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THE PROPERTY OF MAXIMAL TRANSCENDENTALITY IN THE $\mathcal{N} = 4$ SYM

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We show results for the universal anomalous dimension $\gamma_{\rm uni}(j)$ of the Wilson twist-2 operators in the $\mathcal{N} = 4$ Supersymmetric Yang–Mills theory in the first three orders of perturbation theory. These expressions are obtained by extracting the most complicated contributions from the corresponding anomalous dimensions in QCD.

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INTRODUCTION

The anomalous dimensions (ADs) of the Wilson twist-2 operators govern the Bjorken scaling violation for parton distributions in the framework of Quantum Chromodynamics (QCD) [1, 2]. Now they are known up to the next-to-next-to-leading order (NNLO) of the perturbation theory [3].

The QCD expressions for ADs can be transformed to the case of the $\mathcal{N}$-extended Supersymmetric Yang–Mills theories (SYM) [4] if one uses for the Casimir operators C_A, C_F, T_f the following values $C_A = C_F = N_c$, $T_f n_f = \mathcal{N} N_c/2$. For $\mathcal{N} = 2$ and $\mathcal{N} = 4$-extended SYM the ADs of the Wilson operators get also additional contributions coming from scalar particles [5]. These ADs were calculated in the next-to-leading order (NLO) [5, 6] for the $\mathcal{N} = 4$ SYM.

However, it turns out, that the expressions for eigenvalues of the AD matrix in the $\mathcal{N} = 4$ SYM can be derived directly from the QCD anomalous dimensions without tedious calculations by using a number of plausible arguments. The method elaborated in [5] for this purpose is based on special properties of the integral kernel for the Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation [7–9] in this model and a new relation between the BFKL and Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) equations (see [5]).

1. LEADING ORDER AD MATRIX IN $\mathcal{N} = 4$ SYM

In the $\mathcal{N} = 4$ SYM theory [4] one can introduce the following color and $SU(4)$ singlet local Wilson twist-2 operators [5, 6]:

$$\mathcal{O}^g_{\mu_1,\ldots,\mu_j} = \hat{S} G^a_{\rho\mu_1} \mathcal{D}_{\mu_2} \mathcal{D}_{\mu_3} \cdots \mathcal{D}_{\mu_{j-1}} G^a_{\rho\mu_j}, \tag{1}$$

$$\tilde{\mathcal{O}}^{g}_{\mu_1,\ldots,\mu_j} = \hat{S} G^a_{\rho\mu_1} \mathcal{D}_{\mu_2} \mathcal{D}_{\mu_3} \cdots \mathcal{D}_{\mu_{j-1}} \tilde{G}^a_{\rho\mu_j}, \quad (2)$$

$$\mathcal{O}^{\lambda}_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\lambda}^a_i \gamma_{\mu_1} \mathcal{D}_{\mu_2} \cdots \mathcal{D}_{\mu_j} \lambda^{ai}, \quad (3)$$

$$\tilde{\mathcal{O}}^{\lambda}_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\lambda}^a_i \gamma_5 \gamma_{\mu_1} \mathcal{D}_{\mu_2} \cdots \mathcal{D}_{\mu_j} \lambda^{ai}, \quad (4)$$

$$\mathcal{O}^{\phi}_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\phi}^a_r \mathcal{D}_{\mu_1} \mathcal{D}_{\mu_2} \cdots \mathcal{D}_{\mu_j} \phi^a_r, \quad (5)$$

where $\mathcal{D}_\mu$ are covariant derivatives. The spinors λ_i and field tensor $G_{\rho\mu}$ describe gluinos and gluons, respectively, and ϕ_r are the complex scalar fields. For all operators in Eqs. (1)–(5) the symmetrization of the tensors in the Lorentz indices $\mu_1, \ldots, \mu_j$ and a subtraction of their traces is assumed. Due to the fact that all twist-2 operators belong to the same supermultiplet the eigenvalues of AD matrix can be expressed through one *universal AD* $\gamma_{\rm uni}(j)$ with shifted argument. At the leading order (LO), it has the form (8) [11].

