TWO-DIMENSIONAL INTEGRODIFFERENTIAL EQUATIONS FOR UNEQUAL MASS PARTICLE SYSTEMS

S. A. Sofianos, *G. J. Rampho*

Department of Physics, University of South Africa, Pretoria, South Africa

R. M. Adam

Nuclear Energy Corporation of South Africa, Pretoria, South Africa

*E-mail: sofiasa@science.unisa.ac.za

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The two-variable Integrodifferential Equation Approach (IDEA) valid for A nucleons is generalized to describe quantum-mechanical systems consisting of A, unequal mass, particles. The method is based on an expansion of the wave function in Faddeev amplitudes for the various particle pairs and a subsequent expansion of them in terms of potential harmonics. Projecting the resulting Faddeev-type equations on a specific two-body space and spin–isospin channel one obtains coupled, two-variable, integrodifferential equations describing the system. These equations can be readily applied, for example, to hypernuclear systems such as the double hypernucleus $\Lambda\Lambda\alpha$ which can be handled either as a three- or as a six-body problem. We demonstrate our approach by applying it to various single and double hypernuclei and compare the results to those obtained by other methods.

Приближение интегродифференциального уравнения двух переменных для A-нуклонов обобщается с целью описания квантово-механических систем, состоящих из A-частиц с неравными массами. Метод основан на разложении волновой функции по фаддеевским амплитудам для различных пар частиц и последующем их разложении по потенциальным гармоникам. Проектируя полученные в результате уравнения фаддеевского типа на специально выбранное двумерное пространство и спин-изоспиновый канал, можно выписать связанные интегродифференциальные уравнения с двумя переменными для описания системы. Уравнения можно применять, например, к гипернуклонным системам, таким как двойное гиперъядро $\Lambda\Lambda\alpha$, которое можно рассматривать как трех- или шестичастичную задачу. В качестве примера использования нашего приближения приводится его применение к различным одиночным и двойным гиперъядрам, результаты сравниваются с данными, полученными другими методами.

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INTRODUCTION

Multistrange hypernuclei consisting of various hyperons (Λ, Ξ, Σ, etc.) have been the subject of numerous experimental and theoretical works during the last few decades (see, for example, [1–3] and references therein). To handle

*E-mail: sofiasa@science.unisa.ac.za

such systems, various approaches have been employed based on cluster models, G-matrix theory, variational methods, and resonating group model, just to mention a few of them. Thus, Yamada and Ikeda [4] used a microscopic cluster model to study the $^{7}_{\Sigma}$Li system; Hiyama et al. [5] used a variational method with Jacobi-coordinate Gaussian-basis functions to study the $\Lambda - \Sigma$ conversion in $^{4}_{\Lambda}$He and $^{4}_{\Lambda}$H systems. Similar methods were used by Nemura and collaborators [6, 7] to investigate several light Λ and $\Lambda\Lambda$ hypernuclei up to $A = 6$ and by Shoeb et al. [8] who study s- and p-shell double hypernuclei variationally. G-matrix methods were also employed [9].

The methods used, albeit powerful, are quite complicate in applications. A more rigorous approach is the one based on Faddeev equations which, however, are restricted to three- and four-body systems. We recall here the Faddeev calculations for the $\Lambda(\Sigma)NN$ system [10] and the extensive investigations carried out by Filikin and co-workers for various hypernuclei (Λnp, $^{9}_{\Lambda}$Be, $^{13}_{\Lambda}$C, $^{10}_{\Lambda\Lambda}$Be, etc.) considered as three- and four-body systems [11–15]. Going beyond four-particle systems via the Faddeev formalism is not at present feasible and therefore one has to resort to clusters and the use of effective intercluster interactions. The construction of these interactions is not an easy task at all due, mainly, to the limited availability of the relevant spectrum and scattering data. Thus, it is not unusual to have binding energies obtained with various potentials and differing by a few MeV.

An alternative method, based on hyperspherical harmonics, was suggested by Adam and Fiedeldey [16, 17]. After expanding the wave function in Faddeev amplitudes for the various particle pairs and a further expansion in potential harmonics [18], one obtains two-variable integrodifferential equations for the A-particle system by projecting the equations on a specific two-body space and spin–isospin channel. For equal mass particles, the method has been successfully applied in the past by Fabre and collaborators [19–25]. The IDEA, as is nowadays known, includes the two-body correlations into account exactly while higher order correlations are included via the hypercentral approximation quite satisfactorily without increasing the complexity of the equations. Three-body forces can also be included in a straightforward manner [19].

In the present work we describe how to generalize the IDEA method to unequal masses systems with the inclusion of spin and isospin states. We demonstrate the applicability of the method by considering various single and double hypernuclei and compare the results with those obtained by other approaches.

In Sec. 1 we describe the method and give the necessary details. In Sec. 2 we present some applications and compare the results obtained with those of other methods. Conclusions are drawn in Sec. 3 while some technical details concerning hyperspherical coordinates, the projection on the r_{ij} space, and a practical approach to spinology, are given in the Appendices.

1. THE INTEGRODIFFERENTIAL EQUATION APPROACH

We consider $A = N + 1$ particles of mass m_i, $i = 1, \ldots, A$, interacting via pairwise forces. We assume that n_a of them are particles of mass m_a forming the set a, n_b of mass m_b forming the set b, etc. Then one may have $N_a = n_a(n_a - 1)/2$ possible pairs in the set a forming the particle channel A, $N_b = n_b(n_b - 1)/2$ pairs in b forming B, $N_c = n_a n_b$ pairs formed by particles from the set a and b forming the particle channel C and so on. In the description of the formalism we shall assume, for clarity, that we have two types of particles and therefore a maximum of three types of pairs can be formed from them; the description for the general case is straightforward.

In the Faddeev formalism, the total wave function for the system can be expanded as

$$\Psi(\mathbf{x}) = \sum_{ij} \psi_{ij}^A(\mathbf{x}) + \sum_{kl} \psi_{kl}^B(\mathbf{x}) + \sum_{mn} \psi_{mn}^C(\mathbf{x}) \equiv \sum_{ij \in c'} \psi_{ij}^{c'}(\mathbf{x}), \ c' = A, B, C, \ (1)$$

where $\mathbf{x}$ is the coordinates vector; (ij), (kl), and $(mn))$ are the pairs within each set (e.g., $N_i N_j$, $\Lambda_k \Lambda_l$, $N_m \Lambda_n$), and ψ_{ij}^c, $c = A, B, C$, are the Faddeev components satisfying the equation

$$(T - E)\psi_{ij}^c(\mathbf{x}) = -V_{ij}^c(r_{ij}) \sum_{c'=A,B,C} \left[\sum_{kl \in c'} \psi_{kl}^{c'}(\mathbf{x}) \right]. \qquad (2)$$

The corresponding chain of Jacobi coordinates $\xi = \{\xi_1, \xi_2, \ldots, \xi_N, \mathbf{X}\}$ are defined in terms of position coordinates for the particles, $\mathbf{x} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_A\}$,

$$\xi_i = \left[\frac{2A}{M_T} \frac{M_i m_{i+1}}{M_{i+1}} \right]^{1/2} \left(\mathbf{x}_{i+1} - \frac{1}{M_i} \sum_{j=1}^{i} m_j \mathbf{x}_j \right), \quad i = 1, \ldots, N, \qquad (3)$$

where m_i is the mass of particle i, $M_i = \sum_{j=1}^{i} m_j$; $\mathbf{X}$ is the center-of-mass coordinate, and M_T is the total mass. Further details on this system of coordinates are given in Appendices A and B. The first coordinate ξ_1 will also be denoted by $\mathbf{r}_{ij}$ to indicate that the pair (ij) is chosen as the reference particle pair.

For states which are invariant by rotation in the $(D - 3)$-dimensional space and spanned by the $N - 1$ vectors, $\xi_2, \xi_3, \ldots, \xi_N$, the amplitudes $\psi_{ij}^c(\mathbf{x})$ can be written as a function of the collective variable r and $\mathbf{r}_{ij}$

$$\psi_{ij}^c(\mathbf{x}) = F_{ij}^c(\mathbf{r}_{ij}, r), \quad ij \in c. \qquad (4)$$

Therefore,

$$(T - E)F_{ij}^c(\mathbf{r}_{ij}, r) = -V_{ij}^c(r_{ij}) \sum_{c'} \left[\sum_{kl \in c'} F_{kl}^{c'}(\mathbf{r}_{kl}, r) \right], \qquad (5)$$

where the summation includes all pairs (kl) in each channel c'.

To extract the equation for the amplitude $F^c(\mathbf{r}_{ij}, r) \equiv F_{ij}^c(\mathbf{r}_{ij}, r)$ for the pair (ij), we first expand it in potential harmonics $\mathcal{P}_{2K+\ell}^{\ell,m}(\Omega_{ij})$ (see Appendix C)

$$F^c(\mathbf{r}_{ij}, r) = \sum_{K=0}^{\infty} \mathcal{P}_{2K+\ell}^{\ell,m}(\Omega_{ij}) U_K^{c,\ell}(r), \qquad (6)$$

where the radial functions $U_K^{c,\ell}(r)$ are given in terms $F^c(\mathbf{r}_{ij}, r)$ by

$$U_K^{c,\ell}(r) = \int \mathcal{P}_{2K+\ell}^{\ell,m}(\Omega_{ij}) F^{c,\ell}(\mathbf{r}_{ij}, r) \, d\Omega. \qquad (7)$$

Then we project on the $\mathbf{r}_{ij}$ space which requires the evaluation of $\langle \mathbf{r}_{ij} | F^c(\mathbf{r}_{kl}) \rangle$. As is shown in Appendix C this is given by

$$\langle \mathbf{r}_{ij} | F^c(\mathbf{r}_{kl}) \rangle = \sum_K \langle \mathcal{P}_{2K+\ell}^{\ell,m}(\cos 2\phi_{ij}) | \mathcal{P}_{2K+\ell}^{\ell,m}(\cos 2\phi_{kl}) \rangle \mathcal{P}_{2K+\ell}^{\ell,m*}(\Omega_{ij}) \times$$

$$\times \int \mathcal{P}_{2K+\ell}^{\ell,m*}(\Omega_{kl}) F^c(\mathbf{r}_{kl}, r) \, d\Omega. \quad (8)$$

An expression for matrix elements $\langle \mathcal{P}_{2K+\ell}^{\ell,m}(\cos 2\phi_{ij}) | \mathcal{P}_{2K+\ell}^{\ell,m}(\cos 2\phi_{kl}) \rangle$ for equal mass particles has been derived in [18,26] with the help of the kinematical rotation vector while for unequal mass particles in [16] the result being

$$\langle \mathcal{P}_{2K+\ell}^{\ell,m}(\Omega_{ij}) | \mathcal{P}_{2K+\ell}^{\ell,m}(\Omega_{kl}) \rangle = \cos \phi_{ij}^{\ell} \frac{P_K^{\alpha,\beta+\ell}(\cos 2\varphi_{ij}^{kl})}{P_K^{\alpha,\beta+\ell}(1)}. \qquad (9)$$

The angles φ_{ij}^{kl} depend on whether the pair (kl) is identical to (ij), or joint, such as (12) and (14), or disjoint, such as (12) and (45), and can be obtained by (see Appendix B)

$$\cos 2\varphi_{ij}^{kl} = \begin{cases} +1 & \text{if } (kl) = (ij), \\ \dfrac{m_i m_k - m_j(m_i + m_j + m_k)}{m_i m_k + m_j(m_i + m_j + m_k)} & \text{if } (kl) \text{ and } (ij) \text{ are joint,} \\ -1 & \text{if } (kl) \text{ and } (ij) \text{ are disjoint.} \end{cases} \qquad (10)$$

The subscript ij is to remind us that these angles are those corresponding to the case where the first pair in the Jacobi coordinates corresponds to the (ij) pair.

In our applications we consider $\ell = 0$ only and therefore terms with this index will be omitted from now on. The inclusion, however, of higher partial waves under the assumption that the pair (ij) is in that state, is trivial and entails the additional term, $\hat{\ell}^2/(1+z)$ in the Hamiltonian (C.6) and the ℓ-dependent terms in Eq. (C.21).

