_j x_k +$$

$$+ \sum_{j,k,l=1}^{3} G_{i,jkl}(t)x_j x_k x_l + \ldots \quad (151)$$

This series, in principle, is infinite, however one makes the approximation keeping only the first terms and neglecting the remainder. For example, in [6] only the two first terms were kept. It turns out that the G_i do not contribute to the final results due to the triplanar symmetry of considered nuclei, hence

$$\xi_i^\tau(\mathbf{r},t) = \sum_{j=1}^{3} G_{i,j}^\tau(t) x_j.$$

The coefficients $G_{i,j}$ can be expressed analytically in terms of the variables $\bar{\mathcal{R}}_{21}(t)$ and $\bar{\mathcal{L}}_{11}(t)$. Really, small variations $\mathcal{R}_{\lambda\mu}^\tau \equiv \delta R_{\lambda\mu}^\tau$ and $\mathcal{L}_{\lambda\mu}^\tau \equiv \delta L_{\lambda\mu}^\tau$ are naturally expressed in terms of variations of $n^\tau(\mathbf{r},t)$ and $u_i^\tau(\mathbf{r},t)$ (13):

$$\begin{aligned}
\mathcal{R}_{\lambda\mu}^\tau(t) &= \int d^3r\, r_{\lambda\mu}^2 \delta n^\tau(\mathbf{r},t), \\
\mathcal{L}_{\lambda\mu}^\tau(t) &= m\int d^3r\,[(ru_{\mathrm{eq}}^\tau)_{\lambda\mu}\delta n^\tau + (r\delta u^\tau)_{\lambda\mu} n_{\mathrm{eq}}^\tau] = m\int d^3r\,(r\delta u^\tau)_{\lambda\mu} n_{\mathrm{eq}}^\tau.
\end{aligned} \tag{152}$$

In the last equation we have supposed that $u_{\mathrm{eq}}^\tau = 0$, i.e., there is no motion at equilibrium. The variations δn and δu_i are not independent. A relation between them is obtained by means of the continuity equation [10]

$$\delta n = -\sum_{i=1}^{3} \nabla_i(n\xi_i), \quad \delta u_i = \frac{\partial \xi_i}{\partial t}.$$

It is convenient to introduce the «cyclic» combinations of ξ_i analogously to the cyclic variables in (5):

$$\rho_{+1}^\tau = -\frac{1}{\sqrt{2}}(\xi_1^\tau + i\xi_2^\tau), \quad \rho_0^\tau = \xi_3^\tau, \quad \rho_{-1}^\tau = \frac{1}{\sqrt{2}}(\xi_1^\tau - i\xi_2^\tau)$$

and to write them as $\rho_\mu^\tau(\mathbf{r},t) = \sum_{\nu=-1}^{+1} (-1)^\nu S_{\mu,-\nu}^\tau(t) r_\nu$. Then

$$\begin{aligned}
\delta n^\tau &= -\sum_{i=1}^{3} \nabla_i(n^\tau \xi_i^\tau) = -\sum_{\nu=-1}^{+1} (-1)^\nu \nabla_\nu(n^\tau \rho_{-\nu}^\tau), \\
\delta u_\mu^\tau &= \frac{\partial \rho_\mu^\tau}{\partial t} = \sum_{\nu=-1}^{+1} (-1)^\nu \dot{S}_{\mu,-\nu}^\tau(t) r_\nu.
\end{aligned}$$

Using these expressions one finds

$$\mathcal{R}^{\tau}_{\lambda\mu}(t) = -\int d^3r \sum_{\sigma,\nu} C^{\lambda\mu}_{1\sigma,1\nu} r_\sigma r_\nu \sum_{\phi=-1}^{+1} (-1)^\phi \nabla_\phi (n^\tau \rho^\tau_{-\phi}) =$$

$$= \sum_{\sigma,\nu} C^{\lambda\mu}_{1\sigma,1\nu} \int d^3r\, n^\tau_{\rm eq} (\rho^\tau_\sigma r_\nu + \rho^\tau_\nu r_\sigma) =$$

$$= 2\sum_{\phi,\sigma,\nu} C^{\lambda\mu}_{1\sigma,1\nu} (-1)^\phi \int d^3r\, n^\tau_{\rm eq} S^\tau_{\sigma,-\phi} r_\phi r_\nu =$$

$$= 2\sum_{k,\kappa}\sum_{\phi,\sigma,\nu} C^{\lambda\mu}_{1\sigma,1\nu} (-1)^\phi S^\tau_{\sigma,-\phi} C^{k\kappa}_{1\phi,1\nu} R^\tau_{k\kappa}({\rm eq}).$$

Now taking into account the axial symmetry ($\kappa = 0$) one gets

$$\mathcal{R}^{\tau}_{\lambda\mu} = \frac{2}{\sqrt{3}}\Bigg[(\sqrt{2}R^\tau_{20} - R^\tau_{00}) C^{\lambda\mu}_{1\mu,10} S^\tau_{\mu,0} -$$

$$- \left(\frac{1}{\sqrt{2}} R^\tau_{20} + R^\tau_{00}\right)\left(C^{\lambda\mu}_{1\mu+1,1-1} S^\tau_{\mu+1,-1} + C^{\lambda\mu}_{1\mu-1,11} S^\tau_{\mu-1,1}\right)\Bigg].$$

Exactly the same derivation for $\mathcal{L}^\tau_{\lambda\mu}$ leads to the following result:

$$\mathcal{L}^{\tau}_{\lambda\mu} = m\sum_{\sigma,\nu} C^{\lambda\mu}_{1\sigma,1\nu} \int d^3r\, n^\tau_{\rm eq} \dot\rho^\tau_\nu r_\sigma =$$

$$= (-1)^\lambda \frac{m}{\sqrt{3}}\Bigg[(\sqrt{2}R^\tau_{20} - R^\tau_{00}) C^{\lambda\mu}_{1\mu,10} \dot S^\tau_{\mu,0} -$$

$$- \left(\frac{1}{\sqrt{2}} R^\tau_{20} + R^\tau_{00}\right)\left(C^{\lambda\mu}_{1\mu+1,1-1} \dot S^\tau_{\mu+1,-1} + C^{\lambda\mu}_{1\mu-1,11} \dot S^\tau_{\mu-1,1}\right)\Bigg].$$

We are interested in $\bar{\mathcal{R}}_{21}$ and $\bar{\mathcal{L}}_{11}$. Remembering that $R_{00} = -Q_{00}/\sqrt{3}$, $R_{20} = \left(\frac{2}{3}\right)^{3/2} Q_{00}\delta$, and $Q^\tau_{00} = \frac{1}{2}Q_{00}$ (due to approximation (17)), we find

$$\bar{\mathcal{R}}_{2\pm1} = \frac{1}{3\sqrt{2}} Q_{00}\left[\left(1 - \frac{2}{3}\delta\right)\bar S_{0,\pm1} + \left(1 + \frac{4}{3}\delta\right)\bar S_{\pm1,0}\right],$$

$$\bar{\mathcal{L}}_{1\pm1} = \frac{m}{6\sqrt{2}} Q_{00}\left[\left(1 - \frac{2}{3}\delta\right)\dot{\bar S}_{0,\pm1} - \left(1 + \frac{4}{3}\delta\right)\dot{\bar S}_{\pm1,0}\right],$$

where $\bar S_{\sigma,\nu} = S^n_{\sigma,\nu} - S^p_{\sigma,\nu}$ (and $S_{\sigma,\nu} = S^n_{\sigma,\nu} + S^p_{\sigma,\nu}$). Having in mind the $\mathrm{e}^{i\Omega t}$ time dependence (vibrational motion), we can substitute $\dot{\bar S}_{\sigma,\nu}$ by $i\Omega \bar S_{\sigma,\nu}$. Solving

these equations with respect to $\bar{S}_{\sigma,\nu}$, we have

$$\bar{S}_{0,1} = \frac{3}{\sqrt{2}}\left[\bar{\mathcal{R}}_{21} - \frac{2i}{m\Omega}\bar{\mathcal{L}}_{11}\right] \Big/ \left[Q_{00}\left(1 - \frac{2}{3}\delta\right)\right],$$

$$\bar{S}_{1,0} = \frac{3}{\sqrt{2}}\left[\bar{\mathcal{R}}_{21} + \frac{2i}{m\Omega}\bar{\mathcal{L}}_{11}\right] \Big/ \left[Q_{00}\left(1 + \frac{4}{3}\delta\right)\right].$$