2. TRANSCENDENTALITY PRINCIPLE

As it was already pointed out in the Introduction, the universal AD can be extracted directly from the QCD results without finding the scalar particle contribution. This possibility is based on the deep relation between the DGLAP and BFKL dynamics in the $\mathcal{N} = 4$ SYM [5,9].

To begin with, the eigenvalues of the BFKL kernel turn out to be analytic functions of the conformal spin $|n|$ at least in two first orders of perturbation theory [5]. Further, in the framework of the $\overline{\rm DR}$-scheme [12] one can obtain from the BFKL equation (see [9]), that there is no mixing among the special functions of different transcendentality levels i*, i.e., all special functions at the NLO correction contain only sums of the terms $\sim 1/\gamma^i$ $(i = 3)$. More precisely, if we introduce the transcendentality level i for the eigenvalues $\omega(\gamma)$ of integral kernels of the BFKL equations in accordance with the complexity of the terms in the corresponding sums (here Ψ is Riemannian Ψ-function)

$$\Psi \sim 1/\gamma, \quad \Psi' \sim \zeta(2) \sim 1/\gamma^2, \quad \Psi'' \sim \zeta(3) \sim 1/\gamma^3, \quad (6)$$

then for the BFKL kernel in LO and in NLO the corresponding levels are $i = 1$ and $i = 3$, respectively.

Because in $\mathcal{N} = 4$ SYM there is a relation between the BFKL and DGLAP equations (see [5,9]), the similar properties should be valid for the ADs themselves, i.e., the basic functions $\gamma^{(0)}_{\rm uni}(j)$, $\gamma^{(1)}_{\rm uni}(j)$ and $\gamma^{(2)}_{\rm uni}(j)$ are assumed to be of

*Note that similar arguments were used also in [13] to obtain analytic results for contributions of some complicated massive Feynman diagrams without direct calculations.

the types $\sim 1/j^i$ with the levels $i = 1$, $i = 3$ and $i = 5$, respectively. An exception could be for the terms appearing at a given order from previous orders of the perturbation theory. Such contributions could be generated and/or removed by an approximate finite renormalization of the coupling constant. But these terms do not appear in the $\overline{\mathrm{DR}}$-scheme.

It is known, that at the LO and NLO approximations (with the SUSY relation for the QCD color factors $C_F = C_A = N_c$) the most complicated contributions (with $i = 1$ and $i = 3$, respectively) are the same for all LO and NLO ADs in QCD [3] and for the LO and NLO scalar–scalar ADs [6]. This property allows one to find the universal ADs $\gamma^{(0)}_{\mathrm{uni}}(j)$ and $\gamma^{(1)}_{\mathrm{uni}}(j)$ without knowing all elements of the AD matrix [5], which was verified by the exact calculations in [6].

Using above arguments, we conclude, that at the NNLO level there is only one possible candidate for $\gamma^{(2)}_{\mathrm{uni}}(j)$. Namely, it is the most complicated part of the QCD AD matrix (with the SUSY relation for the QCD color factors $C_F = C_A = N_c$). Indeed, after the diagonalization of the AD matrix its eigenvalues should have this most complicated part as a common contribution because they differ from each other only by a shift of the argument and their differences are constructed from less complicated terms. The nondiagonal matrix elements of the AD matrix contain also only less complicated terms (see, for example, AD exact expressions at LO and NLO approximations in [3] for QCD and [6] for $\mathcal{N} = 4$ SYM), and therefore they cannot generate the most complicated contributions to the eigenvalues of AD matrix. Thus, the most complicated part of the NNLO QCD ADs should coincide (up to color factors) with the universal AD $\gamma^{(2)}_{\mathrm{uni}}(j)$.