Instead of the coordinates r_{ij} and r one may introduce the (z, r) set where $z = \cos 2\phi_1 = 2r_{ij}^2/r^2 - 1$ and r. Then, letting

$$f(z, z', \cos 2\varphi_{ij}^{kl}) = W(z') \sum_{K=0}^{\infty} P_K^{\alpha, \beta}(z) P_K^{\alpha, \beta}(z') \frac{P_K^{\alpha, \beta}(\cos 2\varphi_{ij}^{kl})}{P_K^{\alpha, \beta}(1) h_K^{\alpha, \beta}}, \tag{11}$$

where the constant $h_K^{\alpha, \beta}$ is introduce to take care of the normalization,

$$h_K^{\alpha, \beta} = \int_{-1}^{+1} W(z)[P_K^{\alpha, \beta}(z)]^2 \, dz, \tag{12}$$

and $W(z)$ for the weight function

$$W(z) = (1 - z)^\alpha (1 + z)^\beta, \quad \alpha = (D - 5)/2, \ \beta = 1/2, \tag{13}$$

we obtain the projection functions for channel c

$$f^c(z, z') = \sum_{c'=1}^{N_p} \sum_{(kl)\in c'} f_A^c(z, z'; \cos 2\varphi_{ij}^{kl}) = \sum_{c'=1}^{N_p} \mathcal{F}_{c'}^c(z, z') \tag{14}$$

with obvious notation. Finally, letting

$$F^c(r_{ij}, r) = r^{-(D-1)/2} P^c(z, r), \quad z = 2(r_{ij}/r)^2 - 1$$

we obtain the integrodifferential equation for the Faddeev-type components $P^c(z, r)$

$$\left\{ \frac{\hbar^2 A}{M} H_{rz} - E \right\} P^c(z, r) = -V^c\left(\frac{r}{\mu_c}\sqrt{(1+z)/2} \right) \times$$

$$\times \left[P^c(z, r) + \sum_{c'=1}^{N_p} \int_{-1}^{+1} \mathcal{F}_{c'}^c(z, z') P^{c'}(z', r) \, dz' \right], \tag{15}$$

where

$$H_{rz} = -\frac{\partial^2}{\partial r^2} + \frac{\mathcal{L}(\mathcal{L}+1)}{r^2} - \frac{4}{r^2}\frac{1}{W(z)}\frac{\partial}{\partial z}(1 - z^2)W(z)\frac{\partial}{\partial z}, \tag{16}$$

$\mathcal{L} = (D - 3)/2$, and μ^c is the reduced mass for the channel c,

$$\mu^c = \left[\frac{2Am_i m_j}{M(m_i + m_j)} \right]^{1/2}, \quad ij \in c. \tag{17}$$

Equation (15) can be written in a more transparent matrix form (for simplicity we assume that we have two channels only)

$$\begin{pmatrix} H_{rz} + V^A - E & 0 \\ 0 & H_{rz} + V^B - E \end{pmatrix} \begin{pmatrix} P^A \\ P^B \end{pmatrix} =$$

$$= -\begin{pmatrix} V^A & 0 \\ 0 & V^B \end{pmatrix} \begin{pmatrix} \widehat{\mathcal{F}}^{AA} & \widehat{\mathcal{F}}^{AB} \\ \widehat{\mathcal{F}}^{BA} & \widehat{\mathcal{F}}^{BB} \end{pmatrix} \begin{pmatrix} P^A \\ P^B \end{pmatrix}, \tag{18}$$

where the operators $\widehat{\mathcal{F}}^{cc'}$ are such that

$$\widehat{\mathcal{F}}^{cc'} P^{c'} = \int\limits_{-1}^{+1} \mathcal{F}^c_{c'}(z, z') P^{c'}(z', r). \tag{19}$$

Note that in the diagonal operators $\widehat{F}_{cc}$ the $(kl) = (ij)$ elements should be excluded as this term has been shifted to the left.

The reduced equation, Eq. (15), is S-projected and thus it excludes the effects of higher partial waves. These effects can be included, albeit approximately and in an average way, using the hypercentral potential for each channel. This is defined by

$$V^c_0(r) = \frac{1}{h_0} \int\limits_{-1}^{+1} W(z) V^c \left(\frac{r}{\mu_c} \sqrt{(1+z)/2} \right) dz, \tag{20}$$

where the normalization factor h_0 is given by (12) for $K = 0$. Introducing $V^c_0(r)$ into both sides of (15) we obtain the IDEA equation

$$\begin{pmatrix} G^A_{zr} - E & 0 \\ 0 & G^B_{zr} - E \end{pmatrix} \begin{pmatrix} P^A \\ P^B \end{pmatrix} =$$

$$-\begin{pmatrix} \Delta V^A & 0 \\ 0 & \Delta V^B \end{pmatrix} \begin{pmatrix} \widehat{\mathcal{F}}^{AA} & \widehat{\mathcal{F}}^{AB} \\ \widehat{\mathcal{F}}^{BA} & \widehat{\mathcal{F}}^{BB} \end{pmatrix} \begin{pmatrix} P^A \\ P^B \end{pmatrix}, \tag{21}$$

where now

$$G^c_{zr} = \frac{\hbar^2 A}{M} \left[H_{rz} + \sum_{c'} \nu_{c'} V^{c'}_0(r) \right] + \Delta V^c \tag{22}$$

with

$$\Delta V^c = V^c\left(\frac{r}{\mu_c}\sqrt{(1+z)/2}\right) - V_0^c(r) \tag{23}$$

and ν_c is the number of pairs in a given type c. Note that $\sum_c \nu_c = A(A-1)/2$, the total number of pairs.

From the above it is clear that the extension of the IDEA to unequal mass particles results in no major complications and that the equations to be solved are still quite simple and easy to apply to systems consisting of particles with unequal masses. These equations are, however, coupled, and the number of channels in the coupled system is equal to the number of different type of particle pairs present, e.g., NN, $N\Lambda$, and $\Lambda\Lambda$.

The solution of Eq. (21) can be achieved either by solving it as a two-dimensional integrodifferential equation, or via adiabatic approximations [19,23]. In the Extreme Adiabatic Approximation (EAA) we assume that the amplitude can be written as a product

$$P^c(z,r) = P_\lambda^c(z,r)u_\lambda(r). \tag{24}$$

This implies that the orbital motion is very rapid as compared to the radial motion and contains most of the energy. Then Eq. (21) can be split into two equations, namely,

$$\left[\begin{pmatrix} D^A & 0 \\ 0 & D^B \end{pmatrix} + \begin{pmatrix} \Delta V^A & 0 \\ 0 & \Delta V^B \end{pmatrix}\begin{pmatrix} \widehat{\mathcal{F}}^{AA} & \widehat{\mathcal{F}}^{AB} \\ \widehat{\mathcal{F}}^{BA} & \widehat{\mathcal{F}}^{BB} \end{pmatrix}\right]\begin{pmatrix} P^A \\ P^B \end{pmatrix} =$$
$$= U_\lambda(r)\begin{pmatrix} P^A \\ P^B \end{pmatrix} \tag{25}$$

with

$$D^c = \frac{\hbar^2 A}{M}\frac{4}{r^2}\frac{1}{W(z)}\frac{\partial}{\partial z}(1-z^2)W(z)\frac{\partial}{\partial z} + \Delta V^c$$

from which we determine, for each r, the eigenpotential $U_\lambda(r)$ which is used to evaluate the binding energy E_λ^{EAA} from the second equation,

$$\frac{\hbar^2 A}{M}\left[-\frac{d^2}{dr^2} + \frac{\mathcal{L}(\mathcal{L}+1)}{r^2} + \sum_{c'}\nu_{c'}V_0^{(c')}(r) + U_\lambda(r)\right]u_\lambda(r) = E_\lambda^{\text{EAA}}u_\lambda(r). \tag{26}$$

The EAA provides a lower bound and the accuracy achieved for nuclear systems depends of the short range characteristics of the underlying nucleon–nucleon forces. For soft potentials the accuracy is of the order of 0.2% and it can be further improved by using the Uncoupled Adiabatic Approximation (UAA) [19].

1.1. Spin-Dependent Forces. *1.1.1. The System abc.* Let us consider a system of three particles a, b, and c of unequal masses in which spin–isospin states of specific symmetry are present for all three pairs. We denote the three interparticle pairs by $\alpha = (12)$, $\beta = (31)$, and $\gamma = (23)$ and assume that the total potential is of the form

$$V_T(\mathbf{x}) = \sum_{\gamma'} \Big[V^{1+}(r_{\gamma'})P_{\gamma'}^{1+} + V^{3+}(r_{\gamma'})P_{\gamma'}^{3+} =$$

$$+ V^{1-}(r_{\gamma'})P_{\gamma'}^{1-} + V^{3-}(r_{\gamma'})P_{\gamma'}^{3-} \Big] \equiv \sum_{\gamma'} \mathbb{V}_{\gamma'}, \quad (27)$$

where γ' runs over all pairs α, β, γ; and $P_\alpha^{n\pm}$, $n = 1, 3$, are the spin–singlet and spin–triplet projection operators, respectively. The total wave function may be written as

$$\Psi_T(\mathbf{x}, \sigma\tau) = \mathbb{Y}_\alpha + \mathbb{Y}_\beta + \mathbb{Y}_\gamma, \quad (28)$$

where σ (τ) is the three-body spin (isospin), and

$$\mathbb{Y}_{\gamma'} = |\mathcal{A}'_{\gamma'}\rangle\psi_{\gamma'}^{\mathcal{S}'} + |\mathcal{A}\rangle\psi_{\gamma'}^{\mathcal{S}} + |\mathcal{S}'_{\gamma'}\rangle\psi_{\gamma'}^{\mathcal{A}'} + |\mathcal{S}\rangle\psi_{\gamma'}^{\mathcal{A}}, \qquad \gamma' = \alpha, \beta, \gamma, \quad (29)$$

where $\mathcal{S}'$, $\mathcal{S}$, $\mathcal{A}'$, and $\mathcal{A}$, denote the mixed symmetric, fully symmetric, mixed antisymmetric, and fully antisymmetric states while $|\mathcal{A}'\rangle$, $|\mathcal{A}'\rangle$, $|\mathcal{S}'\rangle$, and $|\mathcal{S}_{\gamma'}\rangle$ are the corresponding three-body spin–isospin states.

The Faddeev equations then read

$$(H_0 - E)\mathbb{Y}_\alpha = -\mathbb{V}_\alpha(\mathbb{Y}_\alpha + \mathbb{Y}_\beta + \mathbb{Y}_\gamma),$$
$$(H_0 - E)\mathbb{Y}_\beta = -\mathbb{V}_\beta(\mathbb{Y}_\alpha + \mathbb{Y}_\beta + \mathbb{Y}_\gamma), \quad (30)$$
$$(H_0 - E)\mathbb{Y}_\gamma = -\mathbb{V}_\gamma(\mathbb{Y}_\alpha + \mathbb{Y}_\beta + \mathbb{Y}_\gamma).$$

The projection on the r_α space is independent of the spinology of the system. Thus, we will concentrate on the spin–isospin projection.

