ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
2012. Т. 43. ВЫП. 5

STRINGY DIFFERENTIAL GEOMETRY FOR DOUBLE FIELD THEORY, BEYOND RIEMANN*

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While the fundamental object in Riemannian geometry is a metric, closed string theories call for us to put a two-form gauge field and a scalar dilaton on an equal footing with the metric. Here we propose a novel differential geometry which treats the three objects in a unified manner, manifests not only diffeomorphism and one-form gauge symmetry but also $\mathbf{O}(D,D)$ T-duality, and enables us to rewrite the known low-energy effective action of them as a single term. We comment that the notion of cosmological constant naturally changes.

PACS: 02.40.-k

INTRODUCTION

Symmetry guides the structure of Lagrangians and organizes the physical laws into simple forms. For example, in Maxwell theory, the Abelian gauge symmetry does not allow for an explicit mass term of the vector potential, and Lorentz symmetry unifies the original Maxwell's four equations into two.

In general relativity, where the key quantity is the space-time metric, the diffeomorphism symmetry first demands replacing ordinary derivatives by covariant derivatives which involve a connection. Setting the metric to be covariant constant determines the (torsionless) connection, i.e., the Christoffel symbol, in terms of the metric and its derivatives, and hence diffeomorphism uniquely picks up the scalar curvature as the covariant term which is lowest order in derivatives of the metric.

On the other hand, in string theories, the metric, $g_{\mu\nu}$, accompanies a Kalb–Ramond two-form gauge field, $B_{\mu\nu}$, and a scalar dilation, ϕ, since the three of them complete the bosonic massless sector of a closed string. Their low-energy effective action is of the well-known form:

$$S_{\rm eff} = \int dx^D \sqrt{-g}\, {\rm e}^{-2\phi} \left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12} H_{\lambda\mu\nu}H^{\lambda\mu\nu} \right), \qquad (1)$$

*This proceeding article is based on the author's collaborative works with Imtak Jeon and Kanghoon Lee [1–5].

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where R is the scalar curvature of the metric and $H_{\lambda\mu\nu}$ is the three-form field strength of the two-form gauge field. Here and henceforth we consider an arbitrary space-time dimension, D, without restricting ourselves to the critical values, 10 or 26. Each term in (1) is clearly invariant under the diffeomorphism as well as the one-form gauge symmetry,

$$x^\mu \to x^\mu + \delta x^\mu, \qquad B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu. \tag{2}$$

Moreover, though not manifest, the action enjoys T-duality which mixes the three companions, $g_{\mu\nu}, B_{\mu\nu}, \phi$, in a nontrivial manner, first noted by Buscher [6–8] and further studied in [9–13]: If we redefine the dilaton, $\phi \to d$, and set a $2D \times 2D$ symmetric matrix, $\mathcal{H}_{AB}$ from $g_{\mu\nu}$, $B_{\mu\nu}$ [14], as

$$\mathrm{e}^{-2d} = \sqrt{-g}\,\mathrm{e}^{-2\phi}, \qquad \mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \tag{3}$$

T-duality is conveniently realized by an $\mathbf{O}(D,D)$ rotation which acts on the $2D$-dimensional vector indices, $A, B, \dots$, in a standard manner, while d is taken to be an $\mathbf{O}(D,D)$ singlet. The $\mathbf{O}(D,D)$ group is defined by the invariance of the constant metric of the following form:

$$\mathcal{J}_{AB} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

Throughout the present paper, this metric is being used to freely raise or lower the $2D$-dimensional vector indices. Indeed, Hull and Zwiebach [15,16], later with Hohm [17,18], managed to rewrite the effective action (1) in terms of the redefined dilaton, d, the $2D \times 2D$ matrix, $\mathcal{H}_{AB}$, and their ordinary derivatives, such that the $\mathbf{O}(D,D)$ T-duality structure became manifest, yet the diffeomorphism and the one-form gauge symmetry were not any more. In their approach, the space-time dimension is formally doubled from D to $2D$, with coordinates, $x^\mu \to y^A = (\tilde{x}_\mu, x^\nu)$. The new coordinates, $\tilde{x}_\mu$, may be viewed as the canonical conjugates of the winding modes of closed strings. However, as a field theory counterpart to the level matching condition in closed string theories, it is required that all the fields, as well as all of their possible products, should be annihilated by the $\mathbf{O}(D,D)$ d'Alembert operator, $\partial^2 = \partial_A \partial^A$,

$$\partial^2 \Phi \equiv 0, \qquad \partial_A \Phi_1 \partial^A \Phi_2 \equiv 0. \tag{5}$$

This «level matching constraint» — which we also assume in this paper — actually means that the theory is not truly doubled: there is a choice of coordinates $(\tilde{x}', x')$, related to the original coordinates $(\tilde{x}, x)$, by an $\mathbf{O}(D,D)$ rotation, in which all the fields do not depend on the $\tilde{x}'$ coordinates [17]. Henceforth, the equivalence symbol, «$\equiv$», means an equality up to the constraint (5).

