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# DYNAMICS OF LOCALIZED STATES IN $N = 4$ SUSY QM

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A consistent approach is offered for investigating the temporal dynamics of localized states. It is based on exactly solvable quantum mechanical models with multiwell potentials and the associate propagators. The Hamiltonian states with multiwell potentials form an adequate basis for expanding wave packets (WP) of various types and degrees of localization. Special features of WP tunneling have been studied with due regard to all Hamiltonian states with symmetric and asymmetric potentials.

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In most cases, theoretical analysis of tunneling transitions in double-well potentials is carried out in the two-mode approximation [1]. The significant characteristics in this approach are the energy difference of the ground state and the first excited state ($\Delta = E_1 - E_0$), and also, their wave functions. The use of this approximation provides explanation for the general properties of tunneling processes; however, it fails to give interpretation of many fine effects [2–4]. Besides, the analysis of the processes with multiwell potentials is hampered by the fact that the models employed operate, as a rule, on phenomenological or piecewise potentials (e.g., with cross-linked rectangular wells and barriers or parabolas), which are far from real potentials. It should be noted that there exist exactly solvable models with multiwell potentials [5–7], which can be applied to describe the tunneling processes. The WP time evolution is described in terms of the propagators, which take into account the contribution of the whole Hamiltonian spectrum. In papers [8–10], an approach has been proposed to construct new propagators on the basis of the propagators already known in the context of supersymmetric quantum mechanics. It is of interest to apply the mentioned procedure for obtaining propagators in the exactly solvable quantum mechanical models with multiwell potentials.

The goal of the present work has been to study the peculiarities of localized state dynamics in multiwell potentials in the consistent approach. Exactly solvable models obtained in the $N = 4$ SUSY QM [7] have been used as potentials, both symmetric and asymmetric, and the WP dynamics is described by means of the propagators corresponding to these models and calculated in the approach [8–10].

## 1. $N = 4$ SUSY QM AND MULTIWELL POTENTIALS

The procedure of constructing Hamiltonians with multiwell potentials in the $N = 4$ SUSY QM is considered in detail in [7]. It can be realized through adding additional levels with the energy $\varepsilon$ below the ground state energy $E_0$ of the basic Hamiltonian $H_0$. In this case the multiwell structure of the potentials derived will be most indicative for $(E_0 - \varepsilon)/E_0 \ll 1$. The super-Hamiltonian of $N = 4$ SUSY QM comprises three nontrivial Hamiltonians (for details, see [7]) $H_+^- = H_-^+ = H_0 - \varepsilon$ and $H_-^-$, $H_+^+$, the spectra of which have an additional level below the ground state of the original Hamiltonian, and the others are in coincidence with the states of $H_0 - \varepsilon$. The Hamiltonian $H_-^-$ and its wave functions are related to $H_+^- = H_0 - \varepsilon$ and to the initial wave functions as

$$
\begin{gathered}
H_-^- = H_+^- - \frac{d^2}{dx^2} \ln(\phi_1(x,\varepsilon) + \phi_2(x,\varepsilon)), \\
\psi_-^-(x,E) = \frac{1}{\sqrt{2(E_i - \varepsilon)}} \frac{W\{\psi_+^-(x,E_i), \phi(x,\varepsilon,1)\}}{\phi(x,\varepsilon,1)}, \\
\psi_-^-(x,E=0) = \frac{N^{-1}}{\phi(x,\varepsilon,1)}, \quad \phi(x,\varepsilon,1) = \phi_1(x,\varepsilon) + \phi_2(x,\varepsilon), \\
N^{-2} = -\frac{2W\{\phi_1,\phi_2\}}{\Delta(+\infty,\varepsilon) - \Delta(-\infty,\varepsilon)}\,, \quad \Delta(x,\varepsilon) = \frac{\phi_1(x,\varepsilon) - \phi_2(x,\varepsilon)}{\phi_1(x,\varepsilon) + \phi_2(x,\varepsilon)},
\end{gathered} \tag{1}
$$