Restricting ourselves, for simplicity, to even states only, i.e. to $\psi_{\gamma'}^{\mathcal{S}}$ and $\psi_{\gamma'}^{\mathcal{S}'}$ states, $\gamma' = \alpha, \beta, \gamma$, and projecting with $\langle\mathcal{A}'_\alpha|\langle r_\alpha|$ and $\langle\mathcal{A}|\langle r_\alpha|$ results in the following system for the pair α,

$$(H_0 - E)\psi_\alpha^{\mathcal{A}'} = - \sum_{n=1+,3+} V_\alpha^n \Big[\langle\mathcal{A}'_\alpha|P_\alpha^n|\mathcal{A}'_\alpha\rangle\psi_\alpha^{\mathcal{S}'} + \langle\mathcal{A}'_\alpha|P_\alpha^n|\mathcal{A}\rangle\psi_\alpha^{\mathcal{S}} +$$

$$+ \langle\mathcal{A}'_\alpha|P_\alpha^n|\mathcal{A}'_\beta\rangle I_{\alpha\beta}^{\mathcal{S}'} + \langle\mathcal{A}'_\alpha|P_\alpha^n|\mathcal{A}\rangle I_{\alpha\beta}^{\mathcal{S}} + \langle\mathcal{A}'_\alpha|P_\alpha^n|\mathcal{A}'_\gamma\rangle I_{\alpha\gamma}^{\mathcal{S}'} + \langle\mathcal{A}'_\alpha|P_\alpha^n|\mathcal{A}\rangle I_{\alpha\gamma}^{\mathcal{S}} \Big] \quad (31)$$

and

$$(H_0 - E)\psi_\alpha^{\mathcal{S}} = - \sum_{n=1+,3+} V_\alpha^n \Big[\langle\mathcal{A}|P_\alpha^n|\mathcal{A}'_\alpha\rangle\psi_\alpha^{\mathcal{S}'} + \langle\mathcal{A}|P_\alpha^n|\mathcal{A}\rangle\psi_\alpha^{\mathcal{S}} +$$

$$+ \langle\mathcal{A}|P_\alpha^n|\mathcal{A}'_\beta\rangle I_{\alpha\beta}^{\mathcal{S}'} + \langle\mathcal{A}|P_\alpha^n|\mathcal{A}\rangle I_{\alpha\beta}^{\mathcal{S}} + \langle\mathcal{A}|P_\alpha^n|\mathcal{A}'_\gamma\rangle I_{\alpha\gamma}^{\mathcal{S}'} + \langle\mathcal{A}|P_\alpha^n|\mathcal{A}\rangle I_{\alpha\gamma}^{\mathcal{S}} \Big] \quad (32)$$

with

$$I_{\alpha\beta}^n = \int\limits_{-1}^{+1} f(z, z'; \cos 2\varphi_\alpha^\beta)\psi_\beta^n(z')\,dz', \quad n = \mathcal{S}' \text{ or } \mathcal{S}. \tag{33}$$

Note that, since we are dealing with a three-particle system, in (33) only one term exists for each channel. Similar projections are valid for the $\mathbb{Y}_\beta$ and $\mathbb{Y}_\gamma$ components.

We may use now the recoupling relations (see Appendix D)

$$|\mathcal{A}_\beta'\rangle = -\frac{1}{2}|\mathcal{A}_\alpha'\rangle + \frac{\sqrt{3}}{2}|\mathcal{S}_\alpha'\rangle, \qquad |\mathcal{A}_\gamma'\rangle = -\frac{1}{2}|\mathcal{A}_\alpha'\rangle - \frac{\sqrt{3}}{2}|\mathcal{S}_\alpha'\rangle \tag{34}$$

and the projections

$$\langle\mathcal{A}|P_\alpha^{1+}|\mathcal{A}\rangle = \langle\mathcal{A}|P_\alpha^{1+}|\mathcal{A}_\alpha'\rangle = \langle\mathcal{A}_\alpha'|P_\alpha^{1+}|\mathcal{A}_\alpha'\rangle =$$
$$= \langle\mathcal{A}|P_\alpha^{3+}|\mathcal{A}\rangle = -\langle\mathcal{A}|P_\alpha^{3+}|\mathcal{A}_\alpha'\rangle = \langle\mathcal{A}_\alpha'|P_\alpha^{3+}|\mathcal{A}_\alpha'\rangle = \frac{1}{2} \tag{35}$$

to obtain

$$(-H_0 + E)\psi_\alpha^{\mathcal{S}'} = \mathcal{V}_\alpha^+(\psi_\alpha^{\mathcal{S}'} + \lambda^- I_{\alpha\beta}^{\mathcal{S}'} + \lambda^+ I_{\alpha\gamma}^{\mathcal{S}'}) + \mathcal{V}_\alpha^-(\psi_\alpha^{\mathcal{S}} + I_{\alpha\beta}^{\mathcal{S}} + I_{\alpha\gamma}^{\mathcal{S}}),$$
$$(-H_0 + E)\psi_\alpha^{\mathcal{S}} = \mathcal{V}_\alpha^-(\psi_\alpha^{\mathcal{S}'} + \lambda^- I_{\alpha\beta}^{\mathcal{S}'} + \lambda^+ I_{\alpha\gamma}^{\mathcal{S}'}) + \mathcal{V}_\alpha^+(\psi_\alpha^{\mathcal{S}} + I_{\alpha\beta}^{\mathcal{S}} + I_{\alpha\gamma}^{\mathcal{S}}), \tag{36}$$

where we let

$$\mathcal{V}_\alpha^\pm(z, r) = \frac{V_\alpha^{1+}(r/\mu_\alpha\sqrt{(1+z)/2}) \pm V_\alpha^{3+}(r/\mu_\alpha\sqrt{(1+z)/2})}{2} \tag{37}$$

and $\lambda^\pm = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$. The other channels can be obtained by cyclic permutations.

Equations (36) are reduced to the SIDE equations by letting

$$\psi_\alpha^n((z, r) = r^{-(D-1)/2}P_\alpha^n(z, r) \tag{38}$$

to obtain

$$\left\{\frac{\hbar^2 A}{M}\left[-H_r + \frac{4}{r^2}H_z\right] + E\right\}P_\alpha^{\mathcal{S}'}(z, r) = \mathcal{V}^+\Pi_\alpha^{\mathcal{S}'}(z, r) + \mathcal{V}^-\Pi_\alpha^{\mathcal{S}}(z, r), \tag{39}$$

$$\left\{\frac{\hbar^2 A}{M}\left[-H_r + \frac{4}{r^2}H_z\right] + E\right\}P_\alpha^{\mathcal{S}}(z, r) = \mathcal{V}^-\Pi_\alpha^{\mathcal{S}'}(z, r) + \mathcal{V}^+\Pi_\alpha^{\mathcal{S}}(z, r). \tag{40}$$

The H_r and H_z are given by

$$H_r = -\frac{\partial^2}{\partial r^2} + \frac{\mathcal{L}(\mathcal{L}+1)}{r^2}, \quad H_z = \frac{1}{W(z)}\frac{\partial}{\partial z}(1-z^2)W(z)\frac{\partial}{\partial z}, \tag{41}$$

while

$$\Pi_\alpha^{S'}(z.r) = P_\alpha^{S'}(z,r) + \lambda^- I_{\alpha\beta}^{S'}(z,r) + \lambda^+ I_{\alpha\gamma}^{S'}(z,r), \tag{42}$$

$$\Pi_\alpha^S(z.r) = P_\alpha^S(z,r) + I_{\alpha\beta}^S(z,r) + I_{\alpha\gamma}^S(z,r). \tag{43}$$

For the β and γ channels we have the terms

$$\Pi_\beta^{S'}(z.r) = P_\beta^{S'}(z,r) + \lambda^- I_{\beta\gamma}^{S'}(z,r) + \lambda^+ I_{\beta\alpha}^{S'}(z,r), \tag{44}$$

$$\Pi_\beta^S(z.r) = P_\beta^S(z,r) + I_{\beta\gamma}^S(z,r) + I_{\beta\alpha}^S(z,r) \tag{45}$$

and

$$\Pi_\gamma^{S'}(z.r) = P_\gamma^{S'}(z,r) + \lambda^- I_{\gamma\alpha}^{S'}(z,r) + \lambda^+ I_{\gamma\beta}^{S'}(z,r), \tag{46}$$

$$\Pi_\gamma^S(z.r) = P_\gamma^S(z,r) + I_{\gamma\alpha}^S(z,r) + I_{\gamma\beta}^S(z,r). \tag{47}$$

The IDEA equations are easily obtained:

$$\left\{ \frac{\hbar^2 A}{M} \left[-H_r + \sum_{\gamma'} V_{0,\gamma'}(r) + \frac{4}{r^2} H_z \right] + E \right\} P_\alpha^{S'}(z,r) =$$
$$= \left(\mathcal{V}^+(z,r) - V_{0,\alpha} \right) \Pi_\alpha^{S'}(z,r) + \mathcal{V}^-(z,r)\Pi_\alpha^S(z,r), \tag{48}$$

$$\left\{ \frac{\hbar^2 A}{M} \left[-H_r + \sum_{\gamma'} V_{0,\gamma'}(r) + \frac{4}{r^2} H_z \right] + E \right\} P_\alpha^S(z,r) =$$
$$= \mathcal{V}^-(z,r)\Pi_\alpha^{S'}(z,r) + \left(\mathcal{V}^+(z,r) - V_{0,\alpha} \right) \Pi_\alpha^S(z,r) \tag{49}$$

and similarly for the other two pair-channels β and γ.

For equal mass particles of mass m, the Faddeev components are the same and the above system reduces to

$$\left\{ \frac{\hbar^2}{m} \left[-H_r + 3V_0(r) + \frac{4}{r^2} H_z \right] + E \right\} P^{S'}(z,r) =$$
$$= \left(\mathcal{V}^+(z,r) - V_0 \right) \Pi^{S'}(z,r) + \mathcal{V}^-(z,r)\Pi^S(z,r), \tag{50}$$

$$\left\{ \frac{\hbar^2}{m} \left[-H_r + 3V_0(r) + \frac{4}{r^2} H_z \right] + E \right\} P^S(z,r) =$$
$$= \mathcal{V}^-(z,r)\Pi^{S'}(z,r) + \left(\mathcal{V}^+(z,r) - V_0 \right) \Pi^S(z,r), \tag{51}$$

i.e., one recovers the three-nucleon equations [22].

1.1.2. The System aab. Here we have again the $\alpha = (12)$, $\beta = (31)$, and $\gamma = (23)$ channels but the latter two give the same projections. Assuming, as in the abc case, that there are only singlet and triplet even states, the system (36) is written as

$$(-H_0 + E)\psi_\alpha^{\mathcal{S}'} = \mathcal{V}_\alpha^+ \left[\psi_\alpha^{\mathcal{S}'} - I_{\alpha\beta}^{\mathcal{S}'}\right] + \mathcal{V}_\alpha^- \left[\psi_\alpha^{\mathcal{S}} + 2I_{\alpha\beta}^{\mathcal{S}}\right],$$

$$(-H_0 + E)\psi_\alpha^{\mathcal{S}} = \mathcal{V}_\alpha^- \left[\psi_\alpha^{\mathcal{S}'} - I_{\alpha\beta}^{\mathcal{S}'}\right] + \mathcal{V}_\alpha^+ \left[\psi_\alpha^{\mathcal{S}} + 2I_{\alpha\beta}^{\mathcal{S}}\right],$$

$$(-H_0 + E)\psi_\beta^{\mathcal{S}'} = \mathcal{V}_\beta^+ \left[\psi_\beta^{\mathcal{S}'} + \lambda^- I_{\beta\gamma}^{\mathcal{S}'} + \lambda^+ I_{\beta\alpha}^{\mathcal{S}'}\right] + \mathcal{V}_\beta^- \left[\psi_\beta^{\mathcal{S}} + I_{\beta\gamma}^{\mathcal{S}} + I_{\beta\alpha}^{\mathcal{S}}\right],$$

$$(-H_0 + E)\psi_\beta^{\mathcal{S}} = \mathcal{V}_\alpha^- \left[\psi_\beta^{\mathcal{S}'} + \lambda^- I_{\beta\gamma}^{\mathcal{S}'} + \lambda^+ I_{\beta\alpha}^{\mathcal{S}'}\right] + \mathcal{V}_\beta^+ \left[\psi_\beta^{\mathcal{S}} + I_{\beta\gamma}^{\mathcal{S}} + I_{\beta\alpha}^{\mathcal{S}}\right],$$

$$(-H_0 + E)\psi_\gamma^{\mathcal{S}'} = \mathcal{V}_\gamma^+ \left[\psi_\gamma^{\mathcal{S}'} + \lambda^- I_{\gamma\alpha}^{\mathcal{S}'} + \lambda^+ I_{\gamma\beta}^{\mathcal{S}'}\right] + \mathcal{V}_\gamma^- \left[\psi_\gamma^{\mathcal{S}} + I_{\gamma\alpha}^{\mathcal{S}} + I_{\gamma\beta}^{\mathcal{S}}\right],$$

$$(-H_0 + E)\psi_\gamma^{\mathcal{S}} = \mathcal{V}_\gamma^- \left[\psi_\gamma^{\mathcal{S}'} + \lambda^- I_{\gamma\alpha}^{\mathcal{S}'} + \lambda^+ I_{\gamma\beta}^{\mathcal{S}'}\right] + \mathcal{V}_\gamma^+ \left[\psi_\gamma^{\mathcal{S}} + I_{\gamma\alpha}^{\mathcal{S}} + I_{\gamma\beta}^{\mathcal{S}}\right].$$