Combining the two types of the parameters,

$$X^A = (\Lambda_\mu, \delta x^\nu),$$

the diffeomorphism and the one-form gauge transformations (2) can be expressed in a unified fashion,

$$\begin{aligned}\delta_X\mathcal{H}_{AB} \equiv X^C\partial_C\mathcal{H}_{AB} + 2\partial_{[A}X_{C]}\mathcal{H}^C{}_B + 2\partial_{[B}X_{C]}\mathcal{H}_A{}^C,\\ \delta_X\left(\mathrm{e}^{-2d}\right) \equiv \partial_A\left(X^A\mathrm{e}^{-2d}\right).\end{aligned} \tag{6}$$

These expressions can be identified as the generalized Lie derivatives whose commutator leads to the Courant bracket [13, 19, 20]. In fact, in our previous work [1], starting from the observation that $\mathcal{H}_{AB}$ given in (3) assumes a generic form of a symmetric $\mathbf{O}(D,D)$ element*, we constructed a certain differential operator which can be made compatible with the gauge transformations (6), being characterized by a projection:

$$P_{AB} = P_{BA} = \frac{1}{2}(\mathcal{J}+\mathcal{H})_{AB}, \qquad P_A{}^BP_B{}^C = P_A{}^C. \tag{7}$$

In this work, generalizing the results of [1] (and also [21] by Hassan in «undoubled» space-time), we propose a novel differential geometry apt for the unifying description of the closed string massless sector, which manifests all the relevant structures simultaneously:

- $\mathbf{O}(D,D)$ *T-duality*.
- *Gauge symmetry*:
  1. *Double-gauge symmetry*:
     — *Diffeomorphism*,
     — *One-form gauge symmetry*.
  2. *Local Lorentz symmetry*.

In particular, we reformulate the effective action (1) into a single term, like

$$S_{\text{eff}} \equiv \int dy^{2D}\,\mathrm{e}^{-2d}\,\mathcal{H}^{AB}S_{AB}. \tag{8}$$

1. SEMI-COVARIANT DERIVATIVE

Employing the main idea of [1], we start with a differential operator, $\nabla_C = \partial_C + \Gamma_C$, which acts on a generic quantity carrying $\mathbf{O}(D,D)$ vector indices,

$$\nabla_CT_{A_1A_2\cdots A_n} := \partial_CT_{A_1A_2\cdots A_n} - \omega\Gamma^B{}_{BC}T_{A_1A_2\cdots A_n} + \sum_{i=1}^{n}\Gamma_{CA_i}{}^BT_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}, \tag{9}$$

*The expression of $\mathcal{H}_{AB}$ in (3) is the most general form of a $2D{\times}2D$ matrix, satisfying $\mathcal{H}_{AB}=\mathcal{H}_{BA}$, $\mathcal{H}_A{}^B\mathcal{H}_B{}^C=\delta_A{}^C$, and that the upper left $D{\times}D$ block is nondegenerate [1].

where ω denotes the given weight of each field, $T_{A_1 A_2 \cdots A_n}$, and the connection must satisfy,

$$\Gamma_{CAB} + \Gamma_{CBA} = 0, \qquad \Gamma_{ABC} + \Gamma_{CAB} + \Gamma_{BCA} = 0. \tag{10}$$

The only quantity which has a nontrivial weight in this paper is e^{-2d} having $\omega = 1$. Thanks to the symmetric properties (10), the ordinary derivatives in the definition of the generalized Lie derivative [13, 19, 20] can be replaced with our differential operator to give

$$\begin{aligned}\hat{\mathcal{L}}_X T_{A_1 \cdots A_n} := X^B \nabla_B T_{A_1 \cdots A_n} + \omega \nabla_B X^B T_{A_1 \cdots A_n} + \\ + \sum_{i=1}^{n} 2\nabla_{[A_i} X_{B]} T_{A_1 \cdots A_{i-1}}{}^{B}{}_{A_{i+1} \cdots A_n},\end{aligned} \tag{11}$$

since the connection terms cancel.

