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RECONSTRUCTION THEOREM AND CLUSTER PROPERTIES OF WIGHTMAN FUNCTIONS IN NONCOMMUTATIVE QUANTUM FIELD THEORY

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The rigorous definition of quantum field operator is done in any theory where usual product between corresponding test functions is substituted by the star product. The important example of such a theory is noncommutative quantum field theory. Cluster properties of Wightman functions are proved in these theories.

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INTRODUCTION

The Wightman reconstruction theorem [1, 2] asserts that the set of the functions, satisfying some conditions, determines uniquely quantum field theory (QFT). It implies that there exists the space J, in which quantum field operator φ_f is a densely defined operator, $f \equiv f\,(x)$ is a test function. All scalar products in J are determined by Wightman functions

$$W\,(x_1, \ldots, x_n) \equiv \langle \Psi_0, \varphi\,(x_1) \cdots \varphi\,(x_n)\, \Psi_0 \rangle,$$

Ψ_0 is a vacuum vector. It means that the scalar product between arbitrary vectors in J can be approximated with arbitrary accuracy by a finite linear combination of Wightman functions. $W\,(x_1, \ldots, x_n)$ are generalized functions, which in the commutative theory are usually tempered distributions.

Noncommutative quantum field theory (NC QFT), being one of the generalizations of standard QFT, is intensively developed during the last years. Essential interest in NC QFT is connected with the fact that in some cases it is obtained as a low-energy limit from the strings theory.

The simplest and at the same time most studied version of noncommutative theory is based on the following Heisenberg-like commutation relations between coordinates:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\,\theta_{\mu\nu}, \tag{1}$$

where $\theta_{\mu\nu}$ is a constant antisymmetric matrix.

It is known that NC QFT determined by Heisenberg-like commutation relations can be formulated also in a commutative space if we substitute the usual product between the test functions by the $\star$ (Moyal-type) product of test functions. Let us recall that:

$$\varphi(x) \star \varphi(y) = \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^{\mu}}\,\frac{\partial}{\partial y^{\nu}}\right)\varphi(x)\,\varphi(y).$$

Let us stress that the set of the theories defined by the $\star$-product is wider than NC QFT as a theory defined by the $\star$-product is not restricted by the $SO\,(1,1)\otimes SO\,(2)$ invariance, which follows from the Heisenberg-like commutation relations. Here we consider any theory in which usual product between operators (strictly between corresponding test functions) is substituted by the $\star$-product.

As the $\star$-product contains infinitive number of derivatives, the corresponding space of generalized functions cannot be a tempered distribution space. As was proved in our paper [3]*, the test-functions space is a Gelfand–Shilov space S^{β} with $\beta < 1/2$ [5].

The main point in the derivation of the reconstruction theorem is the construction of the space, in which quantum field operator is well defined. This construction has been done in our paper [6].

In this report we give a rigorous definition of quantum field operator in NC QFT and prove the cluster properties of Wightman functions.

1. RIGOROUS DEFINITION OF QUANTUM FIELD OPERATORS IN NC QFT

Let us define rigorously quantum field operator φ_f. To this end we construct a closed and nondegenerate space J such that operators φ_f be well defined on dense domain of J. The difference of noncommutative case from commutative one is that action of the operator φ_f is defined by the $\star$-product.

Construction of space J we shall begin with introduction of the set of breaking sequences of the following kind

$$g = \{g_0, g_1, \ldots, g_k\}, \quad k \text{ depends on } g, \tag{2}$$

where

$$g_0 \in \mathbb{C}, \quad g_1 = g_1^1\,(x_1), x_1 \in \mathbb{R}^4,$$
$$g_i = g_i^1\,(x_1) \star \cdots \star g_i^i\,(x_i), \quad x_j \in \mathbb{R}^4, \quad 1 \leqslant j \leqslant i.$$

*The similar result was also obtained by M. A. Soloviev [4].

The every possible finite sums of the sequences belonging to the set under consideration form space J_0' on which action of the operator φ_f, $f = f(x)$, $x \in \mathbb{R}^4$ will be determined.