Since $\psi_\beta^n = \psi_\gamma^n$, by summing up the equations for the β and γ channels we finally obtain

$$(-H_0 + E)\mathbf{1}\begin{pmatrix}\psi_\alpha^{\mathcal{S}'} \\ \psi_\alpha^{\mathcal{S}} \\ \psi_\beta^{\mathcal{S}'} \\ \psi_\beta^{\mathcal{S}}\end{pmatrix} =$$

$$= \begin{pmatrix} \mathcal{V}_\alpha^+ & \mathcal{V}_\alpha^- & -\mathcal{V}_\alpha^+ \hat{I}_{\alpha\beta} & 2\mathcal{V}_\alpha^- \hat{I}_{\alpha\beta} \\ \mathcal{V}_\alpha^- & \mathcal{V}_\alpha^+ & -\mathcal{V}_\alpha^- \hat{I}_{\alpha\beta} & 2\mathcal{V}_\alpha^+ \hat{I}_{\alpha\beta} \\ \mathcal{V}_\beta^+\left(-\frac{1}{2}\hat{I}_{\beta\alpha}\right) & \mathcal{V}_\beta^- \hat{I}_{\beta\alpha} & \mathcal{V}_\beta^+\left(1 - \frac{1}{2}\hat{I}_{\beta\beta}\right) & \mathcal{V}_\beta^-(1 + \hat{I}_{\beta\beta}) \\ \mathcal{V}_\beta^-\left(-\frac{1}{2}\hat{I}_{\beta\alpha}\right) & \mathcal{V}_\beta^+ \hat{I}_{\beta\alpha} & \mathcal{V}_\beta^-\left(1 - \frac{1}{2}\hat{I}_{\beta\beta}\right) & \mathcal{V}_\beta^+(1 + \hat{I}_{\beta\beta}) \end{pmatrix}\begin{pmatrix}\psi_\alpha^{\mathcal{S}'} \\ \psi_\alpha^{\mathcal{S}} \\ \psi_\beta^{\mathcal{S}'} \\ \psi_\beta^{\mathcal{S}}\end{pmatrix}. \quad (52)$$

1.2. Presence of Spin States Only. It is clear from the above analysis that the form of the final equations depends on the spin–isospin states which prescribe the symmetries for the wave function components $\mathbb{Y}_{\gamma'}$ as in Eq. (29). There are cases where only the spin states play a role. One such system is the $\Lambda\Lambda\,^3$He system, where the isospin of Λ is $t = 0$. Furthermore, the Λ–Λ interaction is $^1S_0$ and while that of $\Lambda\,^3$He consists of a singlet and triplet with $m_\sigma = 1/2$. Thus, one may assume the ansatz

$$\Psi = |\sigma_\alpha^a\rangle\Psi_\alpha^{1+} + |\sigma_\beta^a\rangle\Psi_\beta^{1+} + |\sigma_\beta^s\rangle\Psi_\beta^{3+} + |\sigma_\gamma^a\rangle\Psi_\gamma^{1+} + |\sigma_\gamma^s\rangle\Psi_\gamma^{3+}, \quad (53)$$

where the subscripts α, β, and γ stand for the pairs $\alpha = \Lambda_1\Lambda_2$, $\beta = \Lambda_1\,^3$He, and $\gamma = \Lambda_2\,^3$He. The states $|\sigma_\alpha^a\rangle$ and $|\sigma_\alpha^s\rangle$ are the mixed antisymmetric and mixed symmetric three-body spin states (see Appendix D).

Then, for the Ψ_α^{1+}, we have the projection

$$(H_0 - E)\Psi_\alpha^{1+} =$$

$$= -V_\alpha^{1+} \langle \sigma_\alpha^a | \mathcal{P}_\alpha^{1+} \left[|\sigma_\alpha^a\rangle \Psi_\alpha^{1+} + |\sigma_\beta^a\rangle I_{\alpha\beta}^{1+} + |\sigma_\beta^s\rangle I_{\alpha\beta}^{3+} + |\sigma_\gamma^a\rangle I_{\alpha\gamma}^{1+} + |\sigma_\gamma^s\rangle I_{\alpha\gamma}^{3+} \right] =$$

$$= -V_\alpha^{1+} \left[\Psi_\alpha^{1+} - \frac{1}{2} I_{\alpha\beta}^{1+} + \frac{\sqrt{3}}{2} I_{\alpha\beta}^{3+} - \frac{1}{2} I_{\alpha\gamma}^{1+} - \frac{\sqrt{3}}{2} I_{\alpha\gamma}^{3+} \right].$$

Note that the notation (α, β) designates the projection with the r_α pair of the component β. This corresponds to the angle φ_α^β.

For the Ψ_β^{1+} projection we have

$$(H_0 - E)\Psi_\beta^{1+} =$$

$$= -V_\beta^{1+} \langle \sigma_\beta^a | \mathcal{P}_\beta^{1+} \left[|\sigma_\alpha^a\rangle I_{\beta\alpha}^{1+} + |\sigma_\beta^a\rangle \Psi_\beta^{1+} + |\sigma_\beta^s\rangle \Psi_\beta^{3+} + |\sigma_\gamma^a\rangle I_{\beta\gamma}^{1+} + |\sigma_\gamma^s\rangle I_{\beta\gamma}^{3+} \right] =$$

$$= -V_\beta^{1+} \left[\Psi_\beta^{1+} - \frac{1}{2} I_{\beta\alpha}^{1+} - \frac{1}{2} I_{\beta\gamma}^{1+} + \frac{\sqrt{3}}{2} I_{\beta\gamma}^{3+} \right],$$

similarly, for Ψ_β^{3+},

$$(H_0 - E)\Psi_\beta^{3+} =$$

$$= -V_\beta^{3+} \langle \sigma_\beta^s | \mathcal{P}_\beta^{3+} \left[|\sigma_\alpha^a\rangle I_{\beta\alpha}^{1+} + |\sigma_\beta^a\rangle \Psi_\beta^{1+} + |\sigma_\beta^s\rangle \Psi_\beta^{3+} + |\sigma_\gamma^a\rangle I_{\beta\gamma}^{1+} + |\sigma_\gamma^s\rangle I_{\beta\gamma}^{3+} \right] =$$

$$= -V_\beta^{3+} \left[\Psi_\beta^{3+} + \frac{\sqrt{3}}{2} I_{\beta\alpha}^{1+} - \frac{\sqrt{3}}{2} I_{\beta\gamma}^{1+} - \frac{1}{2} I_{\beta\gamma}^{3+} \right].$$

For the γ channel we have

$$(H_0 - E)\Psi_\gamma^{1+} = -V_\gamma^{1+} \left[\Psi_\gamma^{1+} - \frac{1}{2} I_{\gamma\alpha}^{1+} - \frac{1}{2} I_{\gamma\beta}^{1+} - \frac{\sqrt{3}}{2} I_{\gamma\beta}^{3+} \right],$$

$$(H_0 - E)\Psi_\gamma^{3+} = -V_\gamma^{3+} \left[\Psi_\gamma^{3+} - \frac{\sqrt{3}}{2} I_{\gamma\alpha}^{1+} + \frac{\sqrt{3}}{2} I_{\gamma\beta}^{1+} - \frac{1}{2} I_{\gamma\beta}^{3+} \right].$$

From the choice (53) we must have $\Psi_\beta^{1+} = \Psi_\gamma^{1+}$ and $\Psi_\beta^{3+} = -\Psi_\gamma^{3+}$. We also have $V_\beta = V_\gamma$. Using these symmetries, we obtain the system

$$(H_0 + V_\alpha^{1+} - E)\Psi_\alpha^{1+} = -V_\alpha^{1+} \left[-I_{\alpha\gamma}^{1+} - \sqrt{3} I_{\alpha\gamma}^{3+} \right],$$

$$(H_0 + V_\gamma^{1+} - E)\Psi_\gamma^{1+} = -\frac{1}{2} V_\gamma^{1+} \left[-I_{\gamma\alpha}^{1+} - I_{\gamma\beta}^{1+} + \frac{\sqrt{3}}{2} I_{\gamma\beta}^{3+} \right], \qquad (54)$$

$$(H_0 + V_\gamma^{3+} - E)\Psi_\gamma^{3+} = -\frac{1}{2} V_\gamma^{3+} \left[-\sqrt{3} I_{\gamma\alpha}^{1+} + \sqrt{3} I_{\gamma\beta}^{1+} + I_{\gamma\beta}^{3+} \right].$$

In a similar way we may construct equations for any other system of up to four particles.

2. APPLICATIONS

The first question addressed was about the accuracy of the adiabatic approximation in the case where the systems consist of unequal mass particles. For this we choose, as a testing, the $^5_\Lambda$He and the $^6_{\Lambda\Lambda}$He systems where exact calculations were performed by Adam and Fiedeldey [16] using a variety of potentials. These systems are considered as five-body and six-body systems, respectively. As input potentials we choose the Bassichis–Gal ΛN (c) potential [30]; for $\Lambda\Lambda$, the Dalitz potential [31], while for NN we employed three interactions of different characteristics, namely, the soft core Volkov potential V7 [32], the hard core S3 potential of Afnan and Tang [33], and the Malfliet Tjon MTV potential as modified by Zabolitzky [34]. The results obtained are given in Table 1.

Table 1. **Results for the $^5_\Lambda$He system and the $^6_{\Lambda\Lambda}$He system considered as five- and six-body systems, respectively. The $V_{\Lambda N}$ is that of Bassichis–Gal (c) [30] and the $V_{\Lambda\Lambda}$ is that of Dalitz [31]**

V_{NN}	$^5_\Lambda$He		$^6_{\Lambda\Lambda}$He	
	E_{EAA}	E_{exact} [16]	E_{EAA}	E_{exact} [16]
Volkov-7 [32]	32.16	31.60	42.52	42.93
S3 [33]	31.36	30.85	42.13	41.99
MTV [34]	34.55	33.61	46.16	44.36

It is seen that the accuracy of the EAA is of the order of 1% for all potentials. This difference can be attributed to the EAA but also to numerical inaccuracies creeping in, especially for the MTV potential. This difference can be further reduced by using the UAA [19]. Similar calculations can be performed for any other system in which the particles interact via Wigner forces. Let us treat again the $^6_{\Lambda\Lambda}$He considered as a three-body $\Lambda\Lambda - \alpha$ system previously treated by Filikin and Gal within the Faddeev formalism [12] as a three-body problem. We also consider the double hypernucleus $^{10}_{\Lambda\Lambda}$Be assumed to be a four-body $\Lambda\Lambda\alpha\alpha$ system. The $\Lambda\Lambda$ interaction is a singlet $^1S_0$ potential of the form

$$V_{\Lambda\Lambda}(r) = \sum_i^3 v_i \exp\left(-\frac{r^2}{\beta_i^2}\right), \tag{55}$$

the parameters being adjusted so that (55) is the phase equivalent to the Nijmegen hard core interactions [35]. For convenience we present these parameters in Table 2. In the same table we give also the parameters of the $\Lambda\alpha$ and $\alpha\alpha$ interactions which have the same form as (55) but with two Gaussian terms [36]. Using these interactions we obtained results in the uncoupled adiabatic approximation which are given in Table 3.