We fix the connection by requiring

$$\nabla_A P_{BC} = 0, \quad \nabla_A \bar{P}_{BC} = 0, \quad \nabla_A d := \partial_A d + \frac{1}{2}\Gamma^B{}_{BA} = 0, \tag{12}$$

where $\bar{P}_{AB} = (\mathcal{J} - P)_{AB}$ corresponds to the «antichiral» projection which is complementary to the «chiral» projection, P_{AB} in (7). Further, $\nabla_A d$ is defined by the relation,

$$\nabla_A(\mathrm{e}^{-2d}) = -2(\nabla_A d)\,\mathrm{e}^{-2d}. \tag{13}$$

It follows that

$$\nabla_A \mathcal{J}_{BC} = 0, \qquad \nabla_A \mathcal{H}_{BC} = 0. \tag{14}$$

That is to say, our differential operator thoroughly annihilates the closed string massless sector represented by d and $\mathcal{H}_{AB}$, which indicates that we are on a right track to achieve a unifying description of the massless modes.

In terms of P, $\bar{P}$, d and their derivatives, the connection reads explicitly (cf. [1]),

$$\begin{aligned}\Gamma_{CAB} = 2\big(P\partial_C P\bar{P}\big)_{[AB]} + 2\big(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\big)\partial_D P_{EC} - \\ - \frac{4}{D-1}\big(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D}\big)\big(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\big).\end{aligned} \tag{15}$$

Furthermore, if we set

$$\begin{aligned}\mathcal{P}_{CAB}{}^{DEF} &:= P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \frac{2}{D-1} P_{C[A} P_{B]}{}^{[E} P^{F]D}, \\ \bar{\mathcal{P}}_{CAB}{}^{DEF} &:= \bar{P}_C{}^D \bar{P}_{[A}{}^{[E} \bar{P}_{B]}{}^{F]} + \frac{2}{D-1} \bar{P}_{C[A} \bar{P}_{B]}{}^{[E} \bar{P}^{F]D},\end{aligned} \tag{16}$$

which satisfy

$$\begin{gathered}\mathcal{P}_{CABDEF}=\mathcal{P}_{DEFCAB}=\mathcal{P}_{C[AB]D[EF]},\\ \mathcal{P}_{CAB}{}^{DEF}\mathcal{P}_{DEF}{}^{GHI}=\mathcal{P}_{CAB}{}^{GHI},\\ \mathcal{P}^{A}{}_{ABDEF}=0,\quad P^{AB}\mathcal{P}_{ABCDEF}=0,\quad \text{etc.},\end{gathered}\tag{17}$$

the connection (15) belongs to the kernel of these rank six-projectors,

$$\mathcal{P}_{CAB}{}^{DEF}\Gamma_{DEF}=0,\qquad \bar{\mathcal{P}}_{CAB}{}^{DEF}\Gamma_{DEF}=0.\tag{18}$$

Under the double-gauge transformations (6), the connection and the derivative (9) transform as

$$\begin{gathered}(\delta_X-\hat{\mathcal{L}}_X)\Gamma_{CAB}\equiv 2\big[(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE}-\delta_C^F\delta_A^D\delta_B^E\big]\partial_F\partial_{[D}X_{E]},\\ (\delta_X-\hat{\mathcal{L}}_X)\nabla_C T_{A_1\cdots A_n}\equiv\sum_i 2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BFDE}\partial_F\partial_{[D}X_{E]}T_{\cdots B\cdots}.\end{gathered}\tag{19}$$

Hence, they are not double-gauge covariant. We say, a tensor is double-gauge covariant if and only if its double-gauge transformation agrees with the generalized Lie derivative. Nonetheless, the characteristic property of our derivative, ∇_A, is that, combined with the projections, it can generate various $\mathbf{O}(D,D)$ and double-gauge covariant quantities, as follows:

$$\begin{gathered}P_C{}^D\bar{P}_{A_1}{}^{B_1}\bar{P}_{A_2}{}^{B_2}\cdots\bar{P}_{A_n}{}^{B_n}\nabla_D T_{B_1B_2\cdots B_n},\\ \bar{P}_C{}^D P_{A_1}{}^{B_1}P_{A_2}{}^{B_2}\cdots P_{A_n}{}^{B_n}\nabla_D T_{B_1B_2\cdots B_n},\\ P^{AB}\bar{P}_{C_1}{}^{D_1}\bar{P}_{C_2}{}^{D_2}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_A T_{BD_1D_2\cdots D_n},\\ \bar{P}^{AB}P_{C_1}{}^{D_1}P_{C_2}{}^{D_2}\cdots P_{C_n}{}^{D_n}\nabla_A T_{BD_1D_2\cdots D_n},\\ P^{AB}\bar{P}_{C_1}{}^{D_1}\bar{P}_{C_2}{}^{D_2}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_A\nabla_B T_{D_1D_2\cdots D_n},\\ \bar{P}^{AB}P_{C_1}{}^{D_1}P_{C_2}{}^{D_2}\cdots P_{C_n}{}^{D_n}\nabla_A\nabla_B T_{D_1D_2\cdots D_n}.\end{gathered}\tag{20}$$

Here, the latter second order derivatives actually follow from the recurrent applications of the former first order derivatives. The index n can be trivial, such that the covariant quantities include $P^{AB}\nabla_A T_B$ and $\bar{P}^{AB}\nabla_A T_B$.

The above result suggests us to call the differential operator, ∇_A, a «semi-covariant» derivative.

2. CURVATURES

Straightforward computation can show that the usual curvature,

$$\mathcal{R}_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^{E} \Gamma_{BED} - \Gamma_{BC}{}^{E} \Gamma_{AED}, \tag{21}$$

set by the connection (15), is not double-gauge covariant, yet it satisfies

$$\mathcal{R}_{CDAB} = \mathcal{R}_{[CD][AB]}, \qquad P_C{}^I \bar{P}_D{}^J \mathcal{R}_{IJAB} = 0. \tag{22}$$

We define, as for a key quantity in our formalism,

$$S_{ABCD} := \frac{1}{2} \left(\mathcal{R}_{ABCD} + \mathcal{R}_{CDAB} - \Gamma^{E}{}_{AB} \Gamma_{ECD} \right), \tag{23}$$

which can be shown, by brute force computation, to meet

$$\begin{gathered} S_{ABCD} = S_{\{ABCD\}} := \frac{1}{2}(S_{[AB][CD]} + S_{[CD][AB]}), \quad S_{A[BCD]} = 0, \\ P_I{}^A P_J{}^B \bar{P}_K{}^C \bar{P}_L{}^D S_{ABCD} \equiv 0, \\ P_I{}^A \bar{P}_J{}^B P_K{}^C \bar{P}_L{}^D S_{ABCD} \equiv 0, \\ P_I{}^A \bar{P}_J{}^C \mathcal{H}^{BD} S_{ABCD} \equiv 0, \end{gathered} \tag{24}$$

and have a connection to a commutator,

$$P_I{}^A \bar{P}_J{}^B [\nabla_A, \nabla_B] T_C \equiv 2 P_I{}^A \bar{P}_J{}^B S_{CDAB} T^D. \tag{25}$$

Under the double-gauge transformations (6), we get

$$(\delta_X - \hat{\mathcal{L}}_X) S_{ABCD} \equiv 4 \nabla_{\{A} \left[(\mathcal{P} + \bar{\mathcal{P}})_{BCD\}}{}^{EFG} \partial_E \partial_{[F} X_{G]} \right], \tag{26}$$

from which double-gauge and $\mathbf{O}(D, D)$ T-duality covariant, rank two-tensor as well as scalar follow,

$$P_I{}^A \bar{P}_J{}^B S_{AB}, \qquad \mathcal{H}^{AB} S_{AB}. \tag{27}$$

Here we set

$$S_{AB} = S_{BA} := S^{C}{}_{ACB}, \tag{28}$$

which turns out to be, from direct computation, traceless,

$$S^{A}{}_{A} \equiv 0. \tag{29}$$

Especially, the covariant scalar constitutes the effective action (8) as

$$\mathcal{H}^{AB} S_{AB} \equiv R + 4\Box\phi - 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu}, \tag{30}$$

and this is consistent with [1, 18].