where $\phi_i(x,\varepsilon)\,, i = 1,2$ are the two linearly-independent solutions of the auxiliary equation $H_0\phi(x) = \varepsilon\phi(x)$, which are non-negative and show an asymptotic behavior at $x \to -\infty$ $\phi_1(x) \to +\infty$ ($\phi_2(x) \to 0$), and at $x \to +\infty$ $\phi_1(x) \to 0$ ($\phi_2(x) \to +\infty$); $\psi_+^-(x,E_i)$ are the normalized wave functions of the initial

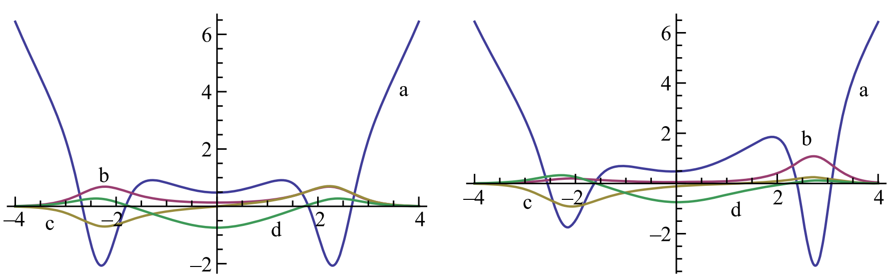


Fig. 1. *a*) Potential $U_-^-(\xi)$ ($\omega = 1$, $\nu = -0.02$) (a), wave functions of the ground state (b), first (d) and second (c) excited states. *b*) Potential $U_+^+(\xi, \lambda + 1)$ ($\omega = 1$, $\nu = -0.02$, $\lambda = -0.95$)

Hamiltonian, and $W\{\phi_1, \phi_2\}$ is the Wronskian. At this choice of solutions to the auxiliary equation we have $N^{-2} = W\{\phi_1, \phi_2\}$.

Using the property of form invariance between $H_+^+$ and $H_-^-$ [7], we obtain from (1) the relationships for $H_+^+$, $\psi_+^+$ by replacement $\phi(x, \varepsilon, 1) \to \phi(x, \varepsilon, \lambda+1)$ $= \phi_1(x, \varepsilon) + (\lambda + 1)\phi_1(x, \varepsilon)$, where $\lambda$ is the parameter limited by the condition $\lambda > -1$, and the normalization ground-state constant of $H_+^+$ is equal to $N_{\lambda+1}^{-2} = (1 + \lambda)\, N^{-2}$.

We shall use the harmonic oscillator (HO) model as a basic Hamiltonian. For a certain choice of the parameters, $(\nu, \omega, \lambda)$, $U_-^-(\xi)$, and $U_+^+(\xi, \lambda)$ as well as the corresponding wave functions are presented in Fig. 1. It should be noted that in terms of the dimensionless variable $\xi = \sqrt{\omega}\, x$ the only way to vary the form of the potential is by varying $0 < \bar{\varepsilon} < 1$ and $-1 < \lambda$. In the case of natural units $x$, additionally, the form of the potential (in particular, position of local minima) can be changed by variation of $\omega$.

## 2. TIME EVOLUTION OF STATES IN THE $N = 4$ SUSY QM

If at the initial instant $t = 0$ the Gaussian WP $\Phi(x) = \left(\dfrac{\omega\, \mathrm{e}^{2R}}{\pi}\right)^{1/4} \times$ $\times \exp\left(-\dfrac{\omega}{2}(x - a_i)^2\, \mathrm{e}^{2R}\right)$, $R$ being the compression parameter, is localized at the point $x = a_i$, then its time and space evolution can be described by the relation [11]

$$\begin{gathered}\Phi\,(x, t) = \int\limits_{-\infty}^{+\infty} K\,(x, t;\; x_0, 0)\; \Phi(x_0)\; dx_0, \\ K\,(x, t;\; x_0, 0) = \sum_{n=0}^{\infty} \psi_n(x)\; \psi_n^*(x_0)\; \mathrm{e}^{-iE_n t}.\end{gathered} \tag{2}$$

