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$(\hbar, k)$-DYNAMICS AS SOME GENERALIZATION OF EQUILIBRIUM QUANTUM STATISTICAL MECHANICS

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We show that the quantum statistical mechanics (QSM) describing quantum and thermal properties of objects has only the sense of a particular semiclassical approximation. We propose a more general (than QSM) microdescription of objects in a heat bath taking into account a vacuum as an object environment; we call it $(\hbar, k)$-dynamics ($\hbar kD$). We construct a new model of thermostat, namely, a quantum heat bath, and study its properties including the cases of «cold» and «thermal» vacua. We introduce a new generative operator, Schroedingerian, or stochastic action operator, and show its fundamental role in the determination of such macroquantities as internal energy, effective temperature, and effective entropy. We establish that in $\hbar kD$ the ratio of effective action to effective entropy at zero temperature equals the universal constant $\hbar/2k$. This result corresponds to experimental data taken recently under studying of a new matter state — nearly perfect fluid.

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1. THE LIMITATIONS OF EQUILIBRIUM QUANTUM STATISTICAL MECHANICS

The conviction that equilibrium quantum statistical mechanics (QSM) is not only an adequate description of microobjects in a heat bath but also forms a basis for the corresponding macrodescription has predominated for a fairly long time. At the same time, it is well known that there exist such macroparameters, temperature, for example, whose analogies have not yet been studied on the microlevel.

QSM is based on the notion of the density matrix (operator), which in the energy representation has the form of the Gibbs–von Neumann quantum canonical distribution,

$$w_n = \exp \frac{F - \varepsilon_n}{\Theta}, \tag{1}$$

where ε_n is the spectrum of the object energy; F is the free energy determined by the normalization condition, and Θ is the modulus of the distribution.

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The QSM limitations are manifested in three ways.

First, distribution (1) is insensitive to including the contribution of the zero oscillation energy $\varepsilon_0 = \hbar\omega/2$ to the object energy because such an inclusion is automatically compensated by a change in the expression for the free energy F in view of the normalization condition:

$$\left.\begin{array}{l} \varepsilon_n \Rightarrow \varepsilon_n' = \varepsilon_n + \varepsilon_0 \\ F_n \Rightarrow F_n' = F_n + \varepsilon_0 \end{array}\right\} F' - \varepsilon_n' = F - \varepsilon_n.$$

Moreover, as can be seen in the ground state of the quantum oscillator with

$$\Delta p_0^2 = \frac{\hbar m\omega}{2}, \quad \Delta q_0^2 = \frac{\hbar}{2m\omega},$$

we have

$$\Delta p_0 \Delta q_0 = \frac{\hbar}{2} = \frac{\varepsilon_0}{\omega}, \tag{2}$$

which confirms the direct relation between the quantities ε_0 and $\hbar/2$. (It is principal in quantum physics!)

Second, it is assumed in QSM that

the Zero Law has the form $T = T_0$,

where T_0 is a temperature of heat bath. So, the object temperature does not fluctuate ($\Delta T = 0$). However, we know that the temperature fluctuations in low-temperature experiments are sufficiently noticeable for small objects, including nanoparticles.

Third, according to QSM,

the Third Law has the form $S_{\min} = 0$,

but the assertion that the minimum entropy equals zero (automatically following from distribution (1)) is currently very doubtful.

Fourth, we also note another inconsistency inherent in QSM. As is known, the modulus Θ of distribution (1) has the form

$$\Theta_{\text{cl}} = k_B T = \mathcal{E}_{\text{cl}}. \tag{3}$$

This corresponds to choosing the classical model of the heat bath as a set of infinity number of weakly coupled classical oscillators with average energy $\mathcal{E}_{\text{cl}}$. However, distribution (1) is used in QSM for any objects at any temperatures. But a microobject with quantized energy can be placed in such a heat bath even under the conditions $k_B T < (\varepsilon_n - \varepsilon_{n-1})$.

In other words, equilibrium QSM based on distribution (1) is not a consistent theory for either quantum or thermal phenomena. Therefore, it can now be considered only a particular semiclassical approximation with respect to their quantum and thermal characteristics defined by the respective Planck and Boltzmann constants $\hbar$ and k_B.

So, on the base of QSM it is not possible to construct thermodynamics, suitable for describing of low-temperature area and small objects.

2. STATISTICAL THERMODYNAMICS BY EINSTEIN AS THE FIRST THEORY WITH THERMAL FLUCTUATIONS

Wishing go out of the frame of QSM, we proceed from the fact that no objects are isolated in nature. In other words, we follow the Feynman model, according to which any system can be represented as a set of the object under study and its environment (the «rest of the Universe» as a system of infinity number of freedom degrees). The environment can exert both regular and stochastic actions on the object. Here, we study only the stochastic action. Two types of action, namely, quantum and thermal actions characterized by the respective Planck and Boltzmann constants, can be assigned to it.

It is important to note that every object responds to the stochastic action in the form of fluctuations of its characteristics. The first theory that took into account thermal fluctuations was created by Einstein. It was nonquantum Statistical Thermodynamics. Correspondingly, in it Boltzmann's constant is only used.

