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BOGOLYUBOV'S THEORY OF SUPERFLUIDITY

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The Bogolyubov model of liquid helium is considered. We derive sufficient conditions which ensure an appearance of the Bose condensate in the model. For some temperatures and some positive values of the chemical potential there is the gapless Bogolyubov spectrum of elementary excitations, leading to the proper microscopic interpretation of the superfluidity.

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Let us consider a system of N spinless identical nonrelativistic bosons of mass m enclosed in a centered cubic box $\Lambda \subset \mathbb{R}^3$ of volume $V = |\Lambda| = L^3$ with periodic boundary conditions for the wave functions. The Hamiltonian of the system can be written in the second quantized form as

$$\hat{H}_\Lambda(\mu) \equiv \hat{H}_\Lambda - \mu\hat{N}_\Lambda = \sum_{k\in\Lambda^*}(\epsilon_k - \mu)\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2V}\sum_{p,q,k\in\Lambda^*}\nu(k)\hat{a}_p^\dagger\hat{a}_q^\dagger\hat{a}_{p+k}\hat{a}_{q-k}. \quad (1)$$

Here $\hat{a}_p^\# = \{\hat{a}_p^\dagger \text{ or } \hat{a}_p\}$ are the usual boson creation (annihilation) operators for the one-particle state $\psi_p(x) = V^{-1/2}\exp(ipx), p \in \Lambda^*, x \in \Lambda$, acting on the Fock space $F_\Lambda = \oplus_{n=0}^\infty \mathcal{H}_\mathrm{B}^{(n)}$, where $\mathcal{H}_\mathrm{B}^{(n)} \equiv [L^2(\Lambda^n)]_\mathrm{sym}$ is the symmetrized n-particle Hilbert space appropriate for bosons, and $\mathcal{H}_\mathrm{B}^{(0)} = \mathbb{C}$. The sums in (1) run over the dual set

$$\Lambda^* = \left\{p \in \mathbb{R}^3 : p_\alpha = \frac{2\pi}{L}n_\alpha, n_\alpha = 0, \pm1, \pm2, \ldots, \alpha = 1, 2, 3\right\},$$

$\epsilon_p = |p|^2/(2m)$ is the one-particle energy spectrum of free bosons in the modes $p \in \Lambda^*$ (we propose $\hbar = 1$), $\hat{N}_\Lambda = \sum_{k\in\Lambda^*}\hat{a}_k^\dagger\hat{a}_k$ is the total particle-number operator, μ is the chemical potential, $\nu(k)$ is the Fourier transform of the interaction pair potential $\Phi(x)$. We suppose that $\Phi(x) = \Phi(|x|) \in L^1(\mathbb{R}^3)$ and $\nu(k)$ is a real function with a compact support such that $0 \leqslant \nu(k) = \nu(-k) \leqslant \nu(0)$ for all $k \in \mathbb{R}^3$.

So long as the rigorous analysis of the Hamiltonian (1) is very knotty problem, Bogolyubov introduced the model Hamiltonian of the superfluidity theory [1, 2].

He proposed to disregard the terms of the third and fourth orders in operators $\hat{a}_p^{\#}, p \neq 0$ in the Hamiltonian (1),

$$\hat{H}_\Lambda^{\mathrm{B}}(\mu) = \sum_{k\in\Lambda^*} (\epsilon_k - \mu)\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2V}\sum_{k\neq 0}\nu(k)(\hat{a}_k^\dagger \hat{a}_{-k}^\dagger \hat{a}_0\hat{a}_0 + \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_{-k}\hat{a}_k)+ \\ + \frac{1}{V}\hat{a}_0^\dagger\hat{a}_0\sum_{k\neq 0}\nu(k)\hat{a}_k^\dagger\hat{a}_k + \frac{\nu(0)}{V}\hat{a}_0^\dagger\hat{a}_0\sum_{k\neq 0}\hat{a}_k^\dagger\hat{a}_k + \frac{\nu(0)}{2V}\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0. \quad (2)$$