$K\,(x, t;\; x_0, 0)$ is the propagator, the knowledge of which makes it possible to investigate the localized state dynamics in terms of the potentials of any complexity. We shall briefly run through the method of deriving the propagators for exactly solvable models with multiwell potentials in a closed form in the $N = 4$ SUSY QM, starting from the exactly solvable model with the confinement potential. We introduce the notation for the propagators $K_{\sigma_1}^{\sigma_2}\,(x, t;\; x_0, 0)$ $(\sigma_i = \pm)$ that correspond to the Hamiltonians $H_{\sigma_1}^{\sigma_2}$ $N = 4$ SUSY QM. Now consider the derivation of $K_+^+\,(x, t;\; x_0, 0)$ for the Hamiltonian $H_+^+$. The relationship of $K_+^+\,(x, t;\; x_0, 0)$ with the original $K_+^-\,(x, t;\; x_0, 0)$ for the exactly solvable model with the single-well potential has the form

$$K_+^+(x,t;\ y,0) = \frac{1}{2} L_x L_y \int_{-\infty}^{+\infty} dz K_+^-(x,t;\ z,0)\, G_+^-(z,y,\varepsilon) + \\ + \frac{N_\Lambda^{-2}\,\mathrm{e}^{-i\varepsilon t}}{\phi(x,\varepsilon,\Lambda)\,\phi(y,\varepsilon,\Lambda)}, \quad (3)$$

where $L_x = \left(\frac{d}{dx} - \frac{\phi'(x,\varepsilon,\Lambda)}{\phi(x,\varepsilon,\Lambda)}\right)$, $\Lambda = \lambda + 1$, $G_+^-(z,y,\varepsilon)$ is the Green function of the auxiliary equation at energy $\varepsilon$

$$G_+^-(x,y,\varepsilon) = -\frac{2}{W\{f_l, f_r\}} \left(f_l(x,\varepsilon)\ f_r(y,\varepsilon)\theta(y-x) + \right. \\ \left. + f_l(y,\varepsilon)\ f_r(x,\varepsilon)\theta(x-y)\right).$$

According to the earlier introduced notation $f_l(x,\varepsilon) = \phi_2(x,\varepsilon)$, $f_r(x,\varepsilon) = \phi_1(x,\varepsilon)$. After action of the operator $L_y$ and making some small algebraic manipulations we obtain

$$K_+^+(x,t;y,0) = \frac{-1}{\phi(y,\varepsilon,\Lambda)} \times \\ \times L_x \left[\Lambda \int_{-\infty}^{y} dz\, K_+^-(x,t;z,0)\ \phi_2(z,\varepsilon) - \int_{y}^{\infty} dz\, K_+^-(x,t;z,0)\ \phi_1(z,\varepsilon)\right] + \\ + \frac{N_\Lambda^{-2}\,\mathrm{e}^{-i\varepsilon t}}{\phi(x,\varepsilon,\Lambda)\phi(y,\varepsilon,\Lambda)}. \quad (4)$$