Let us define quantum field as follows:

$$\varphi_f \, g = \{fg_0, f \star g_1, \ldots, f \star g_k\}, \tag{3}$$

where $f \star g_i = f(x) \star g_i^1(x_1) \star \cdots \star g_i^i(x_i)$.

As any vector of space J_0' is the sum of the vectors belonging to set in question, the operator φ_f is determined on any vector of space J_0' and $\varphi_f \Phi \in J_0', \forall\, \Phi \in J_0'$.

We define a scalar product of vectors in J_0' with the help of Wightman functions $W(x_1, \ldots, x_n)$. We consider a chain of vectors: vacuum vector $\Psi_0 = \{1, 0, \ldots, 0\}$, $\Phi_1 = \varphi_{f_1}\Psi_0$, $\Phi_k = \varphi_{f_k} \cdots \varphi_{f_1}\Psi_0$, $f_i = f_i(x_i)$, $x_i \in \mathbb{R}^4$. Evidently, $\Phi_k = \{0, \ldots, f_k \star \ldots \star f_1, 0 \cdots 0\}$ and

$$\Psi_n = \varphi_{f_{k+1}} \cdots \varphi_{f_n}\Psi_0 = \{0, \ldots, f_{k+1} \star \ldots \star f_n, 0 \cdots 0\}.$$

It is obvious that J_0' is a span of the vectors of such a type.

Scalar product of vectors Φ_k and Ψ_n is

$$\begin{aligned}\langle \Phi_k, \Psi_n \rangle = \langle \Psi_0, \varphi_{\bar{f}_1} \cdots \varphi_{\bar{f}_k} \varphi_{f_{k+1}} \cdots \varphi_{f_n} \Psi_0 \rangle = \\ = \int dx_1 \cdots dx_n W(x_1, \ldots, x_n) \times \\ \times \overline{f_1(x_1)} \star \cdots \star \overline{f_k(x_k)} \star f_{k+1}(x_{k+1}) \star \cdots \star f_n(x_n).\end{aligned} \tag{4}$$

Here we consider only Hermitian (real) operators, but the construction can be easily extended to complex fields.

As in commutative case, we need to pass from J_0' to nondegenerate and closed space J.

The space J_0' can contain isotropic, i.e., orthogonal to J_0' vectors which, as is known, form subspace. Designating isotropic space as $\tilde{J}_0$ and passing to factor-space $J_0 = J_0'/\tilde{J}_0$, we obtain nondegenerate space, i.e., the space which does not contain isotropic vectors. For closure of space J_0 we assume, as well as in commutative case, that J_0 is a normalized space. $\bar{J}_0$ (a closure of J_0) is carried out with the help of standard procedure — closure to the introduced norm. This space, in turn, can contain isotropic subspace $\tilde{J}$. Factor-space $J = \bar{J}_0/\tilde{J}$, obviously, will be nondegenerate space.

Thus, we constructed the closed and nondegenerate space J such that operators φ_f are determined on dense domain J_0. Hence, every vector of J can be approximated with arbitrary accuracy by the vectors of the type

$$\varphi_{f_1} \cdots \varphi_{f_n} \Psi_0, \tag{5}$$

where Ψ_0 is a vacuum vector. In other words, the vacuum vector Ψ_0 is cyclic, i.e., the axiom of cyclicity of vacuum is fulfilled.

2. CLUSTER PROPERTIES OF WIGHTMAN FUNCTIONS

Cluster properties of Wightman functions are very important properties of quantum field theory. Let us recall that cluster properties of Wightman functions imply that

$$W(x_1, \ldots, x_k, x_{k+1} + \lambda a, \ldots, x_n + \lambda a) \to \\ \to W(x_1, \ldots, x_k)\, W(x_{k+1}, \ldots, x_n). \quad (6)$$

It is very interesting to consider cluster properties in NC QFT. In what follows, we show that cluster properties are valid in NC QFT as well as in commutative QFT.

First, let us consider the case when time commutes with spatial variables. It is known that in this case there exists one spatial variable which commutes with all other variables. Thus in this case we have two commutative variables and two noncommutative ones.