We recall that among the important statements of this theory there are the next:

1) the model of environment is not change — it is a classical heat bath with energy

$$\mathcal{E}_{\text{cl}} \equiv k_B T;$$

2) the Zero Law takes another form than in QSM and classical Thermodynamics

$$T = (T_0) \pm \Delta T,$$

where

$$(\Delta T)^2 = \frac{k_B}{C_V} T_0^2$$

is a dispersion of object temperature; $T_0 \equiv \langle T \rangle$ is an average object temperature, and C_V is a heat capacity at constant volume;

3) Gibbs' distribution, in contrast to QSM, is written in the space of macroparameters and takes the form

$$\rho(\mathcal{E}) = \exp \frac{F - \mathcal{E}(T, V)}{\Theta_{\text{cl}}}.$$

To obtain a consistent quantum-thermal description of natural objects, in our opinion, it is possible to use other approaches that are different from QSM and nonquantum Statistical Thermodynamics. They are based on one general idea, namely, replacing the classical model of the environment (the heat bath) with an adequate quantum model or a quantum heat bath (QHB). In this model, the thermal equilibrium between the object and its environment is characterized by the effective temperature T_{eff} with a nonzero constraint from below. This quantity takes the environment stochastic action of quantum and thermal types into account simultaneously.

We suppose that it is possible to do it using two different approaches: both macro- and microdescriptions.

3. THE FIRST APPROACH — PHENOMENOLOGICAL MACROTHEORY (QUANTUM STATISTICAL THERMODYNAMICS)

In the first of our approaches, we can modify the macrodescription of objects in the heat bath by taking quantum effects into account in the framework of Einstein statistical thermodynamics with an inclusion of temperature fluctuations but without using the operator formalism. In this case, based on intuitive considerations and physical analogies, we obtain quantum statistical thermodynamics as a phenomenological theory for the macrodescription of natural objects under the conditions of equilibrium with the environment (including the case $T = 0$).

As a result, we obtain a quantum version of statistical thermodynamics. It is based on the quantum model of environment, named quantum heat bath (QHB), as a set of infinity number of weakly coupled quantum oscillators with average energy

$$\mathcal{E}_{\text{Pl}} = \frac{\hbar\omega}{2} \coth \frac{\hbar\omega}{2k_B T},$$

following from experiment.

According to this expression we can introduce a principally new characteristic — the effective temperature

$$T \to T_{\text{eff}} \equiv \frac{\mathcal{E}}{k_B} = \frac{\hbar\omega}{2k_B} \coth \frac{\hbar\omega}{2k_B T}.$$

It is important that

$$T_{\text{eff}}^{\min} = \frac{\hbar\omega}{2k_B} \neq 0.$$

Quantum thermodynamics as a phenomenological macrotheory is characterized by the next positions:

1. The generalized Zero Law:

$$T_{\text{eff}} = (T_0)_{\text{eff}} \pm \Delta T_{\text{eff}},$$

where T_{eff} is the temperature of QHB.

2. The generalized canonical distribution:

$$\rho(\mathcal{E}) = \exp \frac{F_{\text{eff}} - \mathcal{E}(T,V)}{\Theta_{qu}},$$

where, in the distinction from formula (1), the modulus of distribution $\Theta_{\text{cl}} \equiv k_B T$ is changed to $\Theta_{qu} \equiv k_B T_{\text{eff}}$.

3. In this case, we can generalize the notion of entropy and introduce the effective entropy in the form

$$S_{\text{eff}} = k_B \left\{ 1 + \ln \coth \frac{\hbar\omega}{2k_B T} \right\}.$$

So, the third law corresponds to the Nernst theorem

$$S_{\text{eff}}^{\min} = k_B \neq 0.$$

4. FUNDAMENTAL MICROTHEORY: $(\hbar, k)$-DYNAMICS

But we note that the second approach, in which we can modify the fundamental microdescription of the same objects under thermal equilibrium conditions, can also be assumed. For these purposes, we propose formulating a quantum-thermal dynamics or, briefly, $(\hbar, k)$-dynamics ($\hbar k D$), as a modification of standard quantum mechanics taking thermal effects into account. The principal distinction of such a theory from QSM is that in it the state of a microobject under the conditions of contact with the QHB is generally described not by the density matrix but by a temperature-dependent complex wave function.

We note that this is not a «technical sleight-of-hand». Using the wave function, we thereby suppose we should consider pure and mixt states simultaneously in the frame of Gibbs' ensemble. It is in principle differs from Boltzmann's assembly used in QSM.

A general idea of our investigation: to construct a theory it is necessary

1) to change $\hat{\rho}(T) \Rightarrow \Psi_T(q)$;

2) to introduce (except of Hamiltonian) a new operator — the stochastic action operator $\hat{j}$;

3) to use an idea of heat bath at $T = 0$ («cold» heat bath) also;

4) to use an idea of vacuum at $T > 0$ («thermal» vacuum) also.

This theory is based on a new microparameter, namely, the stochastic action operator. In this case, we demonstrate that averaging the corresponding microparameters over the temperature-dependent wave function, we can find the most important effective macroparameters, including internal energy, temperature, and entropy. They have the physical meaning of the standard thermodynamic quantities using in the phenomenological macrodescription.

4.1. The Model of the QHB: A Case of the «Cold» Vacuum. To describe the environment with the holistic stochastic action that was previously called the thermal field vacuum by Umezawa, we introduce a concrete model, the QHB. According to this, the QHB is a set of weakly coupled quantum oscillators with all possible frequencies. The equilibrium thermal radiation can serve as a preimage of such a model in nature.

The specific feature of our understanding of this model is that we assume that we must apply it to both the «thermal» ($T \neq 0$) and the «cold» ($T = 0$) vacua. Thus, in the sense of Einstein, we proceed from a more general understanding of the thermal equilibrium, which can, in principle, be established for any type of environmental stochastic action (purely quantum, quantum-thermal, and purely thermal).