3. UNIVERSAL AD FOR $\mathcal{N} = 4$ SYM

The final three-loop result for the universal AD $\gamma_{\mathrm{uni}}(j)$ for $\mathcal{N} = 4$ SYM is [10]

$$\gamma(j) \equiv \gamma_{\mathrm{uni}}(j) = \hat{a}\gamma^{(0)}_{\mathrm{uni}}(j) + \hat{a}^2\gamma^{(1)}_{\mathrm{uni}}(j) + \hat{a}^3\gamma^{(2)}_{\mathrm{uni}}(j) + \ldots, \quad \hat{a} = \frac{\alpha N_c}{4\pi}, \tag{7}$$

where*

$$\frac{1}{4}\gamma^{(0)}_{\mathrm{uni}}(j+2) = -S_1, \tag{8}$$

$$\frac{1}{8}\,\gamma^{(1)}_{\mathrm{uni}}(j+2) = \Big(S_3 + \overline{S}_{-3}\Big) - 2\,\overline{S}_{-2,1} + 2\,S_1\Big(S_2 + \overline{S}_{-2}\Big), \tag{9}$$

*Note that in accordance with [8] our normalization of $\gamma(j)$ contains the extra factor $-1/2$ in comparison with the standard normalization (see [5]) and differs by sign in comparison with one from [3].

$$\frac{1}{32}\,\gamma_{\rm uni}^{(2)}(j+2) = 2\,\overline{S}_{-3}\,S_2 - S_5 - 2\,\overline{S}_{-2}\,S_3 - 3\,\overline{S}_{-5} + 24\,\overline{S}_{-2,1,1,1} + \\ + 6\Big(\overline{S}_{-4,1} + \overline{S}_{-3,2} + \overline{S}_{-2,3}\Big) - 12\Big(\overline{S}_{-3,1,1} + \overline{S}_{-2,1,2} + \overline{S}_{-2,2,1}\Big) - \\ - \Big(S_2 + 2\,S_1^2\Big)\Big(3\,\overline{S}_{-3} + S_3 - 2\,\overline{S}_{-2,1}\Big) - S_1\Big(8\,\overline{S}_{-4} + \overline{S}_{-2}^2 + \\ + 4\,S_2\,\overline{S}_{-2} + 2\,S_2^2 + 3\,S_4 - 12\,\overline{S}_{-3,1} - 10\,\overline{S}_{-2,2} + 16\,\overline{S}_{-2,1,1}\Big) \quad (10)$$

and $S_a \equiv S_a(j)$, $S_{a,b} \equiv S_{a,b}(j)$, $S_{a,b,c} \equiv S_{a,b,c}(j)$ are harmonic sums

$$S_a(j) = \sum_{m=1}^{j} \frac{1}{m^a}, \quad S_{a,b,c,\cdots}(j) = \sum_{m=1}^{j} \frac{1}{m^a}\,S_{b,c,\cdots}(m), \quad (11)$$

$$S_{-a}(j) = \sum_{m=1}^{j} \frac{(-1)^m}{m^a}, \quad S_{-a,b,c,\cdots}(j) = \sum_{m=1}^{j} \frac{(-1)^m}{m^a}\,S_{b,c,\cdots}(m), \\ \overline{S}_{-a,b,c,\cdots}(j) = (-1)^j\,S_{-a,b,c,\ldots}(j) + S_{-a,b,c,\cdots}(\infty)\Big(1-(-1)^j\Big). \quad (12)$$

The expression (12) is defined for all integer values of arguments but can be easily analytically continued to real and complex j by the method of [5, 14].

The obtained results are very important for the verification of the various assumptions (see [15] and references therein) coming from the investigations of the properties of a conformal operators in the context of AdS/CFT correspondence [16].

CONCLUSION

In this short review we presented the AD $\gamma_{\rm uni}(j)$ for the $\mathcal{N}=4$ supersymmetric gauge theory up to the NNLO approximation. At the first three orders, the univesal AD has been extracted from the corresponding QCD calculations. The four- and five-loop results have been obtained in [17–19] from the long-range asymptotic Bethe equations together with some additional terms, so-called *wrapping corrections*, coming in agreement with Luscher approach*. All the results have been obtained with using of the *transcendentality principle*.

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*The three- and four-loop results for the universal AD have been reproduced also in [20] by solution of so-called Baxter equation, which can be obtained from the long-range asymptotic Bethe equations.