Then, Bogolyubov takes advantage of the macroscopic occupation of the zero momentum one-particle state to replace the corresponding creation and annihilation operators $\hat{a}_0^{\#}$ by c-numbers,

$$\frac{\hat{a}_0^\dagger}{\sqrt{V}} \to \bar{c}, \quad \frac{\hat{a}_0}{\sqrt{V}} \to c, \quad (3)$$

where $c \in \mathbb{C}$ and the bar means complex conjugation. This idea has its roots in the work [3]. In § 63 of this monograph Dirac analyses a many-body system within the framework of second quantization. Bogolyubov developed the Dirac's idea systematically to study Bose condensation and superfluidity in the model (2).

Let $\hat{H}_\Lambda^{\mathrm{B}}(\mu, c)$ be the Hamiltonian (2) after the Bogolyubov approximation (3). This Hamiltonian is a bilinear form in boson operators $\hat{a}_k^{\#}(k \neq 0)$. So, one can diagonalize it by the Bogolyubov canonical transformation. To determine the complex parameter c it is necessary to use some self-consistent procedure.

Bogolyubov considered the Hamiltonian (2) in the case of zero temperature [1, 2]. In the main perturbation order he found that $\mu(\theta = 0) = |c|^2\nu(0)$, where $|c|^2 = \rho_0$ is the density of Bose condensate. In this case the structure of the collective excitation spectrum of the Hamiltonian $\hat{H}_\Lambda^{\mathrm{B}}(\mu, c)$ explains the superfluid properties of the system (2).

It should be noted that the main condition which gives the possibility to replace the Hamiltonian (1) by the model Hamiltonian (2) is

$$\frac{N - N_0}{N} \ll 1, \quad (4)$$

where N_0 is the number of condensate particles. Condition (4) means that the interaction is sufficiently weak and the case of very small temperatures must be considered. Thus, in 1947 Bogolyubov analyzed the model (2) within the framework of (4). The validity of the Bogolyubov approximation (3) has not been rigorously proved.

The rigorous justification for the c-number substitution in the case of the total, correct superstable pair Hamiltonian (1) was done in a classic paper of Ginibre [4]. Recently, this problem was revisited in paper [5]. The authors of [4, 5] did not consider the truncated Bogolyubov's Hamiltonian (2). Nevertheless, Lieb et al. [5] mentioned that their device (based on the Berezin–Lieb inequality) can be used also for the Hamiltonian (2).

Let us first rewrite the Hamiltonian (2) in the following way:

$$\hat{H}^{\mathrm{B}}_{\Lambda}(\mu) = \hat{H}^{\mathrm{B}}_{\Lambda 0}(\mu, c) + \hat{H}^{\mathrm{B}}_{\Lambda 1}(c), \tag{5}$$

where

$$\hat{H}^{\mathrm{B}}_{\Lambda 0}(\mu, c) \equiv \sum_{k\in\Lambda^*}(\epsilon_k - \mu)\hat{a}^{\dagger}_k\hat{a}_k - \frac{\Phi(0)}{2}\hat{a}^{\dagger}_0\hat{a}_0 + \frac{\nu(0)}{2V}\hat{a}^{\dagger}_0\hat{a}^{\dagger}_0\hat{a}_0\hat{a}_0 + \\ + \nu(0)|c|^2\sum_{k\neq 0}\hat{a}^{\dagger}_k\hat{a}_k, \tag{6}$$

$$\hat{H}^{\mathrm{B}}_{\Lambda 1}(c) \equiv \frac{\nu(0)}{2V}(\hat{a}^{\dagger}_0\hat{a}_0 - V|c|^2)\sum_{k\neq 0}\hat{a}^{\dagger}_k\hat{a}_k + \\ + \frac{1}{2V}\sum_{k\neq 0}\nu(k)(\hat{a}^{\dagger}_0\hat{a}_k + \hat{a}_0\hat{a}^{\dagger}_{-k})^{\dagger}(\hat{a}^{\dagger}_0\hat{a}_k + \hat{a}_0\hat{a}^{\dagger}_{-k}). \tag{7}$$

The complex parameter c in formulae (5)–(7) will be defined below. It is easy to see that the Hamiltonian (6) is stable for $\mu \leqslant \nu(0)|c|^2$ and any $c \in \mathbb{C}$.