We begin our presentation by studying the «cold» vacuum and discussing the description of a single quantum oscillator from the number of oscillators forming the QHB model for $T = 0$ from a new standpoint.

But we recall that the lowest state in the energetic ($\Psi_n(q)$) and coherent states (CS) is the same. In the occupation number representation, the «cold» vacuum in which the number of particles is $n = 0$ corresponds to this state. In the q representation, the same ground state of the quantum oscillator is in turn described by the real wave function

$$\Psi_0(q) = [2\pi(\Delta q_0)^2]^{-1/4} \exp\left\{-\frac{q^2}{4(\Delta q_0)^2}\right\}. \tag{4}$$

As is well known, CS are the eigenstates of the non-Hermitian particle annihilation operator $\hat{a}$ with complex eigenvalues. But they include one isolated state $|0_a\rangle = |\Psi_0(q)\rangle$ of the particle vacuum in which eigenvalue of $\hat{a}$ is zero

$$\hat{a}|0_a\rangle = 0|0_a\rangle, \quad \hat{a}|\Psi_0(q)\rangle = 0|\Psi_0(q)\rangle. \tag{5}$$

4.2. Some Features of the «Cold» Vacuum. In what follows, it is convenient to describe the QHB in the q representation. Therefore, we express the annihilation operator $\hat{a}$ and the creation operator $\hat{a}^\dagger$ in terms of the operators $\hat{p}$ and $\hat{q}$ using the traditional method. We have

$$\hat{a} = \frac{1}{2}\left(\frac{\hat{p}}{\sqrt{\Delta p_0^2}} - i\frac{\hat{q}}{\sqrt{\Delta q_0^2}}\right), \quad \hat{a}^\dagger = \frac{1}{2}\left(\frac{\hat{p}}{\sqrt{\Delta p_0^2}} + i\frac{\hat{q}}{\sqrt{\Delta q_0^2}}\right). \tag{6}$$

The particle number operator then becomes

$$\hat{N}_a = \hat{a}^\dagger \hat{a} = \left(\frac{\hat{p}^2}{\Delta p_0^2} - \frac{1}{2}\hat{I} + \frac{\hat{q}^2}{\Delta q_0^2} \right) = \frac{1}{\hbar\omega} \left(\frac{\hat{p}^2}{2m} - \frac{\hbar\omega}{2}\hat{I} + \frac{m\omega^2 \hat{q}^2}{2} \right). \quad (7)$$

The sum of the first and third terms in the parentheses forms the Hamiltonian $\mathcal{H}$ of the quantum oscillator, and after multiplying relation (7) by $\hbar\omega$ on the left and on the right, we obtain the standard interrelation between the expressions for the Hamiltonian in the q and n representations:

$$\hat{\mathcal{H}} = \frac{\hat{p}^{\,2}}{2m} + \frac{m\omega^2 \hat{q}^2}{2} = \hbar\omega \left(\hat{N}_a + \frac{1}{2}\hat{I} \right), \quad (8)$$

where $\hat{I}$ is the unit operator.

From the thermodynamics standpoint, we are concerned with the internal energy of the quantum oscillator in equilibrium with the «cold» QHB. Its value is equal to the mean of the Hamiltonian calculated over the state $|0_a\rangle \equiv |\Psi_0(q)\rangle$:

$$U_0 = \langle \Psi_0(q)|\hat{\mathcal{H}}|\Psi_0(q)\rangle = \hbar\omega \langle \Psi_0(q)|\hat{N}_a|\Psi_0(q)\rangle + \frac{\hbar\omega}{2} = \frac{\hbar\omega}{2} = \varepsilon_0. \quad (9)$$

It follows from formula (9) that in the given case, the state without particles coincides with the state of the Hamiltonian with the minimum energy ε_0. The quantity ε_0, traditionally treated as the energy of zero oscillations, takes the physical meaning of the internal energy U_0 of the quantum oscillator in equilibrium with the «cold» vacuum.

4.3. Passage to the «Thermal» Vacuum. We can pass from the «cold» to the «thermal» vacuum in the spirit of Umezawa using the Bogoliubov (u, v)-transformation with the temperature-dependent coefficients

$$\begin{aligned} u &= \left(\frac{1}{2} \coth \frac{\hbar\omega}{2k_B T} + \frac{1}{2} \right)^{1/2} \mathrm{e}^{i(\pi/4)}, \\ v &= \left(\frac{1}{2} \coth \frac{\hbar\omega}{2k_B T} - \frac{1}{2} \right)^{1/2} \mathrm{e}^{-i(\pi/4)}. \end{aligned} \quad (10)$$

In the given case, this transformation is canonical but leads to a unitarily nonequivalent representation because the QHB at any temperature is a system with an infinite number of freedom degrees.

In the end, such a transformation reduces to passing from the set of quantum oscillator CS to a more general set of states called the thermal correlated coherent states (TCCS). They are selected because they ensure that the Schroedinger coordinate-momentum uncertainties relation is saturated at any temperature. From the second-quantization apparatus standpoint, the Bogoliubov

(u, v)-transformation ensures the passage from the original system of particles with the «cold» vacuum $|0_a\rangle$ to the system of quasiparticles described by the annihilation operator $\hat{b}$ and the creation operator $\hat{b}^\dagger$ with the «thermal» vacuum $|0_b\rangle$.