Now we use the set of Eqs. (24) to find that $\bar{\mathcal{L}}_{11} = -\dfrac{i}{\Omega} m\bar{\omega}^2\delta(1-\alpha)\bar{\mathcal{R}}_{21}$ and, as a result,

$$\bar{\mathcal{R}}_{21} \mp \frac{2i}{m\Omega}\bar{\mathcal{L}}_{11} = \left[1 \mp 2\frac{\bar{\omega}^2}{\Omega^2}(1-\alpha)\delta\right]\bar{\mathcal{R}}_{21}.$$

Introducing the notation

$$A = \frac{3}{\sqrt{2}}\left[1 - 2\frac{\bar{\omega}^2}{\Omega^2}(1-\alpha)\delta\right] \Big/ \left[Q_{00}(1 - \frac{2}{3}\delta)\right],$$

$$B = \frac{3}{\sqrt{2}}\left[1 + 2\frac{\bar{\omega}^2}{\Omega^2}(1-\alpha)\delta\right] \Big/ \left[Q_{00}(1 + \frac{4}{3}\delta)\right], \tag{153}$$

we finally get

$$\bar{S}_{0,1} = A\bar{\mathcal{R}}_{21}, \quad \bar{S}_{1,0} = B\bar{\mathcal{R}}_{21}.$$

A similar analysis of $\bar{\mathcal{R}}_{2-1}$ and $\bar{\mathcal{L}}_{1-1}$ allows us to write immediately

$$\bar{S}_{0,-1} = A\bar{\mathcal{R}}_{2-1}, \quad \bar{S}_{-1,0} = B\bar{\mathcal{R}}_{2-1}.$$

So we have for isovector «cyclic» displacements:

$$\begin{aligned}
\bar{\rho}_{+1} &= \bar{S}_{1,0}r_0 = B\bar{\mathcal{R}}_{21}x_3, \\
\bar{\rho}_{-1} &= \bar{S}_{-1,0}r_0 = B\bar{\mathcal{R}}_{2-1}x_3, \\
\bar{\rho}_0 &= -\bar{S}_{0,1}r_{-1} - \bar{S}_{0,-1}r_{+1} = \sqrt{2}A(\bar{\mathcal{J}}_{13}x_1 + \bar{\mathcal{J}}_{23}x_2),
\end{aligned}$$

where $\bar{\mathcal{J}}_{13} = (\bar{\mathcal{R}}_{2-1} - \bar{\mathcal{R}}_{21})/2$, $\bar{\mathcal{J}}_{23} = i(\bar{\mathcal{R}}_{2-1} + \bar{\mathcal{R}}_{21})/2$. The variable $\mathcal{J}^{\tau}_{ij}$ is a small variation of the tensor $J^{\tau}_{ij} = \displaystyle\int d\{\mathbf{p},\mathbf{r}\}x_i x_j f^{\tau}(\mathbf{r},\mathbf{p},t)$. Isovector cartesian displacements are found by elementary means:

$$\begin{aligned}
\bar{\xi}_1 &= \frac{1}{\sqrt{2}}(\bar{\rho}_{-1} - \bar{\rho}_{+1}) = \sqrt{2}B\bar{\mathcal{J}}_{13}x_3, \\
\bar{\xi}_2 &= \frac{i}{\sqrt{2}}(\bar{\rho}_{-1} + \bar{\rho}_{+1}) = \sqrt{2}B\bar{\mathcal{J}}_{23}x_3, \\
\bar{\xi}_3 &= \bar{\rho}_0 = \sqrt{2}A(\bar{\mathcal{J}}_{13}x_1 + \bar{\mathcal{J}}_{23}x_2).
\end{aligned} \tag{154}$$

By definition, the infinitesimal displacements ξ_i are the differentials ($\xi_1 = dx, \xi_2 = dy, \xi_3 = dz$). This fact allows one to construct the differential equations for current fields. For example, for the current field in the plane $x = 0$ we have

$$\frac{dy}{dz} = \frac{B}{A}\frac{z}{y} \to ydy - \frac{B}{A}zdz = 0. \tag{155}$$

Integrating this equation we find

$$y^2 + \sigma z^2 = \text{const} \equiv c \to \frac{y^2}{c} + \frac{z^2}{c/\sigma} = 1, \tag{156}$$

where $\sigma = -B/A$. Depending on the sign of σ this curve will be either an ellipse or a hyperbola. The careful analysis [6] of the δ dependence of σ shows that in the case of the scissors mode $\sigma > 0$ for all permitted δ (i.e., $-3/4 < \delta < 3/2$) and in the case of IVGQR $\sigma < 0$ for all permitted δ. Therefore the curve (156) is an ellipse for the scissors mode and it is a hyperbola for IVGQR (see Figs. 1, 2).

One can easily see in Fig. 1 that the main constituent of the scissors-mode motion is the rotation (out of phase rotation of neutrons and protons). It is also seen that the rotation is accompanied by the distortion of the nuclear shape — at least it is evident that the long semiaxis becomes smaller. To get a quantitative measure for the contribution of each kind of motion, it is sufficient to write the displacement $\boldsymbol{\xi}$ as the superposition of a rotational component with the coefficient a and an irrotational one with the coefficient b [4]:

$$\boldsymbol{\xi} = a\mathbf{e}_x \times \mathbf{r} + b\nabla(yz) = a(0, -z, y) + b(0, z, y).$$

Comparing the components $\xi_y = (b-a)z$, $\xi_z = (b+a)y$ with ξ_2, ξ_3 in (154), we find

$$b - a = \sqrt{2}\bar{J}_{23}B, \quad b + a = \sqrt{2}\bar{J}_{23}A \to a = \eta(1+\sigma), \quad b = \eta(1-\sigma),$$

where $\eta = \bar{J}_{23}A/\sqrt{2}$. So, for the scissors mode in the small δ limit we have

$$a = 2\eta\left(1 - \frac{3}{4}\delta\right), \quad b = \frac{3}{2}\eta\delta, \quad \frac{b}{a} \simeq \frac{3}{4}\delta\left(1 + \frac{3}{4}\delta\right) \approx \frac{3}{4}\delta, \tag{157}$$

i.e., the current of the scissors mode is dominated by the rotational motion. The contributions of the two kinds of motion to the IVGQR are

$$a = \frac{1}{2}\eta\delta, \quad b = 2\eta\left(1 - \frac{1}{4}\delta\right), \quad \frac{a}{b} \simeq \frac{1}{4}\delta\left(1 + \frac{1}{4}\delta\right) \approx \frac{1}{4}\delta, \tag{158}$$

i.e., the current of the IVGQR is dominated by the irrotational motion.