Denoting

$$\delta\hat{a}_0 \equiv \hat{a}_0 - c\sqrt{V}, \quad \delta\hat{a}^{\dagger}_0 \equiv \hat{a}^{\dagger}_0 - \bar{c}\sqrt{V}, \quad \hat{A}_k \equiv \hat{a}^{\dagger}_0\hat{a}_k + \hat{a}_0\hat{a}^{\dagger}_{-k}, k \neq 0,$$

we can write (7) in the form

$$\hat{H}^{\mathrm{B}}_{\Lambda 1}(c) = \frac{\nu(0)}{2V}\sum_{k\neq 0}\hat{a}^{\dagger}_k\hat{a}_k(\delta\hat{a}^{\dagger}_0\delta\hat{a}_0 + c\sqrt{V}\delta\hat{a}^{\dagger}_0 + \bar{c}\sqrt{V}\delta\hat{a}_0) + \frac{1}{2V}\sum_{k\neq 0}\nu(k)\hat{A}^{\dagger}_k\hat{A}_k. \tag{8}$$

Let us prove that

$$\lim_{V\to\infty}\frac{1}{V}\langle\hat{H}^{\mathrm{B}}_{\Lambda 1}(c)\rangle_{\hat{H}^{\mathrm{B}}_{\Lambda}(\mu)} \geqslant 0, \tag{9}$$

where c is a solution of the equation

$$|c|^2 = \frac{1}{V}\langle\hat{a}^{\dagger}_0\hat{a}_0\rangle_{\hat{H}^{\mathrm{B}}_{\Lambda}(\mu)}.$$

From the Bogolyubov inequality for pressures

$$p[\hat{H}^{\mathrm{B}}_{\Lambda}(\mu)] \leqslant p[\hat{H}^{\mathrm{B}}_{\Lambda 0}(\mu, c)] - \frac{1}{V}\langle\hat{H}^{\mathrm{B}}_{\Lambda 1}(c)\rangle_{\hat{H}^{\mathrm{B}}_{\Lambda}(\mu)}$$

we then obtain that the Hamiltonian (2) is stable for

$$\mu \leqslant \nu(0)|c|^2. \tag{10}$$

Let us introduce the Hamiltonian

$$\hat{H}_{\Lambda}^{\mathrm{B}}(\mu,\nu) \equiv \hat{H}_{\Lambda}^{\mathrm{B}}(\mu) - \sqrt{V}(\bar{\nu}\hat{a}_0 + \nu\hat{a}_0^{\dagger})$$

with sources $\nu \in \mathbb{C}$ breaking the symmetry of $\hat{H}_{\Lambda}^{\mathrm{B}}(\mu)$. Using the Cauchy inequality, we get the estimate

$$|\langle \delta\hat{a}_0^{\dagger}\hat{N}' \rangle_{\hat{H}_{\Lambda}^{\mathrm{B}}(\mu,\nu)}| \leqslant \left[\langle \delta\hat{a}_0^{\dagger}\delta\hat{a}_0 \rangle_{\hat{H}_{\Lambda}^{\mathrm{B}}(\mu,\nu)} \langle \hat{N}'^2 \rangle_{\hat{H}_{\Lambda}^{\mathrm{B}}(\mu,\nu)}\right]^{1/2} \leqslant \rho V \langle \delta\hat{a}_0^{\dagger}\delta\hat{a}_0 \rangle_{\hat{H}_{\Lambda}^{\mathrm{B}}(\mu,\nu)},$$

where $\hat{N}' \equiv \sum\limits_{k\neq 0} \hat{a}_k^{\dagger}\hat{a}_k$.