To obtain from «cold» vacuum «thermal» one using the Bogoliubov (u, v)-transformations it is necessary to pass:

1) from CS to TCCS:

$$\Psi_0(q) \Rightarrow \Psi_T(q), \quad |0_a\rangle \Rightarrow ||0_b\rangle;$$

2) from particles to quasiparticles:

$$\hat{a} \Rightarrow \hat{b} = \hat{b}(T).$$

In this case, the choice of transformation coefficients (10) is fixed by the requirement that for any method of description, the expression for the mean energy of the quantum oscillator in thermal equilibrium be defined by the Planck formula, which can be obtained from experiments:

$$\mathcal{E}_{\text{Pl}} = \hbar\omega \left(\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1 \right)^{-1} + \frac{\hbar\omega}{2} = \frac{\hbar\omega}{2} \coth \frac{\hbar\omega}{2k_B T}. \tag{11}$$

Earlier was shown by us, the state of the «thermal» vacuum $|0_b\rangle \equiv |\Psi_T(q)\rangle$ in the q representation corresponds to the complex wave function

$$\Psi_T(q) = [2\pi(\Delta q)^2]^{-1/4} \exp\left\{ -\frac{q^2}{4(\Delta q)^2}(1 - i\alpha) \right\}, \tag{12}$$

where

$$(\Delta q)^2 = \frac{\hbar}{2m\omega} \coth \frac{\hbar\omega}{2k_B T}, \quad \alpha = \left[\sinh \frac{\hbar\omega}{2k_B T} \right]^{-1}. \tag{13}$$

For its Fourier transform $\Psi_T(p)$ a similar expression with the same coefficient α and

$$(\Delta p)^2 = \frac{\hbar m\omega}{2} \coth \frac{\hbar\omega}{2k_B T} \tag{14}$$

holds.

We note that the expressions for the probability densities $\rho_T(q)$ and $\rho_T(p)$ have already been obtained by Bloch, but the expressions for the phase that depend on the parameter α play a very significant role and were not previously known. It is also easy to see that as $T \to 0$, the parameter $\alpha \to 0$ and the function $\Psi_T(q)$ from the set of TCCS passes to the function $\Psi_0(q)$ from the set of CS.

4.4. Some Features of the «Thermal» Vacuum. Of course, the states from the set of TCCS are the eigenstates of the non-Hermitian quasiparticle annihilation operator $\hat{b}$ with complex eigenvalues. They also include one isolated state of the quasiparticle vacuum in which eigenvalue of $\hat{b}$ is zero,

$$\hat{b}|0_b\rangle = 0|0_b\rangle, \quad \hat{b}|\Psi_T(q)\rangle = 0|\Psi_T(q)\rangle. \tag{15}$$

Using condition (15) and expression (12) for the wave function of the «thermal» vacuum, we obtain the expression for the operator $\hat{b}$ in the q representation:

$$\hat{b} = \frac{1}{2}\sqrt{\coth\frac{\hbar\omega}{2k_BT}}\left[\frac{\hat{p}}{\sqrt{\Delta p_0^2}} - i\frac{\hat{q}}{\sqrt{\Delta q_0^2}}\left(\coth\frac{\hbar\omega}{2k_BT}\right)^{-1}(1 - i\alpha)\right]. \tag{16}$$

The corresponding quasiparticle creation operator has the form

$$\hat{b}^\dagger = \frac{1}{2}\sqrt{\coth\frac{\hbar\omega}{2k_BT}}\left[\frac{\hat{p}}{\sqrt{\Delta p_0^2}} + i\frac{\hat{q}}{\sqrt{\Delta q_0^2}}\left(\coth\frac{\hbar\omega}{2k_BT}\right)^{-1}(1 + i\alpha)\right]. \tag{17}$$

We can verify that as $T \to 0$, the operators $\hat{b}^\dagger$ and $\hat{b}$ for quasiparticles pass to the operators $a^\dagger$ and $\hat{a}$ for particles and

$$|0_b\rangle \Rightarrow |0_a\rangle, \quad \Psi_T(q) \Rightarrow \Psi_0(q).$$

Acting just as above, we obtain the expression for the quasiparticle number operator in the q representation

$$\hat{N}_b = \hat{b}^\dagger\hat{b} = \\ = \frac{1}{4}\coth\frac{\hbar\omega}{2k_BT}\left[\frac{\hat{p}^2}{\Delta p_0^2} - 2\left(\coth\frac{\hbar\omega}{2k_BT}\right)^{-1}\left(\hat{I} + \frac{\alpha}{\hbar}\{\hat{p},\hat{q}\}\right) + \frac{\hat{q}^2}{\Delta q_0^2}\right], \tag{18}$$

where we take $1 + \alpha^2 = \coth^2 \hbar\omega/2k_BT$ into account when calculating the last term.

4.5. Hamiltonian in TCCS. Passing from the quasiparticle number operator to the original Hamiltonian and multiplying by $\hbar\omega$, we obtain

$$\hat{\mathcal{H}} = \hbar\omega\left(\coth\frac{\hbar\omega}{2k_BT}\right)^{-1}\left[\hat{N}_b + \frac{1}{2}\left(\hat{I} + \frac{\alpha}{\hbar}\{\hat{p},\hat{q}\}\right)\right]. \tag{19}$$

We stress that the operator $\{\hat{p},\hat{q}\}$ in formula (19) can also be expressed in terms of bilinear combinations of the operators $\hat{b}^\dagger$ and $\hat{b}$, but they differ from the quasiparticle number operator N_b. This means that the operators $\hat{\mathcal{H}}$ and $\hat{N}_b$ do not commute and that the wave function of form (12) characterizing the state of the «thermal» vacuum is therefore not the eigenfunction of the Hamiltonian.