In case of choosing $H_+^-$ as the HO Hamiltonian, $K_+^-(x,t;\ y,0)$ has the following form [11]:

$$K_+^-(x,t;\ y,0) = \left(\frac{\omega\,\mathrm{e}^{-i\pi(1/2+n)}}{2\pi\ \sin\omega\tau}\right)^{1/2} \times \\ \times \exp\left\{\frac{i\omega}{2\ \sin\omega\ t}\left[\left(x^2+y^2\right)\ \cos\omega\ t - 2xy\right]\right\}, \quad (5)$$

where ($t = n\pi/\omega + \tau$, $n \in N_0$; $0 < \tau < \pi/\omega$, and $\phi_1(\xi,\bar{\varepsilon}) = D_\nu(\sqrt{2}\xi)$, $\phi_2(\xi,\bar{\varepsilon}) = D_\nu(-\sqrt{2}\xi)$. Relations (1), (2), (3) and (5) are basic for investigating the peculiarities of localized state dynamics in the cases of both symmetric and asymmetric multiwell potentials. In the process, the contributions of all the states of the Hamiltonian that form the localized state $\Phi(x, 0)$ are taken into account. $K_{-}^{-}(x, t;\, y, 0)$ is obtained from (3) at $\Lambda(\varepsilon, \lambda) = 1$ and corresponds to the symmetric potential case.

## 3. LOCALIZED STATE DYNAMICS IN MULTIWELL POTENTIALS

In the present paper, attention focusses on the case, where only a few Hamiltonian levels are below the barrier. The above-discussed approach makes it possible to consider the peculiarities of the localized state dynamics with due regard to all the levels of the exactly solvable Hamiltonian with both symmetric and asymmetric multiwell potentials. We note that the consideration of the WP dynamics cannot be reduced to the two-mode approximation even in the $R = 0$ case. A good approximation of the initial localized state is attained with consideration of eight states of the Hamiltonian $H_{-}^{-}$ (see the Table). We shall compare the

| State number | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\lambda = 0, R = 0$ | 0.668 | –0.664 | 0.018 | 0.017 | 0.048 | –0.146 | 0.203 | –0.184 | 0.110 |
| $\lambda = 0, R = 0.35$ | 0.682 | –0.692 | 0.135 | –0.082 | 0.056 | –0.069 | 0.094 | –0.090 | 0.044 |
| $\lambda = -0.95, l$ | 0.208 | –0.945 | 0.179 | –0.105 | 0.065 | –0.066 | 0.077 | –0.060 | 0.013 |
| $\lambda = -0.95, r$ | 0.941 | 0.213 | 0.038 | 0.011 | –0.007 | 0.011 | 0.066 | 0.128 | 0.153 |

results of exact calculation with $|\Phi(\xi, T)| = \left|\sum\limits_{n=0}^{n_{\max}} c_n\, \psi_{(-)+}^{(-)+}(\xi, E_n) \exp\left(-iE_nT\right)\right|$ $(T = \omega\, t,\ \xi = \sqrt{\omega}\, x)$ to demonstrate the efficiency of the basis of states $H_{-}^{-}\ \left(H_{+}^{+}\right)$ in the problem considered. Figure 2 shows the $|\Phi\left(\xi, T\right)|$ values for the cases, where the parameters of potentials $U_{-}^{-}(\xi)$ and $U_{+}^{+}(\xi, \lambda)$ correspond to the ones given in Fig. 1, and the WP compression parameter is $R = 0.35$, this corresponding to the case of weak localization. Initially $(T = 0)$, the WP concentrates in the left local minimum. The WP dynamics includes slow tunnel transitions of subbarrier states and «beats» specified by above-barrier levels, which form the localized state. In the symmetric potential case, the $|\Phi\left(\xi, T\right)|$ variation (Fig. 2, *a*) has a pronounced «oscillatory» character. In this case, the contribution of higher excited states to $|\Phi\left(\xi, T\right)|$ is comparatively inconsiderable and results in insignificant «beats». The situation is quite different when the WP is originally concentrated in some local minimum of $U_{+}^{+}(\xi, \lambda)$ (Fig. 2, *b*). In the process of time evolution, the fraction of the tunneling WP to the other local minimum is relatively small. This is due to the fact that the highest contribution to $\Phi\left(\xi, 0\right)$ in the left local minimum $(\xi_l = -2.153)$ of $U_{+}^{+}(\xi, \lambda)$ comes from the first excited state of $H_{+}^{+}$, the wave function of which in the right well is very small. In other words, as a result of time evolution, the contribution of this state to the tunnel transition