REFERENCES

1. *Gribov V. N., Lipatov L. N.* // Sov. J. Nucl. Phys. 1972. V. 15. P. 438; 675;
   Lipatov L. N. // Sov. J. Nucl. Phys. 1975. V. 20. P. 94;
   Altarelli G., Parisi G. // Nucl. Phys. B. 1977. V. 126. P. 298;
   Dokshitzer Yu. L. // Sov. Phys. JETP. 1977. V. 46. P. 641.
2. *Andersson Bo et al.* // Eur. Phys. J. C. 2002. V. 25. P. 77.
3. *Moch S. et al.* // Nucl. Phys. B. 2004. V. 688. P. 101;
   Vogt A. et al. // Ibid. V. 691. P. 129.
4. *Brink L. et al.* // Nucl. Phys. B. 1977. V. 121. P. 77;
   Gliozzi F. et al. // Ibid. V. 122. P. 253.
5. *Kotikov A. V., Lipatov L. N.* // Nucl. Phys. B. 2003. V. 661. P. 19; hep-ph/0112346.
6. *Kotikov A. V. et al.* // Phys. Lett. B. 2003. V. 557. P. 114.
7. *Lipatov L. N.* // Sov. J. Nucl. Phys. 1976. V. 23. P. 338;
   Fadin V. S. et al. // Phys. Lett. B. 1975. V. 60. P. 50;
   Kuraev E. A. et al. // Sov. Phys. JETP. 1976. V. 44. P. 443; 1977. V. 45. P. 199;
   Balitsky I. I., Lipatov L. N. // Sov. J. Nucl. Phys. 1978. V. 28. P. 822; JETP Lett. 1979. V. 30. P. 355.
8. *Fadin V. S., Lipatov L. N.* // Phys. Lett. B. 1998. V. 429. P. 127;
   Camici G., Ciafaloni M. // Ibid. V. 430. P. 349.
9. *Kotikov A. V., Lipatov L. N.* // Nucl. Phys. B. 2000. V. 582. P. 19.
10. *Kotikov A. V. et al.* // Phys. Lett. B. 2004. V. 595. P. 521.
11. *Lipatov L. N.* // Proc. of the Intern. Workshop on Very High Multiplicity Physics, Dubna, 2000. P. 159–176; Nucl. Phys. Proc. Suppl. A. 2001. V. 99. P. 175.
12. *Siegel W.* // Phys. Lett. B. 1979. V. 84. P. 193.
13. *Fleischer J. et al.* // Nucl. Phys. B. 1999. V. 547. P. 343; Acta Phys. Polon. B. 1998. V. 29. P. 2611.
14. *Kazakov D. I., Kotikov A. V.* // Nucl. Phys. B. 1988. V. 307. P. 721; 1990. V. 345. P. 299; Phys. Lett. B. 1992. V. 291. P. 171;
    Kotikov A. V. // Phys. At. Nucl. 1994. V. 57. P. 133;
    Kotikov A. V., Velizhanin V. N. hep-ph/0501274.
15. *Staudacher M.* // JHEP. 2005. V. 0505. P. 054;
    Beisert N., Staudacher M. // Nucl. Phys. B. 2005. V. 727. P. 1.
16. *Maldacena J.* // Adv. Theor. Math. Phys. 1998. V. 2. P. 231; Intern. J. Theor. Phys. 1988. V. 38. P. 1113;
    Gubser S. S. et al. // Phys. Lett. B. 1998. V. 428. P. 105;
    Witten E. // Adv. Theor. Math. Phys. 1998. V. 2. P. 253.
17. *Kotikov A. V. et al.* // J. Stat. Mech. 2007. V. 0710. P. 10003.
18. *Bajnok Z. et al.* // Nucl. Phys. B. 2009. V. 816. P. 376.
19. *Lukowski T. et al.* hep-th/0912.1624.
20. *Kotikov A. V. et al.* // Nucl. Phys. B. 2009. V. 813. P. 460;
    Beccaria M. et al. // Nucl. Phys. B. 2010. V. 827. P. 565.