As before, we are interested in the thermodynamic quantity, namely, the internal energy U of the quantum oscillator now in thermal equilibrium with the «thermal» QHB. Calculating it just as earlier, we obtain

$$U = \hbar\omega \left(\coth \frac{\hbar\omega}{2k_BT} \right)^{-1} \times \\ \times \left[\langle \Psi_T(q) | \hat{N}_b | \Psi_T(q) \rangle + \frac{1}{2} + \frac{\alpha}{2\hbar} \langle \Psi_T(q) | \{\hat{p}, \hat{q}\} | \Psi_T(q) \rangle \right] \quad (20)$$

in the q representation. Because we average over the quasiparticle vacuum in formula (20), the first term in it vanishes. At the same time, it was shown earlier by us that

$$\langle \Psi_T(q) | \{\hat{p}, \hat{q}\} | \Psi_T(q) \rangle = \hbar\alpha. \quad (21)$$

As a result, we obtain the expression for the internal energy of the quantum oscillator in the «thermal» QHB in the $\hbar kD$:

$$U = \frac{\hbar\omega}{2(\coth \hbar\omega/2k_BT)}(1+\alpha^2) = \frac{\hbar\omega}{2} \coth \frac{\hbar\omega}{2k_BT} = \mathcal{E}_{\mathrm{Pl}}, \quad (22)$$

where $\mathcal{E}_{\mathrm{Pl}}$ is defined by Planck formula (11). This means that the average energy of the quantum oscillator at $T \neq 0$ has the thermodynamic meaning of its internal energy in the case of equilibrium with the «thermal» QHB. As $T \to 0$, it passes to a similar quantity corresponding to equilibrium with the «cold» QHB.

5. NEW FUNDAMENTAL OPERATOR — SCHROEDINGERIAN

5.1. Schroedinger Uncertainties Relation. Because the original statement of the $\hbar kD$ is the idea of the holistic stochastic action of the QHB on the object, we introduce a new operator in the Hilbert space of microstates to implement it.

We recall the general expression of Schwartz inequality

$$|A|^2 \cdot |B|^2 \geqslant |A \cdot B|^2.$$

Schroedinger uncertainties relations (SUR) coordinate-momentum following from it are:

1) Unsaturated SUR

$$(\Delta p)^2 (\Delta q)^2 > |\tilde{R}_{qp}|^2 \equiv \sigma^2 + \frac{\hbar^2}{4};$$

2) Saturated SUR

$$(\Delta p)^2(\Delta q)^2 = |\tilde{R}_{qp}|^2. \tag{23}$$

For real states $\sigma = 0$.

In the absence of stochastic action $\tilde{R}_{qp} \equiv 0$. As leading consideration, hereinafter we use an analysis of the right-hand side of the saturated SUR coordinate-momentum.

5.2. The Stochastic Action Operator (Schroedingerian). For not only a quantum oscillator in a QHB but also any object, the complex quantity in the right-hand side of (23)

$$\widetilde{R}_{pq} = \langle \Delta p | \Delta q \rangle \quad \text{or} \quad \widetilde{R}_{pq} = \langle \, | \Delta \widehat{p} \, \Delta \widehat{q} \, | \, \rangle \tag{24}$$

has a double meaning. On the one hand, it is the amplitude of the transition from the state $|\Delta q\rangle$ to the state $|\Delta p\rangle$; on the other hand, it can be treated as the Schroedinger quantum correlator calculated over an arbitrary state $| \ \rangle$ of some operator.

As is well known, the nonzero value of quantity (24) is the fundamental attribute of nonclassical theory in which the environmental stochastic action on an object plays a significant role. Therefore, it is quite natural to assume that the averaged operator in formula (24) has a fundamental meaning. In view of dimensional considerations, we call it the stochastic action operator or Schroedingerian,

$$\widehat{j} \equiv \Delta \widehat{p} \, \Delta \widehat{q}. \tag{25}$$

Of course, it should be remembered that the operators $\Delta \hat{q}$ and $\Delta \hat{p}$ do not commute and their product is a non-Hermitian operator.

To analyze further, following Schroedinger, we can express the given operator in the form

$$\widehat{j} = \frac{1}{2} \langle | \Delta \widehat{p} \Delta \widehat{q} + \Delta \widehat{q} \Delta \widehat{p} \, | \rangle + \frac{1}{2} \langle | \Delta \widehat{p} \Delta \widehat{q} - \Delta \widehat{q} \Delta \widehat{p} \, | \rangle = \widehat{\sigma} - i \, \widehat{j}_0. \tag{26}$$

It allows separating the Hermitian part of $\hat{j}$ from the anti-Hermitian one. Then the Hermitian operators $\widehat{\sigma}$ and $\hat{j}_0$ have the form

$$\widehat{\sigma} \equiv \frac{1}{2} \{ \Delta \hat{p}, \ \Delta \hat{q} \}, \quad \hat{j}_0 \equiv \frac{i}{2} \, [\hat{p}, \hat{q} \,] = \frac{\hbar}{2} \, \hat{I}. \tag{27}$$

It is easy to see that the mean $\sigma = \langle \, | \widehat{\sigma} | \, \rangle$ of the operator $\hat{\sigma}$ resembles the expression for the standard correlator of coordinate and momentum fluctuations in classical probability theory. It transforms into this expression if the operators $\Delta \hat{q}$ and $\Delta \hat{p}$ are replaced with c-numbers. It reflects the contribution to the transition amplitude $\tilde{R}_{pq}$ of the environmental stochastic action. Therefore, we call the operator $\hat{\sigma}$ the external action operator in what follows. Previously, the possibility of using a similar operator was discussed by Bogoliubov and Krylov (1939), where it was studied as a quantum analogue of the classical action variable in the set of action-angle variables.