To obtain an upper bound for the average in the last inequality we can apply the usual procedure of the Bogolyubov quasi-average method [6] and Bogolyubov, Jr. technique [7]. Define c by the condition $c = \langle \hat{a}_0 \rangle_{\hat{H}_{\Lambda}^{\mathrm{B}}(\mu,\nu)}/\sqrt{V}$, $|c| \leqslant M < \infty$. By the Harris inequality [8] one gets

$$\frac{1}{2}\langle [\delta\hat{a}_0^{\dagger}, \delta\hat{a}_0]_+ \rangle_{\hat{H}_{\Lambda}^{\mathrm{B}}(\mu,\nu)} \leqslant (\delta\hat{a}_0^{\dagger}, \delta\hat{a}_0)_{\hat{H}_{\Lambda}^{\mathrm{B}}(\mu,\nu)} + \frac{\beta}{12}\langle [\delta\hat{a}_0, [\hat{H}_{\Lambda}^{\mathrm{B}}(\mu,\nu), \delta\hat{a}_0^{\dagger}]] \rangle_{\hat{H}_{\Lambda}^{\mathrm{B}}(\mu,\nu)},$$

where $[\cdot,\cdot]_+$ is the anticommutator and $(\cdot,\cdot)_{\hat{\Gamma}}$ denotes the Bogolyubov inner product (or the Duhamel two-point function) with respect to the Hamiltonian $\hat{\Gamma}$ [9]. Literally reiterating the standard for this method calculations, we see that

$$\frac{1}{V}\langle \delta\hat{a}_0^{\dagger}\delta\hat{a}_0 \rangle_{\hat{H}_{\Lambda}^{\mathrm{B}}(\mu,\nu)} \leqslant \frac{\eta}{\sqrt{V}},$$

where η is some positive constant, independent of V. Therefore, it follows from the last inequality that

$$|\langle \delta\hat{a}_0^{\#}\hat{N}' \rangle_{\hat{H}_{\Lambda}^{\mathrm{B}}(\mu,\nu)}| \leqslant \rho\sqrt{\eta}V^{5/4}.$$

Thus, using the representation of the Hamiltonian $\hat{H}_{\Lambda 1}^{\mathrm{B}}(c)$ in the form (8), one can see that the condition (9) is actually justified. The parameter c should be chosen by the stereotyped for the Bogolyubov–Ginibre technique manner. This parameter is connected with the Bose condensate density as $|c|^2 = \rho_0$.

The above analysis confirms an assertion, that if the system is stable after the c-number substitution (3), then so is the original one [5]. It is necessary to notice that authors of works [10–12] at studying of model (2) have come to the incorrect conclusion about instability of Bogolyubov's Hamiltonian at positive chemical potentials.

In a similar manner as in the work of Ginibre [4], one can prove that the model Hamiltonian $\hat{H}_{\Lambda}^{\mathrm{B}}(\mu)$ is thermodynamically equivalent to the approximating

Hamiltonian

$$\hat{H}^{\mathrm{B}}_{\Lambda}(\mu, c) = \sum_{k\neq 0} [\epsilon_k - \mu + |c|^2(\nu(0) + \nu(k))]\hat{a}^{\dagger}_k \hat{a}_k + \\ + \frac{1}{2}\sum_{k\neq 0} \nu(k)(c^2 \hat{a}^{\dagger}_k \hat{a}^{\dagger}_{-k} + \bar{c}^2 \hat{a}_k \hat{a}_{-k}) + \frac{1}{2}\nu(0)|c|^4 V - \mu|c|^2 V.$$

The self-consistency parameter c in the method is determined by the condition that the approximate pressure $p[\hat{H}^{\mathrm{B}}_{\Lambda}(\mu, c)]$ be maximal. At the same time, the stability condition (10) must be fulfilled (in contrast to the paper [11], where $\mu \leqslant 0$).