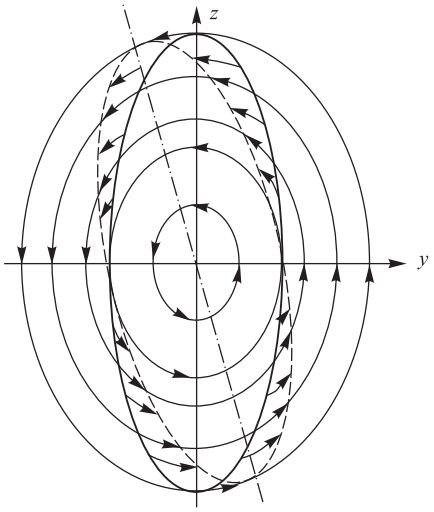

Fig. 1. Schematic picture of isovector displacements for the scissors mode. Thin ellipses are the lines of currents. The thick oval is the initial position of the nucleus' surface (common for protons and neutrons). The dashed oval is the final position of the protons' (or neutrons') surface as a result of infinitesimal displacements shown by the arrows

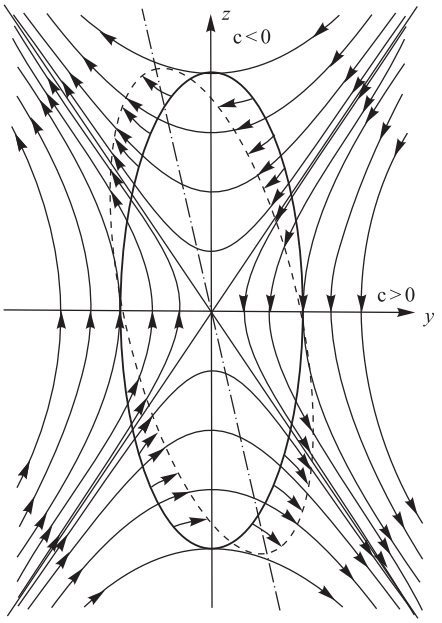

Fig. 2. Schematic picture of isovector displacements for the high-lying mode (IVGQR). The lines of currents are shown by thin lines (hyperbolae). The thick oval is the initial position of the nucleus' surface (common for protons and neutrons). The dashed oval is the final position of the protons' (or neutrons') surface as a result of infinitesimal displacements shown by the arrows

Transition currents are calculated in WFM analogously to transition probabilities. The pole structure of the right-hand side of Eq. (150) tells us that the transition current can be calculated by means of an expression similar to (34):

$$\langle 0|\hat{J}_i(\mathbf{r})|\nu\rangle = \hbar \lim_{\Omega\to\Omega_\nu} (\Omega - \Omega_\nu) n^{\mathrm{eq}}(\mathbf{r}) \overline{\dot{\xi}_i(\mathbf{r},t)\, \exp(i\Omega t)} / \langle \nu|\hat{W}|0\rangle. \qquad (159)$$

For the ξ_i from above we obtain (using formulae (129) and (124))

$$\langle 0|\hat{J}_3(\mathbf{r})|\nu\rangle = -i\Omega_\nu n^{\rm eq}(\mathbf{r})\frac{A}{\sqrt{2}}[\langle 0|\hat{R}_{2-1} - \hat{R}_{21}|\nu\rangle x_1 + i\langle 0|\hat{R}_{2-1} + \hat{R}_{21}|\nu\rangle x_2],$$

$$\langle 0|\hat{J}_2(\mathbf{r})|\nu\rangle = \Omega_\nu n^{\rm eq}(\mathbf{r})\frac{B}{\sqrt{2}}\langle 0|\hat{R}_{2-1} + \hat{R}_{21}|\nu\rangle x_3, \tag{160}$$

$$\langle 0|\hat{J}_1(\mathbf{r})|\nu\rangle = -i\Omega_\nu n^{\rm eq}(\mathbf{r})\frac{B}{\sqrt{2}}\langle 0|\hat{R}_{2-1} - \hat{R}_{21}|\nu\rangle x_3.$$

As is seen, transition currents are proportional to transition probabilities.

If necessary, one can find the next term of the series (151). To calculate the respective coefficients $G_{i,jkl}(t)$ in the WFM method one is obliged to derive (and solve) the set of dynamical equations for higher (fourth) order moments of the Wigner function. Examples of similar calculations for third-rank tensors can be found in [12].

5.2. RPA Method. The procedure of constructing the flow distributions in RPA is more complicated. It is necessary at first to calculate transition currents. Having solutions (72) for X^ν_{mi}, Y^ν_{mi}, one can do it with the help of formula (85):

$$\langle 0|\hat{J}(\mathbf{r})|\nu\rangle = \sum_{mi}(j_{im}X^\nu_{mi} + j_{mi}Y^\nu_{mi}) = K_\nu \sum_{mi}\left\{\frac{j_{im}\mathcal{Q}^*_{im}}{E_\nu - \epsilon_{mi}} - \frac{j_{mi}\mathcal{Q}^*_{mi}}{E_\nu + \epsilon_{mi}}\right\} =$$

$$= K_\nu\left\{\sum_{mi(\Delta N=0)}\left[\frac{j_{im}\mathcal{Q}^*_{im}}{E_\nu - \epsilon_0} - \frac{j_{mi}\mathcal{Q}^*_{mi}}{E_\nu + \epsilon_0}\right] + \right.$$

$$\left. + \sum_{mi(\Delta N=2)}\left[\frac{j_{im}\mathcal{Q}^*_{im}}{E_\nu - \epsilon_2} - \frac{j_{mi}\mathcal{Q}^*_{mi}}{E_\nu + \epsilon_2}\right]\right\}. \tag{161}$$

The operator $\mathcal{Q}$ has a finite number of particle hole matrix elements $\mathcal{Q}_{mi}$, so, in principle, the sums in (161) can be calculated exactly. The same is true for the coefficients c_ν (124). Therefore, in accordance with (150) one could hope to find the exact RPA result for the velocity distribution $\delta\mathbf{u}(\mathbf{r}, t)$. Unfortunately, because of the pole structure of coefficients $c_\nu(\Omega)$, it can be done for any Ω except the required frequency Ω_ν corresponding to the considered mode (resonance). Of course, it is clear that in the case of Ω close enough to Ω_ν the main contribution into $\delta\mathbf{u}$ comes from the single matrix element $\langle 0|\hat{J}(\mathbf{r})|\nu\rangle$. That is why, to get an idea about the distribution of currents in the RPA eigenstate $|\nu\rangle$ it is sufficient to know the transition matrix element $\langle 0|\hat{J}(\mathbf{r})|\nu\rangle$. However, even in this simple model one cannot find a compact analytical expression for sums in (161) — the field of velocities can be constructed only numerically.

As we have already seen it is much more convenient to deal with lines of currents. The differential equation for them can be derived with the help of

formula (150). With a $e^{-i\Omega t}$ time dependence we rewrite it in the more convenient form

$$-n^{\rm eq}(\mathbf{r})i\Omega\xi_i(\mathbf{r}) = \sum_\sigma [\langle 0|\hat{J}_i(\mathbf{r})|\sigma\rangle c_\sigma - \langle\sigma|\hat{J}_i(\mathbf{r})|0\rangle\bar{c}_\sigma]$$

and define the ratio

$$\frac{\xi_2(\mathbf{r})}{\xi_3(\mathbf{r})} = \frac{\sum_\sigma [\langle 0|\hat{J}_2(\mathbf{r})|\sigma\rangle c_\sigma - \langle\sigma|\hat{J}_2(\mathbf{r})|0\rangle\bar{c}_\sigma]}{\sum_\sigma [\langle 0|\hat{J}_3(\mathbf{r})|\sigma\rangle c_\sigma - \langle\sigma|\hat{J}_3(\mathbf{r})|0\rangle\bar{c}_\sigma]}.$$

Remembering the definition of ξ_i and c_σ, multiplying the numerator and the denominator of the right-hand side by $(\Omega - \Omega_\nu)$ and taking the limit $\Omega \to \Omega_\nu$, we arrive to the differential equation

$$\frac{dy}{dz} = \frac{\langle 0|\hat{J}_2(\mathbf{r})|\nu\rangle}{\langle 0|\hat{J}_3(\mathbf{r})|\nu\rangle}, \tag{162}$$

which determines the lines of currents for the resonance state $|\nu\rangle$.