At the same time, the operators $\hat{j}_0$ and $\hat{j}$ were not previously introduced. The operator $\hat{j}_0$ of form (27) reflects a specific peculiarity of the objects to be «sensitive» to the minimum stochastic action of the «cold» vacuum and to respond to it adequately regardless of their states. Therefore, it should be treated as a minimum stochastic action operator. Its mean $J_0 = \hbar/2$ is independent of the choice of the state over which the averaging is performed, and hence it has the meaning of the invariant eigenvalue of the operator $\hat{j}_0$.

5.3. Correspondence between the Constants $\hbar$ and $(1/2)\hbar$. This implies that in the given case, we deal with the universal quantity J_0, which we call the minimal stochastic action. Its fundamental character is already defined by its relation to the Planck world constant $\hbar$. But the problem is not settled yet. Indeed, according to the tradition dating back to Planck, the quantity $\hbar$ is assumed to be called the elementary quantum of the action. At the same time, the factor 1/2 in the quantity J_0 plays a significant role, while half the quantum of the action is not observed in nature. Therefore, the quantities $\hbar$ and $(1/2)\hbar$, whose dimensions coincide, have different physical meanings, and hence must, in our opinion, be named differently. From this standpoint, it would be more natural to call the quantity $\hbar$ the external action quantum.

Hence, the quantity $\hbar$ is this minimum portion of the action transferred to the object from the environment or from another object. Therefore, photons and other quanta of fields being carriers of fundamental interactions are first the carriers of the minimal action equal to $\hbar$. The same is also certainly related to phonons.

Finally, we note that only the quantity $\hbar$ is related to the discreteness of the spectrum of the quantum oscillator energy in the absence of the heat bath. At the same time, the quantity $\hbar/2$ has an independent physical meaning. On the base of formula (9) it specifies the minimum value of the macroparameter — the internal energy U_0 of the quantum oscillator in the «cold» QHB (at $T = 0$).

We evaluate the specific features of the stochastic action operator used in the microdescription below. We recall that this operator is non-Hermitian. This would seemingly contradict the standard requirements imposed on the operators in quantum mechanics, but there is nothing unusual in this. Certainly, the presence of eigenstates and real eigenvalues that are assumed to be compared with observable quantities is characteristic for Hermitian operators. But from the physical standpoint, such eigenvalues are not so interesting, because they characterize something invariable, such as stationary states, for example.

It is quite another matter if we are interested in genuine quantum dynamics, which is naturally associated with transitions from one state to another. In this case, precisely the non-Hermitian operators play an important role. The creation and annihilation operators or, for example, the scattering matrix are among the most well-known of them. The Schroedingerian, or stochastic action operator, also belongs with these operators.

6. EFFECTIVE ACTION AS A FUNDAMENTAL GENERATIVE MACROPARAMETER

6.1. The Mean of the Operator $\widehat{j}$. We now construct the macrodescription of objects using their microdescription in the $\hbar kD$. It is easy to see that the mean $\tilde{J}$ of the operator $\hat{j}$ of form (26) coincides with the complex transition amplitude $\widetilde{R}_{pq}$ or Schroedinger's correlator and, in thermal equilibrium, can be expressed as

$$\tilde{J} = \langle \Psi_T(q) | \, \widehat{j} \, | \Psi_T(q) \rangle = \sigma - iJ_0, \tag{28}$$

where σ and J_0 are the means of the corresponding operators. In what follows, we regard the modulus of the complex quantity $\widetilde{J}$,

$$|\tilde{J}| = \sqrt{\sigma^2 + J_0^2} = \sqrt{\sigma^2 + \frac{\hbar^2}{4}} \equiv J_{\text{eff}} \tag{29}$$

as a new macroparameter and call it the effective action. It has the form

$$J_{\text{eff}} = \frac{\hbar}{2} \coth \frac{\hbar\omega}{2k_B T} \tag{30}$$

for the quantum oscillator and coincides with a similar quantity previously postulated from intuitive considerations.

6.2. Internal Energy and Effective Temperature. We now establish the interrelation between the effective action and traditional thermodynamic quantities. Comparing expression (30) for J_{eff} with (22) for the internal energy U, we can easily see that

$$U = \omega |\tilde{J}| = \omega J_{\text{eff}} \tag{31}$$

for the quantum oscillator. In the high-temperature limit, where

$$\sigma \to J_T = \frac{k_B T}{\omega} \gg \frac{\hbar}{2}, \tag{32}$$

relation (31) becomes

$$U = \omega J_T. \tag{33}$$

Boltzmann previously obtained this formula for macroparameters in classical thermodynamics by generalizing the concept of adiabatic invariants used in classical mechanics.