Table 2. **Parameters of the $\Lambda\Lambda$, $\Lambda\alpha$, and $\alpha\alpha$ interactions. The γ parameter for the $\Lambda\Lambda$ potential assumes the values 0.4804, 0.5463, 0.4672, 1, and 1.2044 corresponding to the NSC97b, NSC97e, NSC97f, ND, and ESC00 variant potentials**

Potential	i	v_i, MeV	β, fm
$\Lambda\Lambda$ [13]	1	–21.49	1.342
	2	$-379.1 \times \gamma$	0.777
	3	9324.0	0.350
$\Lambda\alpha$ [36]	1	450.4	1.25
	2	–404.9	1.41
$\alpha\alpha$ [37]	1	120.0	1.53
	2	–30.18	2.85

Table 3. **UAA bound state results for the $_{\Lambda\Lambda}^{6}$He and $_{\Lambda\Lambda}^{10}$Be considered as a three- ($\Lambda\Lambda\alpha$) and four-body ($\Lambda\Lambda\alpha\alpha$) systems**

Potential	$_{\Lambda\Lambda}^{6}$He			$_{\Lambda\Lambda}^{10}$Be		
	E_{UAA}	Faddeev [12]	Other	E_{UAA}	Faddeev [12]	Other
ESC00	–10.83	–10.7	–10.1 [9]	–16.94	–19.4	–18.7 [9]
ND	–9.33	–9.10	–9.34 [5]	–15.94	–17.7	–17.15 [5]
NSC97b	–7.35	–6.60		–14.28	–15.2	
NSC97e	–7.51	–6.82		–14.50	–15.4	
NSC97f	–7.32			–14.30		
Exp.	–10.9 ± 0.6 [38]			–17.6 ± 0.4 [39]		
	$-7.25 \pm 0.19^{+0.18}_{-0.11}$ [40]			–14.5 ± 0.4 [13]		

The results obtained using the various methods are, in overall, close to each other. However, there is an apparent strong dependence on the potential and the method used for calculations especially for the $\Lambda\Lambda\alpha\alpha$ system.

The effective adiabatic potentials, defined by

$$V_{\mathrm{eff}}(r) = U_{\mathrm{eaa}}(r) + \sum_c \nu_c V_0^c(r) + \frac{\mathcal{L}(\mathcal{L}+1)}{r^2}$$

for the NSC97b, ND, and ESC00 potentials and for the $\Lambda\Lambda\alpha$ system, are shown in Fig. 1, *a*.

We note here that in the adiabatic approximation these potentials go asymptotically to the corresponding binding energy of the $\Lambda\alpha$ system, –3.106 MeV. This asymptotic behavior is an indication that the extracted eigenpotential $U_{\mathrm{EAA}}(r)$ is correct.

Let us consider next the system $\Lambda\Lambda C$, where $C = {}^3$H with $m_\sigma = -1/2$ or $C = {}^3$He with $m_\sigma = 1/2$. Here the ^{1}S$_0$ $\Lambda\Lambda$ interaction is given by potential (55)
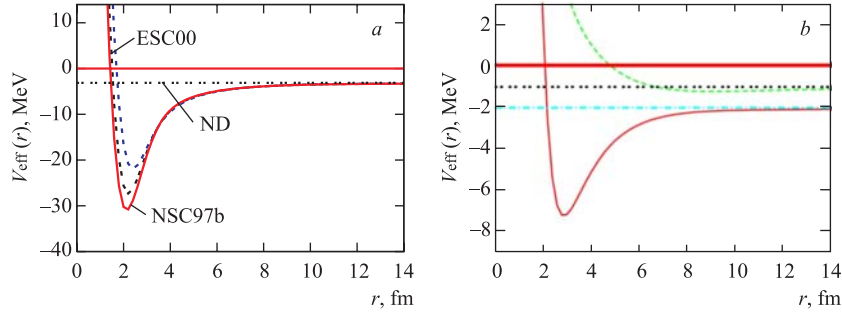
Fig. 1. Effective adiabatic potentials for the $\Lambda\Lambda\alpha$ system: *a*) for the spin-independent forces NSC97b, ND, and ESC00 $\Lambda\alpha$; the $V_{\text{eff}}(r)$ tends to the $\Lambda\alpha$ binding energy of –3.106 MeV; *b*) for the $^5_{\Lambda\Lambda}$He system with spin-dependent forces. The first two eigen-potentials converge, as expected, to the binding energies for the singlet (–2.06 MeV) and triplet (–1.04 MeV) $\Lambda\alpha$ states

which is equivalent to the Nijmegen hard core interaction. The ΛC potential was that of [13], namely:

$$V^{1+}(r) = 450.4 \exp\left(-(r/1.2573)^2\right) - 404.9 \exp\left(-(r/1.41)^2\right),$$

$$V^{3+}(r) = 450.4 \exp\left(-(r/1.2720)^2\right) - 404.9 \exp\left(-(r/1.41)^2\right).$$

The singlet $V^{1+}(r)$ potential supports a bound state at –2.06 MeV; while the triplet $V^{3+}(r)$, at –1.04 MeV.

Applying the EAA to the system (52) we obtain the incremental binding energies $\Delta B_{\Lambda\Lambda}$ (in MeV) [13] for ^{3}H shown in Table 4. These are in fair agreement with those of Filikin et al. [13] and of Nemura et al. [7]. The first two eigenpotentials for the system are shown in Fig. 1, *b*. As expected, they go asymptotically to the binding energies for the singlet and triplet Λ^3H states.

Table 4. **Incremental binding energies $\Delta B_{\Lambda\Lambda}$ (in MeV) for the $^5_{\Lambda\Lambda}$H in the extreme adiabatic approximation. $\Delta B_{\Lambda\Lambda}$ is relative to the $(2J+1)$-weighted average of the $^4_\Lambda$He and $^4_\Lambda$H**

Potential	IDEA	Filikin et al. [13]	Nemura et al. [7]
ESC00	4.3938	3.46	
ND	2.6432	2.11	2.8
NSC97e	0.6628	0.37	
NSC97b	0.4953	0.11	

CONCLUSIONS

We presented a formalism based on expansion of the Faddeev amplitudes for unequal mass particles in terms of the potential harmonics. Such an expansion implies that we choose states which are invariant by rotation in the $(D-3)$-dimensional space spanned by the $N-1$ vectors, $\xi_2, \xi_3, \ldots, \xi_N$, Jacobi coordinates. Therefore the Faddeev amplitudes $\psi_{ij}(\mathbf{x})$ in this space are functions of the collective variable r and the vector $\xi_1 \equiv r_{ij}$ only, $\phi(\mathbf{r}_{ij}, r)$. Expanding in terms of PH and projecting on the same basis, one obtains differential equations which, similarly to the HH expansion case, are not practical for numerical calculations. If, however, the Faddeev-type equations for amplitudes $\Phi^c(\mathbf{r}_{ij}, r)$ are first projected with $\langle \mathbf{r}_{ij} |$ and then expanded, one obtains coupled integrodifferential equations, describing quantum mechanical systems consisting of A, unequal mass, particles. These equations are easy to solve either exactly by treating them as a two-variable system or via the more practical adiabatic approximation. The latter methods can provide us also with the scattering states and in general one may extract the dynamics of the system in an easy and straightforward way.

We first tested the EAA by comparing the results obtain for the $^5_\Lambda$He system and the $^6_{\Lambda\Lambda}$He system considered as five- and six-body systems, respectively. For the three NN considered, the difference between the EAA and the exact results is of the order of 1.5% for $^5_\Lambda$He and about 1% for the double hypernucleus $^6_{\Lambda\Lambda}$He. The somewhat larger value with the MTV potential for the latter system is rather due to numerics as the exact method is sensitive to the way the short range $1/r$ behavior of the potential is handled. Similar results were obtained with other systems and a variety of NN forces, and the accuracy of the EAA was found to be in all cases within 0.5–1.7%. These differences can be even reduced further by using the uncoupled adiabatic approximation. In short, the adiabatic approximations are more than sufficiently good to describe unequal mass systems, the inaccuracies being much smaller than the uncertainties in the input potentials for the various pairs.

The EAA results for spin-dependent interactions for the incremental binding energies $\Delta B_{\Lambda\Lambda}$ for ^{3}H are also in overall good agreement with those of Filikin et al. [13], and Nemura et al. [7]. The corresponding effective eigenpotential serves as a check of our results and numerics as they are expected to go asymptotically to the binding energies of the various $2+1$ channels. This has been demonstrated in Fig. 1 where we recovered the $\Lambda\alpha$ binding energies.

Finally, we stress that the formalism can be extended to include more realistic forces that include tensor and spin-orbit components. This task, however, is beyond the scope of the present work.

Appendix A
JACOBI AND HYPERSPHERICAL COORDINATES

Let us define first the following chain of Jacobi coordinates for a system of $A = N + 1$ particles with masses m_i:

$$\xi_i = a_i \left(\mathbf{x}_{i+1} - \frac{1}{M_i} \sum_{j=1}^{i} m_j \mathbf{x}_j \right), \qquad i = 1, N \tag{A.1}$$

with

$$a_i = \left[\frac{M_T}{2A} \frac{M_{i+1}}{M_i m_{i+1}} \right]^{1/2}, \quad M_i = \sum_{j=1}^{i} m_j, \tag{A.2}$$

$$M_T = \sum_{j=1}^{N+1} m_j, \quad \mathbf{X} = \frac{1}{M_T} \sum_{j=1}^{N+1} m_j \mathbf{x}_j, \tag{A.3}$$

where m_i is the mass of the particle i; M_T is the total mass, and $\mathbf{X}$ is the center of mass. The diagrammatic representation of these Jacobi coordinates is shown in Fig. 2. We may rewrite the system (A.2) in a compact form as follows:

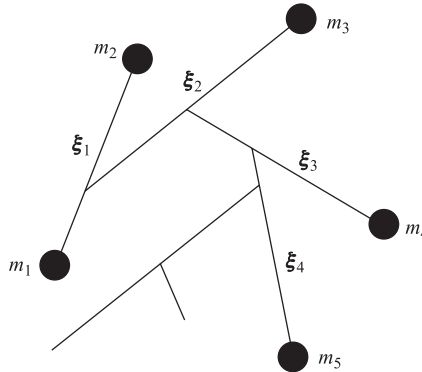$$\tilde{\xi}_i = \sum_{j=1}^{N+1} \mathbb{A}_{ij} \mathbf{x}_j, \tag{A.4}$$

Fig. 2. Diagrammatic representation of the Jacobi coordinates

where we introduced the vector $\tilde{\xi} \equiv (\zeta, X)$, $\zeta_i = \xi_i/a_i$, and

$$
\mathbb{A} = \begin{bmatrix}
-1 & 1 & 0 & \cdots & 0 \\
-\dfrac{m_1}{M_2} & -\dfrac{m_2}{M_2} & 1 & \cdots & 0 \\
-\dfrac{m_1}{M_3} & -\dfrac{m_2}{M_3} & -\dfrac{m_3}{M_3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\dfrac{m_1}{M_N} & -\dfrac{m_2}{M_N} & -\dfrac{m_3}{M_N} & \cdots & 1 \\
\dfrac{m_1}{M_A} & \dfrac{m_2}{M_A} & \dfrac{m_3}{M_A} & \cdots & \dfrac{m_A}{M_A}
\end{bmatrix}. \tag{A.5}
$$

From the matrix (A.5) one can extract the position of each particle, i.e.,

$$\mathbf{x} = \mathbb{A}^{-1}\tilde{\xi}$$

with

$$
\mathbb{A}^{-1} = \begin{bmatrix}
-\dfrac{m_2}{M_2} & -\dfrac{m_3}{M_3} & -\dfrac{m_4}{M_4} & \cdots & -\dfrac{m_N}{M_N} & 1 \\
\dfrac{M_1}{M_2} & -\dfrac{m_3}{M_3} & -\dfrac{m_4}{M_4} & \cdots & -\dfrac{m_N}{M_N} & 1 \\
0 & \dfrac{M_2}{M_3} & -\dfrac{m_4}{M_4} & \cdots & -\dfrac{m_N}{M_N} & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \\
0 & 0 & 0 & \cdots & \dfrac{M_{N-1}}{M_N} & 1
\end{bmatrix}. \tag{A.6}
$$