5.3. Green's Function Method. The distribution function being known, one can calculate the distribution of nuclear currents $\mathbf{j}^\tau(\mathbf{r},t) = mn^\tau(\mathbf{r},t)\mathbf{u}^\tau(\mathbf{r},t)$. There are no any currents in the equilibrium state, so we have

$$\begin{aligned} j_x^\tau(\mathbf{r},t) = mn^\tau(\mathbf{r},t)\delta u_x^\tau(\mathbf{r},t) = \int \frac{2d^3p}{(2\pi\hbar)^3}\, p_x f_1^\tau(\mathbf{r},\mathbf{p},t) = \\ = -\frac{z}{2m\omega_x}[C_+^\tau(t)\omega_+ + C_-^\tau(t)\omega_-]\int \frac{2d^3p}{(2\pi\hbar)^3}\, p_x^2\delta(\epsilon_F - H_0) = \\ = -\frac{z}{\omega_x}\frac{2\pi}{3}[2m\epsilon_F - m^2\omega_x^2(x^2+y^2) - m^2\omega_z^2 z^2]^{3/2}[C_+^\tau(t)\omega_+ + C_-^\tau(t)\omega_-] = \\ = -\frac{z}{2m\omega_x} n_0^\tau(\mathbf{r})[C_+^\tau(t)\omega_+ + C_-^\tau(t)\omega_-], \end{aligned} \tag{163}$$

where the following notation is introduced:

$$C_\pm^\tau(t) = \int_{-\infty}^{t} dt'\, Z_1^\tau(t')\, \cos\,\omega_\pm(t'-t).$$

Deriving (163) we used the approximation (17) which means, in particular, that $\omega_i^n = \omega_i^p$ and $n_0^n = n_0^p = n_0/2$. Another component of the flow is

$$j_z^\tau(\mathbf{r},t) = \int \frac{2d^3p}{(2\pi\hbar)^3}\, p_z f_1^\tau(\mathbf{r},\mathbf{p},t) = -\frac{x+y}{2m\omega_z} n_0^\tau(\mathbf{r})[C_+^\tau(t)\omega_+ - C_-^\tau(t)\omega_-]. \tag{164}$$

With the help of the isovector counterpart of formulae (138) the functions $C^\tau_\pm(t)$ can be written via dynamical variables

$$C^\tau_-(t) = \left[L^\tau_1(t) - \frac{\omega_+}{\omega_-} I^\tau_y(t)\right]/\zeta, \quad C^\tau_+(t) = -\left[L^\tau_1(t) - \frac{\omega_-}{\omega_+} I^\tau_y(t)\right]/\zeta$$

with $\zeta = \dfrac{2\pi^3\epsilon_F^4}{3m\omega_x^3\omega_z^2}\dfrac{2}{(2\pi\hbar)^3} = \beta m\omega_x\omega_z/6\kappa_0$, and the required combinations are

$$\begin{aligned} C^\tau_+(t)\omega_+ + C^\tau_-(t)\omega_- &= -2\omega_z[L^\tau_1(t) + I^\tau_y(t)]/\zeta, \\ C^\tau_+(t)\omega_+ - C^\tau_-(t)\omega_- &= -2\omega_x[L^\tau_1(t) - I^\tau_y(t)]/\zeta. \end{aligned} \tag{165}$$

We are interested in isovector flows $\bar{j}_x = j^n_x - j^p_x$ and $\bar{j}_z = j^n_z - j^p_z$. With the help of the first and last equations of (25) we find

$$\begin{aligned} \bar{C}_+(t)\omega_+ + \bar{C}_-(t)\omega_- &= -2\omega_z i\Omega m\left[1 + 2\frac{\bar{\omega}^2}{\Omega^2}(1-\alpha)\delta\right]\frac{\bar{Q}_1}{\zeta}, \\ \bar{C}_+(t)\omega_+ - \bar{C}_-(t)\omega_- 7 &= -2\omega_x i\Omega m\left[1 - 2\frac{\bar{\omega}^2}{\Omega^2}(1-\alpha)\delta\right]\frac{\bar{Q}_1}{\zeta}. \end{aligned} \tag{166}$$

As a result, we have the explicit expressions for currents

$$\begin{aligned} \bar{j}_z(\mathbf{r},t) &= \frac{i\Omega}{2\zeta}\left[1 - 2\frac{\bar{\omega}^2}{\Omega^2}(1-\alpha)\delta\right]\bar{Q}_1(t)n_0(\mathbf{r})\frac{\omega_x}{\omega_z}(x+y), \\ \bar{j}_x(\mathbf{r},t) &= \frac{i\Omega}{2\zeta}\left[1 + 2\frac{\bar{\omega}^2}{\Omega^2}(1-\alpha)\delta\right]\bar{Q}_1(t)n_0(\mathbf{r})\frac{\omega_z}{\omega_x}z. \end{aligned} \tag{167}$$

Following the recipe of Subsec. 5.2 (formula (162)) we can derive the differential equation for lines of currents, for example, in the plane $x = 0$:

$$\frac{dy}{dz} = \frac{\bar{j}_y}{\bar{j}_z} \to \frac{dy}{dz} = \frac{z}{y}\frac{\omega_z^2}{\omega_x^2}\frac{1 + 2\dfrac{\bar{\omega}^2}{\Omega^2}(1-\alpha)\delta}{1 - 2\dfrac{\bar{\omega}^2}{\Omega^2}(1-\alpha)\delta} = \frac{z}{y}\frac{B}{A} \tag{168}$$

with A and B defined by (153). Obviously, this expression coincides exactly with formula (155). It is necessary to emphasize the principal point: the result (168) is obtained from the GF method in a direct way, whereas deriving formula (155) we made the strong approximation about truncating the expansion (151) which parameterizes the displacements. The agreement of both expressions is not surprising — at the end of Sec. 5 we have shown that in the case of the harmonic oscillator with multipole–multipole residual interactions the WFM and GF methods give identical results.

5.4. Summary of Flow Calculations. In conclusion, in full RPA one must calculate the currents numerically leading to fine details (shell effects) whereas in WFM and GF treatments one obtains their gross structure with analytical formulas. The latter feature is quite important in order to understand the real character of the motion under study since current patterns produced numerically from complicated formulas with a lot of summations like in (161) can hardly be interpreted physically. A good example is the interplay of the scissors mode and the isovector giant quadrupole resonance. Looking only at the flow patterns (see Figs. 1, 2) one would not be able to tell that the former is mostly rotational with a small amount of an irrotational component and the other way round for the latter, as can be seen from Eqs. (157), (158).

CONCLUSION

In this paper, we made an exhaustive comparison of different methods to treat collective excitations in nuclei, like the scissors mode, isovector and isoscalar giant quadrupole resonances. This comparison was exemplified by the harmonic oscillator plus separable quadrupole–quadrupole force model, but it has a more general character.

We investigated the WFM, RPA, and Green's Function (GF) methods. Under certain circumstances all three methods give essentially the same results. For example, all methods give in our model the same analytical expressions for energies and transition probabilities for all the excitations considered. It turned out that the WFM and GF methods are very close to one another. Contrary to the RPA, both work in phase space and incorporate semiclassical aspects, with no need to introduce a single particle basis. Finally, both the methods yield identical sets of dynamical equations for the moments. However, in the case of realistic forces the GF method loses its simplicity and a more complicated pseudoparticle method [39] has to be applied, whereas the WFM method continues to be a convenient and powerful tool for the description of the collective motions, as was demonstrated in [14–17] by employing Skyrme forces.

To show the analytical equivalence between the WFM and RPA methods one needs to introduce the dynamical equations for the transition matrix elements. They can be derived either from the RPA equations for the amplitudes X_{kq}, Y_{kq} or from the WFM dynamical equations for the moments. This proves the identity of eigenvalues in both methods under the condition that a complete basis is used in both the cases. However, both the methods behave differently when the dimension of the space is reduced. Actually, the WFM is designed to use only rather a few moments of low rank, which play the role of collective coordinates

of the model. The restricted number of eigenvalues approximate the collective states in an optimal way, representing, e.g., their centroid positions, as this was shown in [14, 15, 17, 34]. In this sense, the WFM has similarity with the sum rule approach [9] which works, however, only in the cases when practically all strength is exhausted by one state, whereas the WFM method works also in situations when the strength is distributed among several excitations. On the contrary, in the RPA one needs in general rather large space to correctly account for the collectivity of, e.g., the giant resonances. This demonstrates very well the difference between the two approaches: the RPA describes the fine structure of collective excitations, whereas the WFM method yields the corresponding gross structure.