Relation (31) also allows expressing the interrelation between the effective action and the effective temperature $T_{\rm eff}$ in explicit form:

$$T_{\rm eff} = \frac{\omega}{k_B} J_{\rm eff}. \tag{34}$$

This implies that

$$T_{\rm eff}^{\min} = \frac{\omega}{k_B} J_0 = \frac{\hbar\omega}{2k_B} \neq 0. \tag{35}$$

Finally, we note that using formulas (31), (14), and (13), we can rewrite the Schroedinger uncertainties relation for the quantum oscillator for $T \neq 0$ as an expression similar to expression (2) for the case $T = 0$:

$$\Delta p \cdot \Delta q = |\tilde{J}| = \frac{U}{\omega}. \tag{36}$$

6.3. A Geometrical Sense of the Effective Action. To stress the role of the macroparameter $J_{\rm eff}$, we treat formula (29) geometrically. We recall that we pass from the ground state of the quantum oscillator belonging to the set of CS to the state of the «thermal» QHB belonging to the set of TCCS using the Bogoliubov (u, v)-transformation. It forms the Lie group in the states space locally isomorphic to the Lorenz group in the two-dimensional world of events. This means that for TCCS, we can regard the set $(J_{\rm eff}, \sigma)$ as a two-dimensional time-like vector in the pseudo-Euclidean states space and the quantity $J_0 = \hbar/2$ as the length of this vector or an invariant of the corresponding group:

$$J_{\rm eff}^2 - \sigma^2 = J_0^2 = \frac{\hbar^2}{4} = \text{inv}. \tag{37}$$

The role of the traditional Lorenz multipliers β and γ in the given case is played by the quantities

$$\beta_T = \left[\cosh \frac{\hbar\omega}{2k_B T}\right]^{-1}, \quad \gamma_T = \coth \frac{\hbar\omega}{2k_B T}. \tag{38}$$

It is easy to see that the vector of the effective action $(J_{\rm eff}, \sigma)$ is an analogue of the two-dimensional momentum-energy vector $(\mathcal{E}, pc)$ in relativistic mechanics:

$$\varepsilon^2 - p^2c^2 = \varepsilon_0^2 = m^2c^4. \tag{39}$$

In particular, for the quantum oscillator in the TCCS, the quantity $J_{\rm eff}$ is an analogue of $\mathcal{E}$ and σ is an analogue of pc. If interrelation (31) between the internal energy of the quantum oscillator and the effective action is taken into account, then expression (37) becomes

$$U^2 - \sigma^2\omega^2 = U_0^2 = \left(\frac{\hbar\omega}{2}\right)^2. \tag{40}$$

The physical meaning of formula (40) and that of (37) are completely similar to that of relation (39) characterizing the so-called mass shell in the two-dimensional space of events.

6.4. Effective Entropy in the $\hbar kD$. The possibility of introducing entropy in the $\hbar kD$ is also based on using the wave function instead of the density operator. Using the dimensionless expressions for $\rho(q) = |\Psi(q)|^2$ and $\rho(p) = |\Psi(p)|^2$, we propose defining a formal coordinate — momentum entropy S_{qp} by the equality

$$S_{qp} = -k_B \left\{ \int \tilde{\rho}(\tilde{q}) \ln \tilde{\rho}(\tilde{q}) d\tilde{q} + \int \tilde{\rho}(\tilde{p}) \ln \tilde{\rho}(\tilde{p}) d\tilde{p} \right\}. \quad (41)$$

Substituting the corresponding expressions for $\tilde{\rho}(\tilde{q})$ and $\tilde{\rho}(\tilde{p})$ in (41), we obtain

$$S_{qp} = k_B \left\{ \left(1 + \ln \frac{2\pi}{\delta}\right) + \ln \coth \frac{\hbar\omega}{2k_B T} \right\}. \quad (42)$$

Obviously, the final result depends on the choice of the constant δ.

Choosing $\delta = 2\pi$, we can interpret expression (42) as the quantum-thermal entropy or, briefly, the QT-entropy $S_{\rm QT}$ because it coincides exactly with the effective entropy $S_{\rm eff}$ obtained earlier by us in the macrotheory framework

$$S_{\rm QT} \equiv S_{\rm eff} = k_B \left\{ 1 + \ln \frac{J_{\rm eff}}{J_0} \right\} = k_B \left\{ 1 + \ln \Omega \right\}, \quad (43)$$

where Ω according to Boltzmann is a number of microstates in the given macrostate. This ensures the consistency between the main results of our proposed micro- and macrodescriptions and their correspondence to experiments.

We can approach the modification of formal expression S_{qp} in another way. Combining both terms in it, we can represent it in the form

$$-k_B \int dp\, dq \{\rho(p)\rho(q)\} \ln\{\rho(p)\rho(q)\}. \quad (44)$$

It is easy to see that the expression in braces is the Wigner function for the quantum oscillator in a heat bath:

$$W(p,q) = \rho(p)\rho(q) = \{2\pi(\Delta q)^2(\Delta p)^2\}^{-2} \exp\left\{ -\frac{p^2}{2(\Delta p)^2} - \frac{q^2}{2(\Delta q)^2} \right\}. \quad (45)$$

The change of variables in the phase space and the passage to the dimensionless Wigner function with its normalization taken into account allow rewriting expression (44) in the form of the QT-entropy (43):

$$S_{\rm QT} = -k_B \int d\tilde{\mathcal{E}}\, \widetilde{W}(\tilde{\mathcal{E}}) \ln \widetilde{W}(\tilde{\mathcal{E}}), \quad (46)$$

where

$$\widetilde{W} = \frac{\hbar\omega/2}{k_B T_{\rm eff}} \exp\left\{ -\frac{\mathcal{E}}{k_B T_{\rm eff}} \right\}. \quad (47)$$

7. QUANTUM STATISTICAL THERMODYNAMICS ON THE BASE OF THE EFFECTIVE ACTION

The above presentation shows that using the $\hbar kD$ developed here, we can introduce the effective action J_{eff} as a new fundamental macroparameter. The advantage of this quantity is that it has a microscopic preimage, namely, the stochastic action operator $\hat{j}$, or Schroedingerian, which has an obvious physical sense. Moreover, we can in principle express the main thermodynamic characteristics of objects in thermal equilibrium in terms of it. As is well known, temperature and entropy are the most fundamental of them.