Another useful system of coordinates is that of Zernike and Brinkman (ZB) [29] in which one defines

$$
\begin{aligned}
\xi_1 &= r\cos\varphi_1, \\
\xi_2 &= r\sin\varphi_1\cos\varphi_2 \\
&\;\;\vdots \\
\xi_j &= r\sin\varphi_1\cdots\sin\varphi_{j-1}\cos\varphi_j, \\
&\;\;\vdots \\
\xi_{N-1} &= r\sin\varphi_1\cdots\sin\varphi_{N-2}\cos\varphi_{N-1}, \\
\xi_N &= r\sin\varphi_1\cdots\sin\varphi_{N-1},
\end{aligned} \tag{A.7}
$$

where we choose $\cos\varphi_N = 0$

In the Zernike–Brinkman system of coordinates, the Ω coordinates are separated into two parts: first, the $z = \cos 2\varphi$, ($\varphi = \varphi_1$), and $\omega = \omega_1$, the angular

coordinates of ξ_1; second, the Ω_{N-1} for the other hyperspherical coordinates (φ_i, ω_i), $i > 1$, where ω_i are the angular coordinates of ξ_i. Let $d\Omega_{N-1}$ be the surface element of the unit hypersphere $r = 1$ in the $(D - 3)$-dimensional space spanned by the Jacobi coordinates ξ_i, $i > 1$. The volume element is then given by

$$d^{3N}\xi = r^{D-1}\, dr\, d\Omega \tag{A.8}$$

with

$$
\begin{aligned}
\mathrm{d}\Omega &= (\sin \varphi)^{D-4} \cos^2 \varphi\, d\varphi\, d\omega\, d\Omega_{N-1} = \\
&= \frac{1}{2^{D/2}}(1 - z)^{(D-5)/2}(1 + z)^{1/2}\, dz\, d\omega\, d\Omega_{N-1} = W(z)\, dz\, d\omega\, d\Omega_{N-1}, \tag{A.9}
\end{aligned}
$$

where we let

$$\cos \varphi = \frac{r_{12}}{r} = \frac{\xi}{r}, \quad z = \cos 2\varphi = 2\frac{r_{12}^2}{r^2} - 1.$$

The $W(z)$ is known as «weight function».

Appendix B

KINEMATIC ROTATION VECTOR

In expanding the wave function in terms of HPs and projecting with $\langle \mathbf{r}_{ij}|$, one requires the extraction of matrix elements related to projection on (kl) pairs in the D-dimensional space. This can be achieved by using the kinematic rotation vector for the ZB coordinates which is defined in terms of a set of N parameters $\varphi_1, \varphi_2, \ldots, \varphi_N$ of Eq. (A.7), according to

$$
\begin{aligned}
v(\varphi) &= \cos \varphi_1\, \zeta_1 + \sin \varphi_1 \cos \varphi_2\, \zeta_2 + \ldots \\
&\quad + \sin \varphi_1 \cdots \sin \varphi_{i-1} \cos \varphi_i\, \zeta_i + \ldots + \sin \varphi_1 \cdots \sin \varphi_{N-1}\, \zeta_N, \tag{B.1}
\end{aligned}
$$

where we choose $\varphi_N = 0$. This vector represents a rotation in the $D = 3N$ dimensional space.

Let Ξ be any linear combination of Jacobi coordinates

$$\Xi = \sum_{i=1}^{N} c_i \xi_i = C \sum_{i=1}^{N} \frac{c_i}{C} \xi_i, \tag{B.2}$$

where $C^2 = \sum_{1}^{N} c_i^2$. The angles between the vector Ξ and the ξ_n, $n = 1, \ldots, N$ vectors, as defined by (3) in which the first pair is chosen to be the ij pair, are then extracted by comparing the coefficients of (B.2) and (B.1) starting from

$$\cos \varphi_1^{\Xi} = \frac{a_1}{C}.$$

Let us demonstrate the procedure for the three- and four-body cases. For the three-body system, one always has connected pairs, i.e., any two pairs always have a common index as, for example, the (1,2) and (2,3) pairs. In the four-body case, however, one may also have the so-called disconnected pairs such as the (1,2) and (3,4) pairs.

From the above matrices, (A.5) and (A.6), we have for three particles

$$x_1 = -\frac{m_2}{M_2}\frac{1}{a_1}\xi_1 - \frac{m_3}{M_3}\frac{1}{a_2}\xi_2,$$

$$x_2 = \frac{M_1}{M_2}\frac{1}{a_1}\xi_1 - \frac{m_3}{M_3}\frac{1}{a_2}\xi_2,$$

$$x_3 = \frac{M_2}{M_3}\frac{1}{a_2}\xi_2.$$

Thus

$$x_2 - x_1 = a_1\xi_1,$$

$$x_3 - x_2 = -\frac{M_1}{M_2}a_1\xi_1 + a_2\xi_2, \tag{B.3}$$

$$x_3 - x_1 = -\frac{m_2}{M_2}a_1\xi_1 + a_2\xi_2$$

with

$$a_1 = \left[\mu\frac{M_{12}}{m_1m_2}\right]^{1/2}, \quad a_2 = \left[\mu\frac{M_3}{M_{12}m_3}\right]^{1/2} \tag{B.4}$$

and similarly for the case where the first pair is chosen to be the (23) or the (31)

$$b_1 = \left[\mu\frac{M_{23}}{m_2m_3}\right]^{1/2}, \quad b_2 = \left[\mu\frac{M_3}{M_{23}m_1}\right]^{1/2}, \tag{B.5}$$

$$c_1 = \left[\mu\frac{M_{13}}{m_1m_3}\right]^{1/2}, \quad c_2 = \left[\mu\frac{M_3}{M_{13}m_2}\right]^{1/2} \tag{B.6}$$

with $\mu = M_T/2A$ and $M_{kl} = m_k + m_l$.

For ξ_1^{23} (the superscript denotes explicitly the first pair) we have

$$\xi_1^{23} = -\frac{a_1}{b_1}\frac{m_1}{M_2}\xi_1 + \frac{a_2}{b_1}\xi_2,$$

or using the normalization C,

$$C^2 = \frac{a_1^2}{b_1^2}\frac{m_1^2}{M_2^2} + \frac{a_2^2}{b_1^2},$$

we obtain

$$\xi_1^{23} = [1 + \nu]^{-1/2} \xi_1 + [1 + 1/\nu]^{-1/2} \xi_2, \tag{B.7}$$

where

$$\nu = \frac{M_2^2}{m_1^2} \cdot \frac{a_2^2}{a_1^2} = \frac{m_2 M_3}{m_1 m_3},$$

therefore,

$$\cos \varphi_1^{23} = \left[1 + \frac{m_2 M_3}{m_1 m_3}\right]^{-1/2}. \tag{B.8}$$

Similarly for the (31) we have

$$\cos \varphi_1^{31} = \left[1 + \frac{m_3 M_1}{m_2 m_1}\right]^{-1/2}. \tag{B.9}$$

Thus we have the relations

$$\cos 2\varphi_1^{23} = 2 \cos^2 \varphi_1^{23} - 1 = \frac{m_1 m_3 - m_2 M_3}{m_1 m_3 + m_2 M_3}, \tag{B.10}$$

$$\cos 2\varphi_1^{31} = 2 \cos^2 \varphi_1^{31} - 1 = \frac{m_2 m_3 - m_1 M_3}{m_2 m_3 + m_1 M_3}. \tag{B.11}$$

Let us consider the four-body case in which

$$x_1 = -\frac{m_2}{M_2} a_1 \xi_1 - \frac{m_3}{M_3} a_2 \xi_2 - \frac{m_4}{M_4} a_3 \xi_3,$$

$$x_2 = \frac{M_1}{M_2} a_1 \xi_1 - \frac{m_3}{M_3} a_2 \xi_2 - \frac{m_4}{M_4} a_3 \xi_3,$$

$$x_3 = \frac{M_2}{M_3} a_2 \xi_2 - \frac{m_4}{M_4} a_2 \xi_3,$$

$$x_4 = \frac{M_3}{M_4} a_3 \xi_3,$$

where a_1 and a_2 are the same as in the three-body case and

$$a_3 = \left[\mu \frac{M_4}{M_{13} m_4}\right]^{1/2}.$$

Then,

$$X_{21} = a_1\,\xi_1,$$

$$X_{23} = -\frac{M_1}{M_2}a_1\xi_1 + a_2\xi_2,$$

$$X_{31} = -\frac{m_2}{M_2}a_1\xi_1 + a_2\xi_2,$$

$$X_{41} = \frac{m_2}{M_2}a_1\xi_1 + \frac{m_3}{M_3}a_2\xi_2 + a_3\xi_3,$$

$$X_{42} = \frac{m_1}{M_2}a_1\xi_1 + \frac{m_3}{M_3}a_2\xi_2 + a_3\xi_3,$$

$$X_{43} = -\frac{M_2}{M_3}a_2\xi_2 + a_3\xi_3,$$

where we use the abbreviation $X_{21} = x_2 - x_1$, etc. For the connected pairs (21), (32), and (31) we obtain the same results as in the three-body case. Let us consider the connected pair (41). We have

$$X_{41} = \frac{m_2}{M_2}a_1\xi_1 + \frac{m_3}{M_3}a_2\xi_2 + a_3\xi_3$$

from which

$$\cos\varphi_1^{41} = \left[1 + \left(\frac{m_3 M_{12} a_1}{m_2 M_3^2 a_2}\right)^2 + \left(\frac{M_{12} a_1}{m_2 a_3}\right)^2\right]^{-1/2}.$$

After some straightforward algebra we obtain

$$\cos\varphi_1^{41} = \frac{m_2 m_4 - m_1(m_1 + m_2 + m_4)}{m_2 m_4 + m_1(m_1 + m_2 + m_4)}. \tag{B.12}$$

Similar results are obtained for the (42) pair. For the disconnected pairs we have the term $0\,\xi_1$ which gives $\cos\varphi_1^{\mathrm{disc}} = 0$. We may summarize the above, for the general case $A \geqslant 4$, as follows:

$$\cos 2\varphi_{ij}^{kl} = \begin{cases} +1 & \text{if } (kl) = (ij); \\ \dfrac{m_i m_k - m_j(m_i + m_j + m_k)}{m_i m_k + m_j(m_i + m_j + m_k)} & \text{for connected pairs;} \\ -1 & \text{for disconnected pairs.} \end{cases} \tag{B.13}$$

We remind here that the above result is for the reference pair (ij) for which $X_{ij} = a_i(x_j - x_i)$.

Appendix C

EXPANSIONS

The Faddeev components can be expanded either in terms of Hyperspherical Harmonics (HH) or Potential Harmonics (PH). We present here, briefly, these expansions.