It makes no sense to speak about advantages or disadvantages of one of the two discussed methods — they are complementary. Of course, RPA gives complete, exhaustive information concerning the microscopic (particle–hole) structure of collective excitations. However, sometimes a considerable additional effort is required to understand their physical nature. On the contrary, the WFM method gives direct information on the physical nature of excitations. One should note that from the quantum mechanical point of view Wigner function moments (being the average values of some operators) are the quantum observables, which relieves the physical interpretation of the corresponding excitation. Our results serve as a very good illustration of this situation. What do we learn about the scissors mode and IVGQR from each of the two methods? RPA says that the scissors mode is mostly created by $\Delta N = 0$ particle–hole excitations with a small admixture of $\Delta N = 2$ particle–hole excitations and vice versa for IVGQR. Without further effort — this is about all. One does not even suspect the key role of the relative angular momentum in the creation of the scissors mode. On the other hand, the WFM method directly reveals that the scissors mode appears due to oscillations of the relative angular momentum with a small admixture of the quadrupole mode and vice versa for IVGQR. Further, it informs us about the extremely important role of the Fermi surface deformation in the formation of the scissors mode.

The principal difference between the two methods is revealed in their practical use (applications). The RPA equations (70) are written in the very general and convenient for calculations form: the equations for every new pair of amplitudes X_{kq}, Y_{kq} are obtained simply by changing the indices k, q. As a result, there are no big problems to write the set of equations of arbitrary large dimension which can be solved by modern computers without any difficulties. The situation with moments is quite different. It is difficult to write in general terms the dynamical equation for tensors of arbitrary rank n. The equations for Cartesian tensors of every rank must be derived separately (see the text after formula (4)). Even the procedure of the reduction of Cartesian tensors to irreducible ones becomes practically hopeless for large n. And, as a matter of fact, there is no necessity in further increasing n, because it contradicts the grand idea of the WFM method —

to describe the main features of the phenomenon (its gross structure) by the minimal set of the most essential physical characteristics. Nevertheless, taking into account moments of higher and higher rank, one can produce a more and more detailed description of the phenomenon, achieving (at least in principle) the maximally fragmented picture given by the experiment (and RPA).

Two new mathematical results are obtained for the HO + QQ model. We have proved exactly, without any approximations, the orthogonality of the «spurious» state to all physical states. In this sense, we have generalized the result of Lo Iudice [29] derived in a small deformation approximation. The analytical expressions are derived for the normalization factor of the synthetic scissors state and overlaps of this state with eigenstates of the model. We suggested also the differential equation to construct the current lines in the RPA.

Future work in this direction will deal with superfluidity and spin degrees of freedom.

APPENDIX A

It is known that the deformed harmonic oscillator Hamiltonian can be obtained in a Hartree approximation «by making the assumption that the isoscalar part of the QQ force builds the one-body container well» [21]. In our case it is obtained quite easily by summing the expressions for V^p and V^n (formula (7)):

$$V(\mathbf{r},t) = \frac{1}{2}(V^p(\mathbf{r},t) + V^n(\mathbf{r},t)) = \\ = \frac{1}{2}m\omega^2 r^2 + \kappa_0 \sum_{\mu=-2}^{2} (-1)^\mu Q_{2-\mu}(t) q_{2\mu}(\mathbf{r}). \quad \text{(A.1)}$$

In the state of equilibrium (i.e., in the absence of an external field) $Q_{2\pm1} = Q_{2\pm2} = 0$. Using the definition [37] $Q_{20} = Q_{00}\frac{4}{3}\delta$ and the formula $q_{20} = 2z^2 - x^2 - y^2$ we obtain the potential of the anisotropic harmonic oscillator

$$V(\mathbf{r}) = \frac{m}{2}[\omega_x^2(x^2+y^2) + \omega_z^2 z^2]$$

with oscillator frequencies

$$\omega_x^2 = \omega_y^2 = \omega^2(1+\sigma\delta), \quad \omega_z^2 = \omega^2(1-2\sigma\delta),$$

where $\sigma = -\kappa_0\frac{8Q_{00}}{3m\omega^2}$. The definition of the deformation parameter δ must be reproduced by the harmonic oscillator wave functions, which allows one to fix the value of σ. We have

$$Q_{00} = \frac{\hbar}{m}\left(\frac{\Sigma_x}{\omega_x} + \frac{\Sigma_y}{\omega_y} + \frac{\Sigma_z}{\omega_z}\right), \quad Q_{20} = 2\frac{\hbar}{m}\left(\frac{\Sigma_z}{\omega_z} - \frac{\Sigma_x}{\omega_x}\right),$$

where $\Sigma_x = \sum_{i=1}^{A} \left(n_x + \frac{1}{2} \right)_i$ and n_x is the oscillator quantum number. Using the self-consistency condition [37]

$$\Sigma_x \omega_x = \Sigma_y \omega_y = \Sigma_z \omega_z = \Sigma_0 \omega_0,$$

where Σ_0 and ω_0 are defined in the spherical case, we get

$$\frac{Q_{20}}{Q_{00}} = 2\frac{\omega_x^2 - \omega_z^2}{\omega_x^2 + 2\omega_z^2} = \frac{2\sigma\delta}{1 - \sigma\delta} = \frac{4}{3}\delta.$$

Solving the last equation with respect to σ, we find

$$\sigma = \frac{2}{3 + 2\delta}. \tag{A.2}$$

Therefore, the oscillator frequencies and the strength constant can be written as

$$\omega_x^2 = \omega_y^2 = \bar{\omega}^2 \left(1 + \frac{4}{3}\delta \right), \quad \omega_z^2 = \bar{\omega}^2 \left(1 - \frac{2}{3}\delta \right), \quad \kappa_0 = -\frac{m\bar{\omega}^2}{4Q_{00}} \tag{A.3}$$

with $\bar{\omega}^2 = \omega^2/(1 + \frac{2}{3}\delta)$. The condition for volume conservation $\omega_x \omega_y \omega_z = \mathrm{const} = \omega_0^3$ makes ω δ-dependent

$$\omega^2 = \omega_0^2 \frac{1 + (2/3)\delta}{(1 + (4/3)\delta)^{2/3} (1 - (2/3)\delta)^{1/3}}.$$

So the final expressions for oscillator frequencies are

$$\omega_x^2 = \omega_y^2 = \omega_0^2 \left(\frac{1 + (4/3)\delta}{1 - (2/3)\delta} \right)^{1/3}, \quad \omega_z^2 = \omega_0^2 \left(\frac{1 - (2/3)\delta}{1 + (4/3)\delta} \right)^{2/3}. \tag{A.4}$$

It is interesting to compare these expressions with the very popular [9, 37] parameterization

$$\omega_x^2 = \omega_y^2 = \omega'^2 \left(1 + \frac{2}{3}\delta' \right), \quad \omega_z^2 = \omega'^2 \left(1 - \frac{4}{3}\delta' \right).$$

The volume conservation condition gives

$$\omega'^2 = \frac{\omega_0^2}{(1 + (2/3)\delta')^{2/3} (1 - (4/3)\delta')^{1/3}},$$

so the final expressions for oscillator frequencies are

$$\omega_x^2 = \omega_y^2 = \omega_0^2 \left(\frac{1 + (2/3)\delta'}{1 - (4/3)\delta'} \right)^{1/3}, \quad \omega_z^2 = \omega_0^2 \left(\frac{1 - (4/3)\delta'}{1 + (2/3)\delta'} \right)^{2/3}. \tag{A.5}$$

The direct comparison of expressions (A.4) and (A.5) allows one to establish the following relation between δ and δ':

$$\delta' = \frac{\delta}{1+2\delta}, \quad \delta = \frac{\delta'}{1-2\delta'}.$$

One more parameterization of oscillator frequencies can be found in the review [4]:

$$\omega_x^2 = \omega_y^2 = \frac{\omega''^2}{1-(2/3)\delta''}, \quad \omega_z^2 = \frac{\omega''^2}{1+(4/3)\delta''}.$$

One has from the volume conservation condition

$$\omega''^2 = \omega_0^2 \left(1-\frac{2}{3}\delta''\right)^{2/3} \left(1+\frac{4}{3}\delta''\right)^{1/3},$$

so the final expressions for oscillator frequencies are

$$\omega_x^2 = \omega_y^2 = \omega_0^2 \left(\frac{1+(4/3)\delta''}{1-(2/3)\delta''}\right)^{1/3}, \quad \omega_z^2 = \omega_0^2 \left(\frac{1-(2/3)\delta''}{1+(4/3)\delta''}\right)^{2/3}, \qquad \text{(A.6)}$$

that coincide exactly with (A.4), i.e., $\delta'' = \delta$.