If the notion of effective action is used, these heuristic considerations can acquire an obvious meaning. For this, we turn to expression (34) for $T_{\text{eff}} \sim J_{\text{eff}}$. It follows from it that the effective action is also an *intensive* macroparameter characterizing the stochastic action of the «thermal» QHB.

In view of this, the Zero Law of equilibrium quantum Statistical Thermodynamics can be rewritten as

$$J_{\text{eff}} = J_{\text{eff}}^{\text{therm}} \pm \Delta J_{\text{eff}}, \tag{48}$$

where $J_{\text{eff}}^{\text{therm}}$ is effective action of QHB; J_{eff} and ΔJ_{eff} are the means of the effective action of an object and the standard deviation from it. The state of thermal equilibrium can actually be described in the sense of Newton, assuming that «the stochastic action is equal to the stochastic counteraction» in such cases.

We now turn to the effective entropy. In the absence of a mechanical contact, its differential is

$$dS_{\text{eff}} = \frac{\delta Q_{\text{eff}}}{T_{\text{eff}}} = \frac{dU}{T_{\text{eff}}}. \tag{49}$$

Substituting the expressions for internal energy (31) and effective temperature (34) in this relation, we obtain, as was waited,

$$dS_{\text{eff}} = k_B \cdot d\left(\ln \frac{J_{\text{eff}}}{J_0}\right), \tag{50}$$

that corresponds with formula (43). So, the effective or QT-entropy, being an *extensive* macroparameter, can also be expressed in terms of J_{eff}.

As a result, it turns out that two qualitatively different characteristics of thermal phenomena on the macrolevel, namely, the effective temperature and effective entropy, embody the presence of two sides of the process of stochastizing the characteristics of an object in nature in view of the contact with the QHB. At any temperature, they can be expressed in terms of the only macroparameter, namely, the effective action. This macroparameter has the stochastic action operator, or Schroedingerian, simultaneously dependent on the Planck and Boltzmann constants as a microscopic preimage in the $\hbar kD$.

8. CONNECTION WITH EXPERIMENT

Nowadays there are a number of papers on quantum gravitation and quantum field theory where the ratio of shift viscosity to entropy volume density is the subject of interest. It was shown by us that this quantity can be given by

$$\frac{J^{\text{eff}}}{S^{\text{eff}}} = \frac{J^{\text{eff}}_{\min}}{S^{\text{eff}}_{\min}} \frac{\coth T^{\text{eff}}_{\min}/T}{1 + \ln\coth T^{\text{eff}}_{\min}/T} = \varkappa \frac{\coth \varkappa\omega/T}{1 + \ln\coth \varkappa\omega/T} \to \varkappa. \tag{51}$$

In this expression,

$$\varkappa \equiv \frac{J^{\text{eff}}_{\min}}{S^{\text{eff}}_{\min}} = \frac{\hbar}{2k_B} \tag{52}$$

is the limiting ratio for $T \ll T^{\text{eff}}$.

In our opinion, the quantity

$$\varkappa = 3.82 \cdot 10^{-12}\ \text{K} \cdot \text{c} \tag{53}$$

is not only the notation for one of the possible combinations of the world constants $\hbar$ and k_B. It also has its intrinsic physical meaning. It is contained in definition of the effective temperature

$$T^{\text{eff}} = \varkappa\omega \coth \frac{\varkappa\omega}{T}, \tag{54}$$

and also in the displacement law for equilibrium thermal radiation

$$\frac{T}{\omega_{\max}} = 0.7\varkappa.$$

We are sure that the quantity $\varkappa$ plays the role of a constant essentially characterizing the holistic stochastic action on the object.

The analogical with (51) relation in QSM, in contrast to the one in $\hbar kD$, has the form

$$\frac{J}{S} \to \frac{\hbar \exp(-\hbar\omega/k_BT)}{k_B(\hbar\omega/k_BT)\exp(-\hbar\omega/k_BT)} = \frac{T}{\omega} \to 0. \tag{55}$$

Therefore, it is now possible to compare two theories ($\hbar kD$ and QSM) experimentally by measuring the limiting value of this ratio: whether is equal to $\varkappa$ or zero.

The first indication that the quantity $\varkappa$ plays an important role was obtained in Andronikashvili's experiments (1948) on the viscosity of liquid helium below the λ-point. There is also another area of Physics where the constant $\varkappa$ appears. This is experiments with quark–gluon plasmas (RHIC accelerator, Brookhaven, 2005), where it was obtained that

$$\frac{J_{\min}}{S_{\min}} \neq 0. \tag{56}$$

Now the constant $\varkappa$ is also observed in cold atomic gases and different solids as a characteristic of the fundamentally new state of matter — nearly perfect fluid.

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