C.1. Expansion in Hyperspherical Harmonics. Harmonic Polynomials (HP) $\mathcal{H}_{[L]}(\mathbf{x})$, where $\mathbf{x} \equiv x_1, x_2, \ldots, x_A$ is a set of $A = N + 1$ linear coordinates, are homogeneous polynomials satisfying the Laplace equation

$$\left(\sum_{i=1}^{A} \frac{\partial^2}{\partial x_i^2} \right) \mathcal{H}_{[L]}(\mathbf{x}) \equiv \nabla^2 \mathcal{H}_{[L]}(\mathbf{x}) = 0. \tag{C.1}$$

The index $[L]$ denotes a set of $3A - 1$ quantum numbers including the degree L characterizing the polynomial. The Hyperspherical Harmonic (HH) $Y_{[L]}(\Omega)$, where Ω is the set of N spherical coordinates, is then defined via

$$\mathcal{H}_{[L]}(\mathbf{x}) = r^L Y_{[L]}(\Omega), \tag{C.2}$$

where r is the hyperradius, $r^2 = \sum_{i=1}^{A} x_i^2$. The relation (C.2) implies that the HH are HP on the unit hypersphere $r = 1$ satisfying the eigenequation

$$\left[\hat{L}^2(\Omega) + L(L + D - 2) \right] Y_{[L]}(\Omega) = 0 \tag{C.3}$$

and are normalized according to

$$\int Y_{[L]}^*(\Omega) Y_{[L']}(\Omega) \, d\Omega = \delta_{[L],[L']}. \tag{C.4}$$

The $\hat{L}^2(\Omega)$ is the angular part of the Laplace operator which in polar coordinates (r, Ω) is given by

$$\nabla^2 = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{D-1} \frac{\partial}{\partial r} + \frac{\hat{L}^2(\Omega)}{r^2}. \tag{C.5}$$

$\hat{L}(\Omega)$ is known as the grand orbital (or grand angular momentum) operator and is given by

$$\hat{L}^2(\Omega) = \frac{4}{W(z)} \frac{\partial}{\partial z} (1 - z^2) W(z) \frac{\partial}{\partial z} + 2 \frac{\hat{\ell}^2(\omega_{ij})}{1 + z} + 2 \frac{\hat{L}^2(\Omega_{N-1})}{1 - z}, \tag{C.6}$$

$\hat{\ell}(\omega_{ij})$ being the angular momentum for the pair (ij). In the above, we used the hyperspherical coordinates consisting of the hyperradius r and the hyperangle Ω with

$$(r, \Omega) \equiv (r, \omega_{ij}; \Omega_{N-1}), \quad \omega_{ij} = (\theta, \phi), \quad \cos \phi = r_{ij}/r,$$
$$z = 2(r_{ij}/r)^2 - 1 = \cos 2\phi, \quad r_{ij} = r\sqrt{(1+z)/2}.$$

Note that the subscripts are omitted from θ, ϕ, and z when it is clear that they are referred to (ij) pair.

Any function $f(\mathbf{x})$ can be expanded in term of HH. As an example, consider the Schrödinger equation for A particles

$$[T - V(x)]\psi(x) = E\psi(x). \tag{C.7}$$

The wave function $\psi(x)$ can be expanded as

$$\psi(\mathbf{x}) = \sum_{[L]=0}^{\infty} r^{-(D-1)/2} u_{[L]}(r) Y_{[L]}(\Omega). \tag{C.8}$$

Such a choice results in a system of K coupled differential equations (we use $\hbar^2/2m = 1$, $\mathcal{L} = L + (D-3)/2$),

$$\left[-\frac{d^2}{dr^2} + \frac{\mathcal{L}(\mathcal{L}+1)}{r^2} - E \right] u_{[L]}(r) = \sum_{[L']} \int d\Omega \langle Y_{[L]}^* | V(x) | Y_{[L']} \rangle(\Omega)\, u_{L'}(r) =$$

$$= \sum_{[L']} V_{[L],[L']}(r)\, u_{L'}(r), \quad (C.9)$$

where $V_{[L],[L']}(r)$ is known as «potential matrix» [18].

This approach, albeit straightforward, is not practical for numerical calculations because of the tremendous degeneracy of the HH basis for a given ground orbital L which prevents one to obtain converged solutions. Furthermore, the calculations are cumbersome as one has to solve a huge number of differential equations. Moreover, the convergence could be slow especially when hard core potentials are employed or the number of particles considered is large.

C.2. Expansion in Potential Harmonics. Instead of HH expansion one may use an expansion in terms of the more efficient potential harmonics (PH) [18, 28] $\mathcal{P}_{2K+\ell}^{\ell,m}(\Omega_{ij})$ which form a complete basis for continuous functions depending only on the relative coordinate $r_{ij} \equiv \xi_1$. To describe it we recall first the A-body Faddeev-type equation

$$(T - E)\Psi_{ij}(x) = -V(r_{ij}) \sum_{kl} \Psi_{kl}(x). \tag{C.10}$$

We seek solutions which are invariant under rotation in the $(D-3)$-dimensional space and spanned by the $N-1$ vectors, $\xi_2, \xi_3, \ldots, \xi_N$. For these states one has [28]

$$\hat{L}^2(\Omega_{N-1})\Psi_{ij}(\mathbf{x}) = 0. \tag{C.11}$$

The $\mathcal{P}_{2K+\ell}^{\ell,m}(\Omega_{ij})$ are then defined as the eigenfunctions of $\hat{L}^2(\Omega)$ when $L^2(\Omega_{N-1}) = 0$ and they fulfill the eigenequation

$$\left[\hat{L}^2(\Omega) + L(L + D - 2) \right] \mathcal{P}_{2K+\ell}^{\ell,m}(\Omega_{ij}) = 0, \quad L = 2K + \ell. \tag{C.12}$$

Note that $L^2(\Omega)$ and $L^2(\Omega_{N-1})$ correspond to the spaces $D = 3(A-1) = 3N$ and $D = 3(A-2) = 3(N-1)$, respectively. For systems in which the pair (ij) is in ℓ state, while the other pairs are in an S state, these polynomials are given by [18],

$$\mathcal{P}^{\ell,m}_{2K+\ell}(\Omega_{ij}) = N_{K,\ell}Y_{\ell m}(\omega_{ij})\left(\frac{r_{ij}}{r}\right)^\ell P^{\alpha,\beta+\ell}_K\left(2\frac{r^2_{ij}}{r^2}-1\right), \qquad \text{(C.13)}$$

where $\alpha = (D-5)/2$, $\beta = 1/2$, $D = 3(A-1)$, $Y_{\ell m}(\omega_{ij})$ is the spherical harmonic; $P^{\alpha,\beta+\ell}_K(z)$ is a Jacobi polynomial, while $N_{K,\ell}$ is a normalization constant which can be obtained from

$$\int\limits_{(r=1)} \mathcal{P}^{\ell,m*}_{2K+\ell'}(\Omega_{ij})\mathcal{P}^{\ell',m'}_{2K'+\ell'}(\Omega_{ij})\,d\Omega = \delta_{KK'}\,\delta_{\ell\ell'}\,\delta_{mm'}. \qquad \text{(C.14)}$$

The relation (C.11) implies that $\Psi_{ij}(\mathbf{x})$ must be a function of the collective variable r and $\mathbf{r}_{ij}$ only, i.e.,

$$\Psi_{ij}(\mathbf{x}) \equiv F(\mathbf{r}_{ij}, r). \qquad \text{(C.15)}$$

When the total angular momentum ℓ is preserved we may, as usual, write

$$F(\mathbf{r}_{ij}, r) = Y_{\ell m}(\omega_{ij})F^\ell(r_{ij}, r). \qquad \text{(C.16)}$$

Thus, expanding $F^\ell(r_{ij}, r)$ in terms of the complete polynomial basis $N_{K,\ell}(r_{ij}/r)^{\ell/2}P^{\alpha,1/2+\ell}_K(z)$ we obtain, in terms of the HP,

$$F(\mathbf{r}_{ij}, r) = \sum_K \mathcal{P}^{\ell,m}_{2K+\ell}(\Omega_{ij})U^\ell_K(r), \qquad \text{(C.17)}$$

where

$$U^\ell_K(r) = \int \mathcal{P}^{\ell,m}_{2K+\ell}(\Omega_{ij})F(\mathbf{r}_{ij}, r)\,d\Omega. \qquad \text{(C.18)}$$

Equation (C.17) leads to coupled differential equations for $U^\ell_K(r)$ [28] and one again faces similar problems to those mentioned in the expansion in terms of HH.

However, we may transform the coupled differential equations into integro-differential equations by writing for the projection with $\langle\mathbf{r}_{ij}|$

$$\langle\mathbf{r}_{ij}|F_{kl}\rangle = \int\langle\mathbf{r}_{ij}|\mathbf{r}_{kl}\rangle F(\mathbf{r}_{kl}, r)\,d\Omega =$$

$$= \sum_{K,K'}\langle r_{ij}, K, \ell|\mathcal{P}^{\ell,m*}_{2K+\ell}(\Omega_{ij})|r_{kl}, K'.\ell\rangle\int\mathcal{P}^{\ell,m}_{2K'+\ell}(\Omega_{kl})\mathcal{P}^{\ell,m*}_{2K'+\ell}(\Omega_{kl})\,d\Omega U^\ell_{K'}(r) =$$

$$= \sum_K\langle r_{ij}, K, \ell|r_{kl}, K, \ell\rangle\mathcal{P}^{\ell,m*}_{2K+\ell}(\Omega_{ij})\int\mathcal{P}^{\ell,m*}_{2K+\ell}(\Omega_{kl})\,F^\ell(\mathbf{r}_{kl}, r)\,d\Omega.$$

An expression for the overlap integral $\langle r_{ij}, K, \ell | r_{kl}, K.\ell \rangle$ for equal mass particles has been derived by [18, 26] with the help of the kinematical rotation vector. For unequal mass it is given by [16],

$$\langle r_{ij}, K, \ell | r_{kl}, K.\ell \rangle = \langle \mathcal{P}^{\ell,m}_{2K+\ell}(\Omega_{ij}) | \mathcal{P}^{\ell,m}_{2K+\ell}(\Omega_{kl}) \rangle =$$
$$= \cos \phi^{\ell}_{ij} \frac{P^{\alpha,\beta+\ell}_K(\cos 2\varphi^{kl}_{ij})}{P^{\alpha,\beta+\ell}_K(1)}, \quad \text{(C.19)}$$

where $\cos 2\varphi^{kl}_{ij}$ is given by (10).

The integrodifferential equation for the Faddeev amplitudes for the channel c, are obtained by setting $P^{\ell,c}(z,r) = r^{(D-1)/2} F^{\ell,c}(r_{ij},r)$

$$\frac{\hbar^2}{m} \left[-\frac{\partial^2}{\partial r^2} + \frac{\mathcal{L}_\ell(\mathcal{L}_\ell+1)}{r^2} - \right.$$
$$\left. - \frac{4}{r^2} \frac{1}{W(z)} \frac{\partial}{\partial z}(1-z^2)W(z)\frac{\partial}{\partial z} + 2\frac{\ell(\ell+1)}{1+z} - E \right] P^{\ell,c}(z,r) =$$
$$= -V^c(r_{ij}) \sum_c \sum_{kl \in c} \int f^c(z, z'; \cos 2\phi^{kl}_{ij}) P^{\ell,c}(z', r) \, dz', \quad \text{(C.20)}$$

where

$$f^c(z, z'; \cos 2\phi^{kl}_{ij}) = (1+z)^{\ell/2} \times$$
$$\times \sum_K \frac{P^{\alpha,\beta+\ell}_K(\cos 2\phi^{kl}_{ij})}{P^{\alpha,\beta+\ell}_K(1) \, h^{\alpha,\beta+\ell}_K} (1+z')^{\ell/2} \, P^{\alpha,\beta+\ell}_K(z) P^{\alpha,\beta+\ell}_K(z') W(z') \quad \text{(C.21)}$$

with $\mathcal{L}_\ell = \ell + (D-3)/2$. The function $h^{\alpha,\beta+\ell}_K$ is introduced for the normalization constant of the Jacobi polynomials and is given by

$$h^{\alpha,\beta+\ell}_K = \int\limits_{-1}^{+1} \left[P^{\alpha,\beta+\ell}_K(z) \right]^2 (1+z)^\ell W(z) \, dz. \quad \text{(C.22)}$$

Appendix D
THREE-BODY SPIN STATES

The unequal mass spinology is the same as for nucleons and we recall it here to fix our notation used to describe the expansion of wave function and obtain the relevant projections in an easy way. It is customary to use the Clebsch–Gordan

(CG) coefficients to construct first the two-body states

$$|\hat{s}_1\hat{s}_2; S_{12}m_{S_{12}}\rangle = \sum_{m_1m_2} C^{m_1,m_2,m_{S_{12}}}_{s_1,s_2,S_{12}} |s_1m_1\rangle|s_2m_2\rangle, \qquad (D.1)$$

where $C^{m_1,m_2,m_{S_{12}}}_{s_1,s_2,S_{12}}$ is the CG coefficient (we use the notation of [41]). For $S_{12} = 0$ we have the singlet and for $S_{12} = 1$ the triplet states. The three-body spin states ($\hat{\sigma}^n = \hat{s}_3 + \hat{S}_{12}$) are given by

$$|\sigma^n\rangle = \sum_{m_{S_n}\, m_3\, m_1\, m_2} C^{m_1,m_2,m_n}_{s_1,s_2,S_n} C^{m_n,m_3,m_\sigma}_{S_n,s_3,\sigma} |s_1m_1\rangle|s_2m_2\rangle|s_3m_3\rangle, \qquad (D.2)$$

where the states $|\sigma^n\rangle$ are characterized by the two-body quantum number $S_{12} = S_n$ and the three-body total spin $|\sigma^n\rangle$ and m_σ. The singlet states $S_n = 0$ give rise to two-body antisymmetric spin states (with respect to the exchange of particles (12)) and three-body mixed antisymmetric spin states. Similarly the triplet $S_n = 1$ states give rise to two-body symmetric states and three-body mixed-symmetric states. The above corresponds to the enumeration (12,3). The states corresponding to the enumerations ((31,2) and (23,1) can be obtained using 6j symbols and in the case of four particles, 9j symbols.