It is easy to see that equations (A.4) correspond to the case when the deformed density $n(\mathbf{r})$ is obtained from the spherical density $n_0(r)$ by the scale transformation [20]

$$(x, y, z) \to (x\,\mathrm{e}^{\alpha/2}, y\,\mathrm{e}^{\alpha/2}, z\,\mathrm{e}^{-\alpha})$$

with

$$\mathrm{e}^{\alpha} = \left(\frac{1+(4/3)\delta}{1-(2/3)\delta}\right)^{1/3}, \quad \delta = \frac{3}{2}\frac{\mathrm{e}^{3\alpha}-1}{\mathrm{e}^{3\alpha}+2}, \qquad \text{(A.7)}$$

which conserves the volume and does not destroy the self-consistency, because the density and potential are transformed in the same way.

It is necessary to note that Q_{00} also depends on δ

$$Q_{00} = \frac{\hbar}{m}\left(\frac{\Sigma_x}{\omega_x}+\frac{\Sigma_y}{\omega_y}+\frac{\Sigma_z}{\omega_z}\right) = \frac{\hbar}{m}\Sigma_0\omega_0\left(\frac{2}{\omega_x^2}+\frac{1}{\omega_z^2}\right) = \\ = Q_{00}^0 \frac{1}{(1+(4/3)\delta)^{1/3}(1-(2/3)\delta)^{2/3}},$$

where $Q_{00}^0 = A\dfrac{3}{5}R^2$, $R = r_0 A^{1/3}$. As a result, the final expression for the strength constant becomes

$$\kappa_0 = -\frac{m\omega_0^2}{4Q_{00}^0}\left(\frac{1-(2/3)\delta}{1+(4/3)\delta}\right)^{1/3} = -\frac{m\omega_0^2}{4Q_{00}^0}\,\mathrm{e}^{-\alpha},$$

that coincides with the respective result of [20].

APPENDIX B

To calculate the sums $\mathcal{Q}_0 = \sum_{mi(\Delta N=0)} |\mathcal{Q}_{mi}|^2$ and $\mathcal{Q}_2 = \sum_{mi(\Delta N=2)} |\mathcal{Q}_{mi}|^2$ we employ the sum-rule techniques of Suzuki and Rowe [20]. The well-known harmonic oscillator relations

$$\begin{aligned} x\psi_{n_x} &= \sqrt{\frac{\hbar}{2m\omega_x}}(\sqrt{n_x}\psi_{n_x-1} + \sqrt{n_x+1}\psi_{n_x+1}), \\ \hat{p}_x\psi_{n_x} &= -i\sqrt{\frac{m\hbar\omega_x}{2}}(\sqrt{n_x}\psi_{n_x-1} - \sqrt{n_x+1}\psi_{n_x+1}) \end{aligned} \tag{B.1}$$

allow us to write

$$\begin{aligned} xz\psi_{n_x}\psi_{n_z} &= \frac{\hbar}{2m\sqrt{\omega_x\omega_z}}(\sqrt{n_x n_z}\psi_{n_x-1}\psi_{n_z-1} + \\ &+ \sqrt{(n_x+1)(n_z+1)}\psi_{n_x+1}\psi_{n_z+1} + \sqrt{(n_x+1)n_z}\psi_{n_x+1}\psi_{n_z-1} + \\ &+ \sqrt{n_x(n_z+1)}\psi_{n_x-1}\psi_{n_z+1}), \\ \frac{\hat{p}_x\hat{p}_z}{m^2\omega_x\omega_z}\psi_{n_x}\psi_{n_z} &= -\frac{\hbar}{2m\sqrt{\omega_x\omega_z}}(\sqrt{n_x n_z}\psi_{n_x-1}\psi_{n_z-1} + \\ &+ \sqrt{(n_x+1)(n_z+1)}\psi_{n_x+1}\psi_{n_z+1} - \sqrt{(n_x+1)n_z}\psi_{n_x+1}\psi_{n_z-1} - \\ &- \sqrt{n_x(n_z+1)}\psi_{n_x-1}\psi_{n_z+1}). \end{aligned} \tag{B.2}$$

These formulae demonstrate in an obvious way that the operators

$$P_0 = \frac{1}{2}\left(zx + \frac{1}{m^2\omega_x\omega_z}\hat{p}_x\hat{p}_z\right) \quad \text{and} \quad P_2 = \frac{1}{2}\left(zx - \frac{1}{m^2\omega_x\omega_z}\hat{p}_x\hat{p}_z\right)$$

contribute only to the excitation of the $\Delta N = 0$ and $\Delta N = 2$ states, respectively. Following [20], we express the zx component of $r^2Y_{21} = \sqrt{5/16\pi}\mathcal{Q} = -\sqrt{15/8\pi}z(x+iy)$ as

$$zx = P_0 + P_2.$$

Hence, we have

$$\begin{aligned} \epsilon_0 \sum_{mi(\Delta N=0)} \left|\langle 0|\sum_{s=1}^{A} z_s x_s|mi\rangle\right|^2 &= \epsilon_0 \sum_{mi} \left|\langle 0|\sum_{s=1}^{A} P_0(s)|mi\rangle\right|^2 = \\ &= \frac{1}{2}\langle 0|\left[\sum_{s=1}^{A} P_0(s), \left[H, \sum_{s=1}^{A} P_0(s)\right]\right]|0\rangle, \end{aligned} \tag{B.3}$$

where $\epsilon_0 = \hbar(\omega_x - \omega_z)$. The above commutator is easily evaluated for the Hamiltonian with the potential (A.1) as

$$\langle 0| \left[\sum_{s=1}^{A} P_0(s), \left[H, \sum_{s=1}^{A} P_0(s) \right] \right] |0\rangle = \\ = \frac{\hbar}{2m} \epsilon_0 \left(\frac{\langle 0| \sum_{s=1}^{A} z_s^2 |0\rangle}{\omega_x} - \frac{\langle 0| \sum_{s=1}^{A} x_s^2 |0\rangle}{\omega_z} \right). \quad \text{(B.4)}$$

Taking into account the axial symmetry and using the definitions

$$Q_{00} = \langle 0| \sum_{s=1}^{A} (2x_s^2 + z_s^2)|0\rangle, \quad Q_{20} = 2\langle 0| \sum_{s=1}^{A} (z_s^2 - x_s^2)|0\rangle, \quad Q_{20} = Q_{00} \frac{4}{3} \delta,$$

we transform this expression to

$$\langle 0| \left[\sum_{s=1}^{A} P_0(s), \left[H, \sum_{s=1}^{A} P_0(s) \right] \right] |0\rangle = \\ = \frac{\hbar}{6m} \epsilon_0 Q_{00} \left(\frac{1 + (4/3)\delta}{\omega_x} - \frac{1 - (2/3)\delta}{\omega_z} \right). \quad \text{(B.5)}$$

With the help of the self-consistent expressions for ω_x, ω_z (A.3) one comes to the following result:

$$\langle 0| \left[\sum_{s=1}^{A} P_0(s), \left[H, \sum_{s=1}^{A} P_0(s) \right] \right] |0\rangle = \frac{Q_{00}}{6m} \frac{\epsilon_0^2}{\bar{\omega}^2} = \frac{\hbar^2}{6m} Q_{00}^0 \left(\frac{\omega_0}{\omega_z} - \frac{\omega_0}{\omega_x} \right)^2. \quad \text{(B.6)}$$

By using the fact that the matrix elements for the zy component of $r^2 Y_{21}$ are identical to those for the zx component, because of axial symmetry, we finally obtain

$$\epsilon_0 \sum_{mi(\Delta N = 0)} \left| \langle 0| \sum_{s=1}^{A} r_s^2 Y_{21} |mi\rangle \right|^2 = \frac{5}{16\pi} \frac{Q_{00}}{m\bar{\omega}^2} \epsilon_0^2 = \\ = \frac{5}{16\pi} \frac{Q_{00}^0}{m} \frac{\epsilon_0^2}{\omega_0^2} \left(\frac{1 + (4/3)\delta}{1 - (2/3)\delta} \right)^{1/3}. \quad \text{(B.7)}$$

By calculating the double commutator for the P_2 operator, we find

$$\epsilon_2 \sum_{mi(\Delta N=2)} \left| \langle 0| \sum_{s=1}^{A} r_s^2 Y_{21} |mi\rangle \right|^2 = \frac{5}{16\pi} \frac{Q_{00}}{m\bar{\omega}^2} \epsilon_2^2 = \\ = \frac{5}{16\pi} \frac{Q_{00}^0}{m} \frac{\epsilon_2^2}{\omega_0^2} \left(\frac{1+(4/3)\delta}{1-(2/3)\delta} \right)^{1/3}, \quad \text{(B.8)}$$

where $\epsilon_2 = \hbar(\omega_x + \omega_z)$.