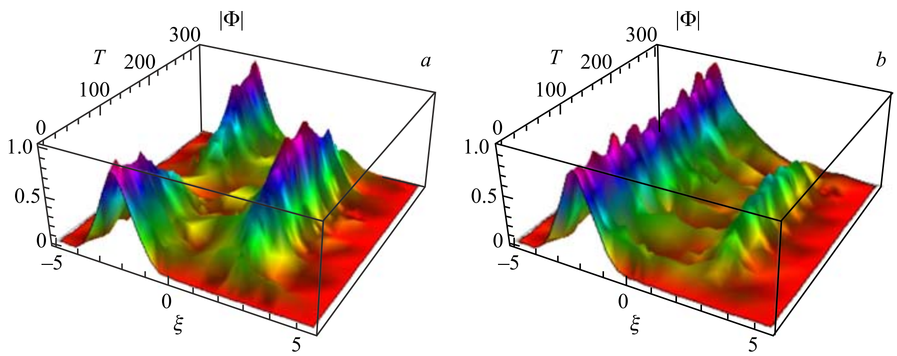


Fig. 2. Behavior of $|\Phi(\xi, T)|$ at the initial localized-state position in the left local minimum with $R = 0.35$: *a*) $(\xi_l = -2.29)$ of the potential $U_-^-(\xi)$, *b*) $(\xi_l = -2.153)$ of the potential $U_+^+(\xi, \lambda)$, $\lambda = -0.95$

is low. A similar situation is observed when originally the WP is concentrated in the right local minimum $(\xi_r = 2.755)$ of $U_+^+(\xi, \lambda)$. In a certain sense, as a result of time evolution, the WP is partially confined in the original well. The mechanism of partial WP «closure» in the original well is rather simple, i.e., if the major contribution to $\Phi(\xi, 0)$ in one of the local minima comes from one of the subbarrier states, then its wave function value in the other well is very low. It follows that its contribution at tunneling to the other well is small. The other subbarrier states contribute little to the tunnel transitions owing to the smallness of their contribution to the $\Phi(\xi, 0)$ formation. Note that here the contribution by higher excited states is more considerable than in the symmetric potential case.

## CONCLUSIONS

The present paper has proposed an approach for investigating the dynamics of initially localized states. It is based on exactly solvable quantum-mechanical models with multiwell potentials and the corresponding exact propagators. With the use of the harmonic oscillator Hamiltonian as an initial one within the framework of $N = 4$ SUSY QM, Hamiltonians with multiwell potentials, both symmetric and asymmetric, as well as the corresponding propagators, have been obtained. The study into the dynamics of initially localized states in this model has demonstrated that the range of applicability of the two-mode approximation for describing the tunneling processes is very limited. It is necessary to point out the adequacy of states of the Hamiltonians $H_-^-$ and $H_+^+$ as the basis for the localized state expansion $\Phi(\xi, 0)$. The WP dynamics includes a slow process of subbarrier state tunneling and fast oscillations (beats) caused by above-barrier states. At low compression parameter values ($R = 0.35$), the amplitude of beats is comparatively small. In symmetric double-well case, variations in $|\Phi(\xi, T)|$ have the «Josephson» character. In the asymmetric potential case, the dynamics of the WP, first localized in one of the $U_+^+(\xi, \lambda)$ minima, has a number of peculiarities. The effect of partial WP «confining» in the original well is observed, which is characterized by suppression of tunneling transitions to the other well. Note that the effect takes place no matter in which of the minima the WP is originally localized. In the case of the initial state, which is uniformly distributed in both the local minima of $U_+^+(\xi, \lambda)$, the phenomenon takes place in the deeper well. That is, the tunnel transitions from a deeper well to a less deep well are far less intense than the inverse process.

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