The above method, albeit straightforward, is not practical. Instead, one may use Young diagrams and symmetrizers [42] to construct the relevant symmetries.

We construct first symmetries for the $m_\sigma = 1/2$ states. For this we use the notation $A_{12} \equiv \alpha_1\alpha_2\beta_3$, where α_1 is simply the state $|s_1m_{s_1}\rangle = |1/2, +1/2\rangle \equiv |+\rangle$ states, etc., with $A_{ij}A_{ij} = 1$ and $A_{ij}A_{kl} = 0$.

The mixed symmetric $|\sigma_{ij}\rangle^s$ and mixed antisymmetric $|\sigma_{ij}\rangle^a \equiv$ are easily obtained from the Young tableaux,

$$|\sigma_{12}; [21]\rangle^s \equiv \left| \sigma\, m_\sigma, \begin{array}{cc} \boxed{1}\boxed{2} \\ \boxed{3} \end{array} \right\rangle^s = \frac{1}{\sqrt{6}}(2A_{12} - A_{31} - A_{23}), \qquad (D.3)$$

$$|\sigma_{31}; [21]\rangle^s \equiv \left| \sigma\, m_\sigma, \begin{array}{cc} \boxed{3}\boxed{1} \\ \boxed{2} \end{array} \right\rangle^s = \frac{1}{\sqrt{6}}(2A_{31} - A_{12} - A_{23}), \qquad (D.4)$$

$$|\sigma_{23}; [21]\rangle^s \equiv \left| \sigma\, m_\sigma, \begin{array}{cc} \boxed{2}\boxed{3} \\ \boxed{1} \end{array} \right\rangle^s = \frac{1}{\sqrt{6}}(2A_{23} - A_{12} - A_{31}), \qquad (D.5)$$

$$|\sigma_{12}; [21]\rangle^a \equiv \left| \sigma\, m_\sigma, \begin{array}{cc} \boxed{1}\boxed{3} \\ \boxed{2} \end{array} \right\rangle^a = \frac{1}{\sqrt{2}}(A_{31} - A_{23}), \qquad (D.6)$$

$$|\sigma_{31}; [21]\rangle^a \equiv \left| \sigma\, m_\sigma, \begin{array}{cc} \boxed{3}\boxed{2} \\ \boxed{1} \end{array} \right\rangle^a = \frac{1}{\sqrt{2}}(A_{23} - A_{12}), \qquad (D.7)$$

$$|\sigma_{23}; [21]\rangle^a \equiv \left| \sigma\, m_\sigma, \begin{array}{cc} \boxed{2}\boxed{1} \\ \boxed{3} \end{array} \right\rangle^a = \frac{1}{\sqrt{2}}(A_{12} - A_{31}). \qquad (D.8)$$

We may express these relations in terms of the (12) pair:

$$|\sigma_{31}\rangle^a \equiv -\frac{1}{2}|\sigma_{12}\rangle^a - \frac{\sqrt{3}}{2}|\sigma_{12}\rangle^s, \qquad |\sigma_{31}\rangle^s \equiv -\frac{1}{2}|\sigma_{12}\rangle^s + \frac{\sqrt{3}}{2}|\sigma_{12}\rangle^a,$$

$$|\sigma_{23}\rangle^a \equiv -\frac{1}{2}|\sigma_{12}\rangle^a + \frac{\sqrt{3}}{2}|\sigma_{12}\rangle^s, \qquad |\sigma_{23}\rangle^s \equiv -\frac{1}{2}|\sigma_{12}\rangle^s - \frac{\sqrt{3}}{2}|\sigma_{12}\rangle^a.$$

(D.9)

The effects of $\sigma_1\cdot\sigma_2$ on the $|\sigma m_\sigma\rangle^s$ and $|\sigma m_\sigma\rangle^a$ can be easily calculated using the relations

$$\sigma_x|\alpha\rangle = |\beta\rangle, \qquad \sigma_x|\beta\rangle = |\alpha\rangle, \tag{D.10}$$

$$\sigma_y|\alpha\rangle = i|\beta\rangle, \qquad \sigma_y|\beta\rangle = -i|\alpha\rangle, \tag{D.11}$$

$$\sigma_z|\alpha\rangle = |\alpha\rangle, \qquad \sigma_z|\beta\rangle = -|\beta\rangle. \tag{D.12}$$

We introduce now the singlet and triplet projection operators

$$\mathcal{P}^{1+} = \frac{1}{2}(1 - P^\sigma), \quad \mathcal{P}^{3+} = \frac{1}{2}(1 + P^\sigma),$$

where P^σ_{12} is the spin-exchange operator

$$P^\sigma_{12} = \frac{1}{2}[1 + \sigma_1\cdot\sigma_2], \quad P^\sigma_{12}|\sigma_{12}\rangle^s = |\sigma_{12}\rangle^s, \quad P^\sigma_{12}|\sigma_{12}\rangle^a = -|\sigma_{12}\rangle^a. \tag{D.13}$$

These operators give

$$\mathcal{P}^{1+}|\sigma_{12}\rangle^a = 1|\sigma_{12}\rangle^a, \qquad \mathcal{P}^{1+}|\sigma_{12}\rangle^s = 0,$$

$$\mathcal{P}^{3+}|\sigma_{12}\rangle^a = 0, \qquad \mathcal{P}^{3+}|\sigma_{12}\rangle^s = 1|\sigma_{12}\rangle^s.$$

(D.14)

The above relations provide us with all projections needed.

For the spin–isospin states we may construct the fully-symmetric $|\mathcal{S}\rangle$, the mixed-symmetric $|\mathcal{S}'_{ij}\rangle$, the mixed-antisymmetric $|\mathcal{A}'_{ij}\rangle$, and the antisymmetric $|\mathcal{A}\rangle$ state, by combining the various spin and isospin symmetries

$$|\mathcal{S}\rangle \equiv |\ \xi; \sigma\tau; [3]\ \rangle^{\mathcal{S}} = \frac{1}{\sqrt{2}}(|\sigma\rangle^s|\tau\rangle^s + |\sigma\rangle^a|\tau\rangle^a), \tag{D.15}$$

$$|\mathcal{S}'_{ij}\rangle \equiv |\xi_{ij}; \sigma\tau; [21]\ \rangle^{\mathcal{S}'} = \frac{1}{\sqrt{2}}(|\sigma_{ij}\rangle^s|\tau_{ij}\rangle^s - |\sigma_{ij}\rangle^a|\tau_{ij}\rangle^a), \tag{D.16}$$

$$|\mathcal{A}'_{ij}\rangle \equiv |\xi_{ij}; \sigma\tau; [21]\ \rangle^{\mathcal{A}'} = -\frac{1}{\sqrt{2}}(|\sigma_{ij}\rangle^s|\tau_{ij}\rangle^a + |\sigma_{ij}\rangle^a|\tau_{ij}\rangle^s), \tag{D.17}$$

$$|\mathcal{A}\rangle \equiv |\xi; \sigma\tau; [111]\ \rangle^{\mathcal{A}} = \frac{1}{\sqrt{2}}(|\sigma\rangle^s|\tau\rangle^a - |\sigma\rangle^a|\tau\rangle^s). \tag{D.18}$$

Note that the $|\mathcal{S}\rangle$ and $|\mathcal{A}\rangle$ states are independent of the (ij). The $|\mathcal{A}'_{31}\rangle$ and $|\mathcal{A}'_{23}\rangle$ states can be expressed in terms of the (12) states using

$$|\mathcal{A}'_{31}\rangle = -\frac{1}{2}|\mathcal{A}'_{12}\rangle + \frac{\sqrt{3}}{2}|\mathcal{S}'_{12}\rangle, \tag{D.19}$$

$$|\mathcal{A}'_{23}\rangle = -\frac{1}{2}|\mathcal{A}'_{12}\rangle - \frac{\sqrt{3}}{2}|\mathcal{S}'_{12}\rangle. \tag{D.20}$$

The singlet and triplet projections are easily obtained (we omit, from now on the (12) subscript):

$$\mathcal{P}^{1+}|\mathcal{S}\rangle = \frac{1}{\sqrt{2}}|\sigma\rangle^a|\tau\rangle^a, \qquad \mathcal{P}^{1+}|\mathcal{S}\rangle = \frac{1}{\sqrt{2}}|\sigma\rangle^s|\tau\rangle^s, \tag{D.21}$$

$$\mathcal{P}^{1+}|\mathcal{S}'\rangle = -\frac{1}{\sqrt{2}}|\sigma\rangle^a|\tau\rangle^a, \qquad \mathcal{P}^{3+}|\mathcal{S}'\rangle = \frac{1}{\sqrt{2}}|\sigma\rangle^s|\tau\rangle^s, \tag{D.22}$$

$$\mathcal{P}^{1+}|\mathcal{A}'\rangle = -\frac{1}{\sqrt{2}}|\sigma\rangle^a|\tau\rangle^s, \qquad \mathcal{P}^{3+}|\mathcal{A}'\rangle = -\frac{1}{\sqrt{2}}|\sigma\rangle^s|\tau\rangle^a, \tag{D.23}$$

$$\mathcal{P}^{1+}|\mathcal{A}\rangle = -\frac{1}{\sqrt{2}}|\sigma\rangle^a|\tau\rangle^s, \qquad \mathcal{P}^{3+}|\mathcal{A}\rangle = \frac{1}{\sqrt{2}}|\sigma\rangle^s|\tau\rangle^a. \tag{D.24}$$

Thus, we have the following projections:

$$\langle\mathcal{A}|\mathcal{P}^{1+}|\mathcal{A}\rangle = \langle\mathcal{A}|\mathcal{P}^{1+}|\mathcal{A}'\rangle = \langle\mathcal{A}|\mathcal{P}^{3+}|\mathcal{A}\rangle = -\langle\mathcal{A}|\mathcal{P}^{3+}|\mathcal{A}'\rangle = \frac{1}{2}, \tag{D.25}$$

$$\langle\mathcal{A}'|\mathcal{P}^{1+}|\mathcal{A}\rangle = \langle\mathcal{A}'|\mathcal{P}^{1+}|\mathcal{A}'\rangle = -\langle\mathcal{A}'|\mathcal{P}^{3+}|\mathcal{A}\rangle = \langle\mathcal{A}'|\mathcal{P}^{3+}|\mathcal{A}'\rangle = \frac{1}{2}. \tag{D.26}$$

Furthermore, using Eqs. (D.19) and (D.20) we obtain

$$\langle\mathcal{A}'|\mathcal{P}^{1+}|A'_{31}\rangle = \langle\mathcal{A}'|\mathcal{P}^{1+}|A'_{23}\rangle = -\frac{1}{4}, \tag{D.27}$$

$$Al\mathcal{P}^{1+}|A'_{31}\rangle = \langle\mathcal{A}|\mathcal{P}^{1+}|A'_{23}\rangle = -\frac{1}{4}, \tag{D.28}$$

$$\langle\mathcal{A}|\mathcal{P}^{3+}|A'_{31}\rangle = \langle\mathcal{A}|\mathcal{P}^{3+}|A'_{23}\rangle = \frac{1}{4}. \tag{D.29}$$

There are cases where only the spin states play a role. In such cases one has to choose an ansatz for the wave function describing the various symmetries similarly to (29).

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