We need also the sums $\mathcal{Q}_0^\tau$ and $\mathcal{Q}_2^\tau$ calculated separately for neutron and proton systems with the mean fields V^n and V^p, respectively. The necessary formulae are easily derivable from the already obtained results. There are no any reasons to require the fulfillment of the self-consistency conditions for neutrons and protons separately, so one has to use formula (B.5). The trivial change of notation gives

$$\langle 0| \left[\sum_{s=1}^{Z} P_0(s), \left[H^p, \sum_{s=1}^{Z} P_0(s) \right] \right] |0\rangle = \\ = \frac{\hbar}{6m} \epsilon_0^p Q_{00}^p \left(\frac{1+(4/3)\delta^p}{\omega_x^p} - \frac{1-(2/3)\delta^p}{\omega_z^p} \right), \quad \text{(B.9)}$$

$$\epsilon_0^p \sum_{mi(\Delta N=0)} \left| \langle 0| \sum_{s=1}^{Z} r_s^2 Y_{21} |mi\rangle \right|^2 = \\ = \frac{5}{16\pi} \frac{\hbar}{m} \epsilon_0^p Q_{00}^p \left(\frac{1+(4/3)\delta^p}{\omega_x^p} - \frac{1-(2/3)\delta^p}{\omega_z^p} \right), \quad \text{(B.10)}$$

$$\epsilon_2^p \sum_{mi(\Delta N=2)} \left| \langle 0| \sum_{s=1}^{Z} r_s^2 Y_{21} |mi\rangle \right|^2 = \\ = \frac{5}{16\pi} \frac{\hbar}{m} \epsilon_2^p Q_{00}^p \left(\frac{1+(4/3)\delta^p}{\omega_x^p} + \frac{1-(2/3)\delta^p}{\omega_z^p} \right). \quad \text{(B.11)}$$

The nontrivial information is contained in oscillator frequencies of the mean fields V^p and V^n (formula (7))

$$(\omega_x^p)^2 = \omega^2 \left[1 - \frac{2}{m\omega^2} (\kappa Q_{20}^p + \bar{\kappa} Q_{20}^n) \right], \quad (\omega_z^p)^2 = \omega^2 \left[1 + \frac{4}{m\omega^2} (\kappa Q_{20}^p + \bar{\kappa} Q_{20}^n) \right], \quad \text{(B.12)}$$

$$(\omega_x^n)^2 = \omega^2 \left[1 - \frac{2}{m\omega^2} (\kappa Q_{20}^n + \bar{\kappa} Q_{20}^p) \right], \quad (\omega_z^n)^2 = \omega^2 \left[1 + \frac{4}{m\omega^2} (\kappa Q_{20}^n + \bar{\kappa} Q_{20}^p) \right].$$

The above-written formulae can also be used to calculate the analogous sums for operators containing various combinations of momenta and coordinates, for example, components of an angular momentum, tensor products $(r\hat{p})_{21}$ and $(\hat{p}^2)_{21}$. By definition $\hat{I}_1 = y\hat{p}_z - z\hat{p}_y$, $\hat{I}_2 = z\hat{p}_x - x\hat{p}_z$. In accordance with (B.1), we have

$$x\hat{p}_z\psi_{n_x}\psi_{n_z} = -i\frac{\hbar}{2}\sqrt{\frac{\omega_z}{\omega_x}}\big(\sqrt{n_x n_z}\psi_{n_x-1}\psi_{n_z-1} - \\ - \sqrt{(n_x+1)(n_z+1)}\psi_{n_x+1}\psi_{n_z+1} + \sqrt{(n_x+1)n_z}\psi_{n_x+1}\psi_{n_z-1} - \\ - \sqrt{n_x(n_z+1)}\psi_{n_x-1}\psi_{n_z+1}\big). \quad \text{(B.13)}$$

Therefore,

$$\hat{I}_2\psi_{n_x}\psi_{n_z} = i\frac{\hbar}{2}\left(\sqrt{\frac{\omega_z}{\omega_x}} - \sqrt{\frac{\omega_x}{\omega_z}}\right)\times \\ \times\big(\sqrt{n_x n_z}\psi_{n_x-1}\psi_{n_z-1} - \sqrt{(n_x+1)(n_z+1)}\psi_{n_x+1}\psi_{n_z+1}\big) + i\frac{\hbar}{2}\left(\sqrt{\frac{\omega_z}{\omega_x}} + \sqrt{\frac{\omega_x}{\omega_z}}\right)\times \\ \times\big(\sqrt{(n_x+1)n_z}\psi_{n_x+1}\psi_{n_z-1} - \sqrt{n_x(n_z+1)}\psi_{n_x-1}\psi_{n_z+1}\big). \quad \text{(B.14)}$$

Having formulae (B.2) and (B.14), one derives the following expressions for matrix elements coupling the ground state with $\Delta N = 2$ and $\Delta N = 0$ excitations:

$$\begin{aligned}
\langle n_x+1, n_z+1|\hat{I}_2|0\rangle &= i\frac{\hbar}{2}\frac{(\omega_x^2-\omega_z^2)}{\omega_x+\omega_z}\sqrt{\frac{(n_x+1)(n_z+1)}{\omega_x\omega_z}}, \\
\langle n_x+1, n_z-1|\hat{I}_2|0\rangle &= i\frac{\hbar}{2}\frac{(\omega_x^2-\omega_z^2)}{\omega_x-\omega_z}\sqrt{\frac{(n_x+1)n_z}{\omega_x\omega_z}}, \\
\langle n_x+1, n_z+1|xz|0\rangle &= \frac{\hbar}{2m}\sqrt{\frac{(n_x+1)(n_z+1)}{\omega_x\omega_z}}, \\
\langle n_x+1, n_z-1|xz|0\rangle &= \frac{\hbar}{2m}\sqrt{\frac{(n_x+1)n_z}{\omega_x\omega_z}}.
\end{aligned} \quad \text{(B.15)}$$

It is easy to see that

$$\langle n_x+1, n_z+1|\hat{I}_2|0\rangle = im\frac{(\omega_x^2-\omega_z^2)}{\omega_x+\omega_z}\langle n_x+1, n_z+1|xz|0\rangle, \\ \langle n_x+1, n_z-1|\hat{I}_2|0\rangle = im\frac{(\omega_x^2-\omega_z^2)}{\omega_x-\omega_z}\langle n_x+1, n_z-1|xz|0\rangle.$$

Due to the degeneracy of the model all particle hole excitations with $\Delta N = 2$ have the same energy ϵ_2 and all particle hole excitations with $\Delta N = 0$ have the energy ϵ_0. This fact allows one to join the last two formulae into one general expression

$$\langle \mathrm{ph}|\hat{I}_2|0\rangle = i\hbar m \frac{(\omega_x^2 - \omega_z^2)}{\epsilon_{\mathrm{ph}}} \langle \mathrm{ph}|xz|0\rangle.$$

Taking into account the axial symmetry we have an analogous formula for $\hat{I}_1$:

$$\langle \mathrm{ph}|\hat{I}_1|0\rangle = -i\hbar m \frac{(\omega_x^2 - \omega_z^2)}{\epsilon_{\mathrm{ph}}} \langle \mathrm{ph}|yz|0\rangle.$$