A necessary condition for $p[\hat{H}^{\mathrm{B}}_{\Lambda}(\mu, c)]$ to be maximum (self-consistency equation) in the case of the Bogolyubov model is

$$\mu - x\nu(0) = \frac{1}{2V}\sum_{k\neq 0}\left[(\nu(0) + \nu(k))\left(\frac{f_k}{E_k}\coth\frac{\beta E_k}{2} - 1\right) - \nu(k)\frac{h_k}{E_k}\coth\frac{\beta E_k}{2}\right],$$

where

$$u_k = \sqrt{\frac{1}{2}\left(\frac{f_k}{E_k} + 1\right)}, \quad v_k = -\sqrt{\frac{1}{2}\left(\frac{f_k}{E_k} - 1\right)},$$

$$f_k = \epsilon_k - \mu + x(\nu(0) + \nu(k)), \quad h_k = x\nu(k), \quad E_k = \sqrt{f_k^2 - h_k^2},$$

and we denote $x \equiv |c|^2$.

It is possible to show that if the potential $\nu(k)$ in the Bogolyubov model of superfluidity (2) satisfies the condition

$$\nu(0) \geqslant \frac{1}{2(2\pi)^3}\int\limits_{\mathbb{R}^3} \frac{d^3k}{\epsilon_k}\nu^2(k),$$

then there exists the domain of stability on the phase diagram $\{0 < \mu \leqslant \mu^*,\ 0 \leqslant \theta \leqslant \theta_0(\mu)\}$, where the nontrivial solution of the self-consistency equation takes place. In this domain there is the nonzero Bose condensate. At the boundary $\theta = \theta_0(\mu)$ of this domain the Bose condensate density equals $\rho_0 = \mu/\nu(0)$. In this case the quasi-particle spectrum of the Bogolyubov's Hamiltonian (2)

$$E_k = \sqrt{\epsilon_k(\epsilon_k + 2\rho_0\nu(k))}$$

has a gapless type and the famous criterion of superfluidity $\min_k(E_k/|k|) > 0$ holds.

As we have noted earlier, the Bogolyubov's theory is a theory of a dilute weakly interacting Bose gas at temperatures far below the λ-point. In contrast with the pair Hamiltonian (1), the Bogolyubov's Hamiltonian (2) is not corresponding to some pair interaction and it is not superstable. Nevertheless, Bogolyubov's approach forms the basis of the systematic application of quantum theory to an interacting system of bosons.

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REFERENCES

1. *Bogolyubov N. N.* // J. Phys. USSR. 1947. V. 11. P. 23.
2. *Bogolyubov N. N.* // Bull. Moscow State Univ. 1947. V. 7. P. 43.
3. *Dirac P. A. M.* The Principles of Quantum Mechanics. 2nd ed. Oxford: Clarendon Press, 1935.
4. *Ginibre J.* // Commun. Math. Phys. 1968. V. 8. P. 26.
5. *Lieb E. H., Seiringer R., Yngvason J.* // Phys. Rev. Lett. 2005. V. 94. P. 080401.
6. *Bogolyubov N. N.* Quasi-Averages in Problems of Statistical Mechanics. Dubna: JINR, 1963.
7. *Bogolyubov N. N., Jr.* // Physica. 1966. V. 32. P. 933.
8. *Harris B.* // J. Math. Phys. 1967. V. 8. P. 1044.
9. *Bogolyubov N. N.* // Physikalishe Abhandlungen Sowjetunion. 1962. V. 6. P. 113.
10. *Angelescu N., Verbeure A., Zagrebnov V. A.* // J. Phys. A. 1992. V. 25. P. 3473.
11. *Bru J. B., Zagrebnov V. A.* // J. Phys. A. 1998. V. 31. P. 9377.
12. *Zagrebnov V. A., Bru J. B.* // Phys. Rep. 2001. V. 350. P. 291.