We admit in correspondence with the standard QFT that Wightman functions are tempered distributions in respect with commutative variables. The cluster properties in this case exist on the same ground as in the commutative theory.

The general case, when all $\theta^{ij} \neq 0$, is much more nontrivial. Nevertheless, we prove that Eq. (6) is valid also in this case.

The crucial point in the derivation of Eq. (6) is the fact that any test function $f_i \equiv f_i(x_i)$ belongs to the Gelfand–Shilov space S^β with $\beta < 1/2$. Precisely it means that if $f_i \in S^\beta$ with $\beta < 1/2$, then

$$f_i \star f_{i+1} \equiv \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x_i^\mu}\,\frac{\partial}{\partial x_{i+1}^\nu}\right) f_i(x_i) f_{i+1}(x_{i+1}) = \\ = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x_i^\mu}\,\frac{\partial}{\partial x_{i+1}^\nu}\right)^n f_i(x_i)\, f_{i+1}(x_{i+1}) \quad (7)$$

converges.

If $f_i \star f_{i+1}$ converges $\forall i$, then $f_1 \star \ldots \star f_n$ converges as well if $f_i \in S^\beta$ with $\beta < 1/2$.

Convergence of the series (7) implies that

$$f_1(x_1) \star \cdots \star f_n(x_n) = f_1^N(x_1) \star \cdots \star f_n^N(x_n) + \varepsilon(N),$$

$$\varepsilon\,(N) \to 0, \text{ if } N \to \infty, \tag{8}$$

$$f_i^N\,(x_i) \star f_{i+1}^N\,(x_{i+1}) = \sum_{n=0}^{N} \frac{1}{n!} \left(\frac{i}{2}\, \theta^{\mu\nu} \frac{\partial}{\partial x_i^{\mu}} \frac{\partial}{\partial x_{i+1}^{\nu}} \right)^n f_i\,(x_i)\, f_{i+1}\,(x_{i+1}).$$

Let us point out that the latter estimation is true for every x_i.

Now let us point out that as $f_i^N\,(x_i) \star f_{i+1}^N$ contains only a finite numbers of derivatives, then the corresponding generalized functions are tempered distributions.

Thus at any N we have cluster properties. This fact gives us the possibility to prove cluster properties in the general case. Actually, on the one hand, for arbitrary λ

$$W\,(x_1, \ldots, x_k, x_{k+1} + \lambda a, \ldots, x_n + \lambda a) \to \\ \to W^N\,(x_1, \ldots, x_k, x_{k+1} + \lambda a, \ldots, x_n + \lambda a) + \varepsilon\,(N), \tag{9}$$

where $\varepsilon\,(N) \to 0$, if $N \to \infty$ and

$$W^N\,(x_1, \ldots, x_n) = \langle \Psi_0,\, f_1^N\,(x_1) \star \cdots \star f_n^N\,(x_n)\, \Psi_0 \rangle.$$

On the other hand, at any N

$$W^N(x_1, \ldots, x_k, x_{k+1} + \lambda a, \ldots, x_n + \lambda a) = \\ = W^N(x_1, \ldots, x_k)\, W^N\,(x_{k+1}, \ldots, x_n) + \varepsilon\,(\lambda), \quad \varepsilon\,(\lambda) \to 0, \text{ if } \lambda \to \infty. \tag{10}$$

Combining Eqs. (9) and (10), we obtain that if $a^2 = -1$, $\lambda \to \infty$

$$W\,(x_1, \ldots, x_k, x_{k+1} + \lambda a, \ldots, x_n + \lambda a) \to W\,(x_1, \ldots, x_k) W\,(x_{k+1}, \ldots, x_n). \tag{11}$$

Thus the cluster properties of Wightman functions are proved in any theory, in which the usual product of test functions is substituted by the $\star$-product. Let us recall that from cluster properties of Wightman functions follows the uniqueness of vacuum state.

CONCLUSIONS

We have rigorously constructed field operators in NC QFT and have proved the cluster properties. Let us stress that our results can be easily extended on the case of arbitrary dimensions. These results are important for any rigorous treatment of the axiomatic approach to NC QFT.

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