The magnetic transition operator (37) is proportional to the angular momentum: $\hat{f}_{1\pm1} = -\dfrac{ie}{4mc}\sqrt{\dfrac{3}{2\pi}}(\hat{I}_2 \mp i\hat{I}_1)$. Therefore, we can write

$$\langle \mathrm{ph}|\hat{f}_{1\pm1}|0\rangle = -\frac{e\hbar}{2c\sqrt{5}} \frac{(\omega_x^2 - \omega_z^2)}{\epsilon_{\mathrm{ph}}} \langle \mathrm{ph}|r^2 Y_{2\pm1}|0\rangle. \tag{B.16}$$

Similar calculations for the tensor product $(r\hat{p})_{21} = -\dfrac{1}{2}[z\hat{p}_x + x\hat{p}_z + i(z\hat{p}_y + y\hat{p}_z)]$ lead to the following relation:

$$\langle \mathrm{ph}|(r\hat{p})_{21}|0\rangle = i\frac{m}{\hbar}\sqrt{\frac{2\pi}{15}}\epsilon_{\mathrm{ph}}\langle \mathrm{ph}|r^2 Y_{2\pm1}|0\rangle = i\frac{m}{2\hbar}\epsilon_{\mathrm{ph}}\langle \mathrm{ph}|r_{21}^2|0\rangle. \tag{B.17}$$

Two kinds of particle hole matrix elements are obtained from the second formula of (B.2):

$$\langle n_x+1, n_z+1|\hat{p}_x\hat{p}_z|0\rangle = -\hbar m\omega_x\omega_z\sqrt{\frac{(n_x+1)(n_z+1)}{2\omega_x 2\omega_z}},$$

$$\langle n_x+1, n_z-1|\hat{p}_x\hat{p}_z|0\rangle = \hbar m\omega_x\omega_z\sqrt{\frac{(n_x+1)n_z}{2\omega_x 2\omega_z}}.$$

Simple comparison with (B.15) shows that

$$\langle n_x+1, n_z+1|\hat{p}_x\hat{p}_z|0\rangle = -m^2\omega_x\omega_z\langle n_x+1, n_z+1|xz|0\rangle,$$
$$\langle n_x+1, n_z-1|\hat{p}_x\hat{p}_z|0\rangle = m^2\omega_x\omega_z\langle n_x+1, n_z-1|xz|0\rangle.$$

With the help of the obvious relations

$$2\omega_x\omega_z = \omega_x^2 + \omega_z^2 - \epsilon_0^2/\hbar^2, \quad -2\omega_x\omega_z = \omega_x^2 + \omega_z^2 - \epsilon_2^2/\hbar^2$$

these two formulae can be joined into one expression

$$\langle \mathrm{ph}|\hat{p}_x\hat{p}_z|0\rangle = \frac{m^2}{2}(\omega_x^2+\omega_z^2-\epsilon_{\mathrm{ph}}^2/\hbar^2)\langle \mathrm{ph}|xz|0\rangle.$$

By definition $\hat{p}_{21}^2 = -\hat{p}_z(\hat{p}_x + i\hat{p}_y)$ and $\hat{r}_{21}^2 = -z(x+iy)$, hence,

$$\langle \mathrm{ph}|\hat{p}_{21}^2|0\rangle = \frac{m^2}{2}(\omega_x^2+\omega_z^2-\epsilon_{\mathrm{ph}}^2/\hbar^2)\langle \mathrm{ph}|r_{21}^2|0\rangle. \tag{B.18}$$

REFERENCES

1. *Hilton R. R.* A Possible Vibrational Mode in Heavy Nuclei // Intern. Conf. on Nuclear Structure, Dubna, June 1976. Unpublished.
2. *Lo Iudice N., Palumbo F.* // Phys. Rev. Lett. 1978. V. 41. P. 1532.
3. *Bohle D. et al.* // Phys. Lett. B. 1984. V. 137. P. 27.
4. *Zawischa D.* // J. Phys. G: Nucl. Part. Phys. 1998. V. 24. P. 683.
5. *Soloviev V. G. et al.* // Nucl. Phys. A. 1996. V. 600. P. 155;
   Soloviev V. G., Sushkov A. V., Shirikova N. Yu. // Phys. Rev. C. 1996. V. 53. P. 1022.
6. *Balbutsev E. B., Schuck P.* // Nucl. Phys. A. 2003. V. 720. P. 293;
   Balbutsev E. B., Schuck P. // Ibid. V. 728. P. 471.
7. *Hamamoto I., Nazarewicz W.* // Phys. Lett. B. 1992. V. 297. P. 25.
8. *Balbutsev E. B., Schuck P.* // Phys. At. Nucl. 2005. V. 68. P. 1497.
9. *Ring P., Schuck P.* The Nuclear Many-Body Problem. Berlin: Springer–Verlag, 1980.
10. *Chandrasekhar S.* Ellipsoidal Figures of Equilibrium. New Haven: Yale Univ. Press, 1969.
11. *Balbutsev E. B., Mikhailov I. N., Vaishvila Z.* // J. Phys. G: Nucl. Phys. 1986. V. 12. P. L185.
12. *Balbutsev E. B., Mikhailov I. N.* // J. Phys. G: Nucl. Phys. 1988. V. 14. P. 545.
13. *Balbutsev E. B., Schuck P.* // Nucl. Phys. A. 1999. V. 652. P. 221.
14. *Balbutsev E. B., Piperova J.* // Sov. J. Nucl. Phys. 1989. V. 50. P. 599.
15. *Balbutsev E. B. et al.* // Nucl. Phys. A. 1994. V. 571. P. 413.
16. *Balbutsev E. B., Molodtsova I. V., Unzhakova A. V.* // Europhys. Lett. 1994. V. 26. P. 499.
17. *Balbutsev E. B.* // Sov. J. Part. Nucl. 1991. V. 22. P. 159.
18. *Varshalovitch D. A., Moskalev A. N., Khersonski V. K.* Quantum Theory of Angular Momentum. L.: Nauka, 1975.
19. *Soloviev V. G.* Theory of Complex Nuclei. M.: Nauka, 1971.

20. *Suzuki T., Rowe D. J.* // Nucl. Phys. A. 1977. V. 289. P. 461.

21. *Hilton R. R.* // Ann. Phys. (N. Y.). 1992. V. 214. P. 258.

22. *Bertsch G. F.* // Nucl. Phys. A. 1975. V. 249. P. 253.

23. *Lipparini E., Stringari S.* // Phys. Rep. 1989. V. 175. P. 103.

24. *Zawischa D., Speth J.* // Z. Phys. A. 1991. V. 339. P. 97.

25. *Lane A. M.* Nuclear Theory. N. Y.: Benjamin, 1964.

26. *Lipparini E., Stringari S.* // Phys. Lett. B. 1983. V. 130. P. 139.

27. *Hilton R. R. et al.* // Proc. of the 1st Intern. Spring Seminar on Nuclear Physics «Microscopic Approaches to Nuclear Structure», Sorrento, 1986 / Ed. A. Covello. Bologna, 1986. P. 357.

28. *Dieperink A. E. L., Moya de Guerra E.* // Phys. Lett. B. 1987. V. 189. P. 267.

29. *Lo Iudice N.* // Nucl. Phys. A. 1996. V. 605. P. 61.

30. *Lo Iudice N.* // Nuovo Cim. 2000. V. 23, No. 9.

31. *Landau L. D., Lifshits E. M.* Quantum Mechanics. M.: Nauka, 1974.

32. *Shlomo S., Bertsch G. F.* // Nucl. Phys. A. 1975. V. 243. P. 507.

33. *Liu K. F., Nguen Van Giai.* // Phys. Lett. B. 1976. V. 65. P. 23.

34. *Da Providĕncia J. P.* // Nucl. Phys. A. 1988. V. 489. P. 111.

35. *Balbutsev E. B. et al.* // Sov. J. Nucl. Phys. 1989. V. 50. P. 788.

36. *Kohl H., Schuck P., Stringari S.* // Nucl. Phys. A. 1986. V. 459. P. 265.

37. *Bohr A., Mottelson B.* Nuclear Structure. N. Y.: Benjamin, 1975. V. 2.

38. *Balbutsev E. B., Mikhailov I. N., Di Toro M.* // Europhys. Lett. 1988. V. 6. P. 317.

39. *Bertsch G. F., Das Gupta S.* // Phys. Rep. 1988. V. 160. P. 190.