Under arbitrary infinitesimal transformations of the dilaton and the projection (of which the latter should obey, from (7), $\delta P = P\delta P\bar{P} + \bar{P}\delta P P$), we get

$$\delta S_{ABCD} = \nabla_{[A}\delta\Gamma_{B]CD} + \nabla_{[C}\delta\Gamma_{D]AB}, \tag{31}$$

where explicitly

$$\begin{aligned}\delta\Gamma_{CAB} = 2P_{[A}{}^{D}\bar{P}_{B]}{}^{E}\nabla_C\delta P_{DE} + 2(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E})\nabla_D\delta P_{EC} - \\ - \frac{4}{D-1}(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D})(\partial_D\delta d + P_{E[G}\nabla^{G}\delta P^{E}{}_{D]}) - \\ - \Gamma_{FDE}\,\delta(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE}.\end{aligned}$$

Now, with (31) and $\nabla_A d = 0$, from the manipulation,

$$\delta S_{\text{eff}} \equiv \int dy^{2D}\, 2\,\mathrm{e}^{-2d}\,\left(\delta P^{AB}S_{AB} - \delta d\,\mathcal{H}^{AB}S_{AB}\right),$$

it is very easy to rederive the equations of motion [18, 22]:

$$P_{(I}{}^{A}\bar{P}_{J)}{}^{B}S_{AB} = 0, \qquad \mathcal{H}^{AB}S_{AB} = 0. \tag{32}$$

3. COMMENTS

In the stringy differential geometry we have proposed, the dilaton, d, appears only explicitly as the overall factor of the action, and its derivatives are completely absorbed into the connection (15), which therefore implies the tight symmetric structure of our formalism. Furthermore, it appears that the natural «cosmological constant term» is nothing but

$$\int dy^{2D}\,\mathrm{e}^{-2d}\Lambda \equiv \int dx^{D}\,\sqrt{-g}\,\mathrm{e}^{-2\phi}\Lambda. \tag{33}$$

As ϕ dynamically grows, this term becomes exponentially suppressed, irrespective of the choice of the frame, i.e., string or Einstein. In this way, the notion of the cosmological constant naturally changes in our stringy differential geometry. This may provide a new spin on the cosmological constant problem.

It has been said that string theory is a piece of 21st century physics that happened to fall into the 20th century. Perhaps, our formalism might provide some clue to a new framework beyond Riemannian geometry.

Acknowledgements. I wish to thank the organizers of the International Workshop «Supersymmetries and Quantum Symmetries» (SQS'2011, July 18–23, 2011), Dubna, Russia. The work was supported by the National Research Foundation of Korea (NRF) grants funded by the Korea government (MEST) with the grant numbers 2005-0049409 (CQUeST) and 2010-0002980.

REFERENCES

1. *Jeon I., Lee K., Park J.-H.* // JHEP. 2011. V. 104. P. 014.
2. *Jeon I., Lee K., Park J.-H.* // Phys. Lett. B. 2011. V. 710. P. 260–264.
3. *Jeon I., Lee K., Park J.-H.* // Phys. Rev. D. 2011. V. 84. P. 044022.
4. *Jeon I., Lee K., Park J.-H.* Incorporation of Fermions into Double Field Theory // JHEP (submitted); arXiv:1109.2035 [hep-th].
5. *Jeon I., Lee K., Park J.-H.* Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity. In preparation.
6. *Buscher T. H.* // Phys. Lett. B. 1985. V. 159. P. 127.
7. *Buscher T. H.* // Phys. Lett. B. 1987. V. 194. P. 59.
8. *Buscher T. H.* // Phys. Lett. B. 1988. V. 201. P. 466.
9. *Giveon A., Rabinovici E., Veneziano G.* // Nucl. Phys. B. 1989. V. 322. P. 167.
10. *Tseytlin A. A.* // Phys. Lett. B. 1990. V. 242. P. 163.
11. *Tseytlin A. A.* // Nucl. Phys. B. 1991. V. 350. P. 395.
12. *Siegel W.* // Phys. Rev. D. 1993. V. 47. P. 5453.
13. *Siegel W.* // Ibid. V. 48. P. 2826.
14. *Duff M. J.* // Nucl. Phys. B. 1990. V. 335. P. 610.
15. *Hull C., Zwiebach B.* // JHEP. 2009. V. 0909. P. 099.
16. *Hull C., Zwiebach B.* // Ibid. P. 090.
17. *Hohm O., Hull C., Zwiebach B.* // JHEP. 2010. V. 1007. P. 016.
18. *Hohm O., Hull C., Zwiebach B.* // Ibid. V. 1008. P. 008.
19. *Courant T.* Dirac Manifolds // Trans. Am. Math. Soc. 1990. V. 319. P. 631–661.
20. *Gualtieri M.* Ph.D. Thesis. Generalized Complex Geometry. arXiv:math/0401221.
21. *Hassan S. F.* // Nucl. Phys. B. 1995. V. 454. P. 86; arXiv:hep-th/9408060.
22. *Kwak S. K.* // JHEP. 2010. V. 1010. P. 047.