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RELATIVISTIC KINETIC MOMENTUM OPERATORS, HALF-RAPIDITIES AND NONCOMMUTATIVE DIFFERENTIAL CALCULUS

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It is shown that the generating function for the matrix elements of irreps of the Lorentz group is the common eigenfunction of the interior derivatives of the noncommutative differential calculus over the commutative algebra generated by the coordinate functions in the Relativistic Configuration Space (RCS). These derivatives commute and can be interpreted as the quantum mechanical operators of the relativistic momentum corresponding to the half of the non-Euclidean distance in the Lobachevsky momentum space (the mass shell).

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Referring the reader to the original paper [1] and more recent articles [2–5], where the notion of the RCS was introduced and developed, we stress here only that the concept of RCS is based on the Fourier expansion on the Lorentz group [6]. In this paper we consider the case with two spatial dimensions corresponding to the signature $(1, 2)$ in the Minkovsky space so that the transitivity surface of the Lorentz group $SO(2, 1)$ is the mass shall of the particle

$$p_\mu p^\mu = {p^0}^2 - \tilde{p}^2 = m^2c^2, \quad p^0 = mc\cosh\chi > 0, \quad \tilde{p} = \sinh\frac{\chi}{2}\,\tilde{n}_p, \quad \tilde{n}_p^2 = 1, \tag{1}$$

$\mu = 0, 1, 2$. RCS is dual in the sense of the above-mentioned Fourier transformation to the Lobachevsky momentum space, realized on the surface (1)

Let us introduce the new momenta $\hat{k}^\mu$, corresponding to the «half distance» $\chi/2$ (or half rapidity in different terminology) in the Lobachevsky momentum space (1) as follows. In the momentum space $\hat{k}$-operators are given by

$$k^0 = 2\,\cosh\frac{\chi}{2}, \quad \tilde{k} = 2\,\sinh\frac{\chi}{2}\,\tilde{n}_p. \tag{2}$$

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The «half distance» — or kinetic k-momenta possess the remarkable properties. In terms of «old» p-momentum the relativistic energy E in the nonrelativistic limit $\|\tilde{p}\| \ll mc$ can be presented approximately as $E = p^0 c = \sqrt{m^2c^4 + \tilde{p}^2c^2} \simeq mc^2 + \tilde{p}^2/2m$. In terms of the «new» k-momentum, the inner energy mc^2 and kinetic energy are explicitly separated without any approximation:

$$E - mc^2 = mc^2 \sinh^2 \frac{\chi}{2} = \frac{\tilde{k}^2}{2m}. \tag{3}$$

Our goal is to show that the corresponding operators of kinetic momentum do exist in RCS and belong to the noncommutative differential calculus [5–7]. To derive the corresponding momentum operators in RCS, we must define the relativistic plane wave as the generating function for the matrix elements of the unitary irreps of the Lorentz group which are expressed in terms of the generalized Jacobi functions [6]:

$$\langle \tilde{\rho}, n | \tilde{k} \rangle = \left(\cosh \frac{\chi}{2} - \sinh \frac{\chi}{2} \, \mathrm{e}^{i(\psi - \phi)} \right)^{-i\rho - 1/2 + n} \times \\ \times \left(\cosh \frac{\chi}{2} - \sinh \frac{\chi}{2} \, \mathrm{e}^{-i(\psi - \phi)} \right)^{-i\rho - 1/2 - n}, \tag{4}$$

$$\langle \tilde{\rho}, 0 | \, \tilde{k} \rangle = \langle \tilde{\rho} | \, \tilde{p} \rangle, \tag{5}$$

$$\langle \tilde{\rho}, n | \, \tilde{k} \rangle = \sum_{m=-\infty}^{\infty} P_{mn}^{-1/2 - i\rho}(\cosh \chi) \, \mathrm{e}^{i(n-m)(\psi - \phi)}, \tag{6}$$

where m, n are simultaneously integer or semi-integer numbers. The desired operators $\hat{\tilde{k}} = (\hat{k}^1, \hat{k}^2)$ in RCS or ρ, ψ, n-representation are obtained using formulae (7)–(9) of Subsubsec. 6.7.3 in [6]:

$$\hat{k}^{\pm} = 2 \, \mathrm{e}^{\pm i\psi} \left\{ \frac{-i\rho - 1/2 \pm n}{i\rho} \sinh \frac{i}{2} \partial_\rho \pm \frac{1}{2\rho} \partial_\psi \, \mathrm{e}^{i/2 \partial_\rho} \right\} \mathrm{e}^{\mp 1/2 \, \partial_n}, \tag{7}$$

$$[\hat{k}^1, \, \hat{k}^2] = \frac{i}{2} [\hat{k}^+, \, \hat{k}^-] = 0, \quad \hat{k}^{\pm} = \frac{\hat{k}^1 \pm i \, \hat{k}^2}{2}. \tag{8}$$

The generalized plane waves (4) are the common eigenfunctions for $\hat{k}^{\pm}$:

$$\hat{k}^{\pm} \, \langle \tilde{\rho}, \, n | \, \tilde{k} \rangle = k^{\pm} \, \langle \tilde{\rho}, \, n | \, \tilde{k} \rangle = 2 \sinh \frac{\chi}{2} \, \mathrm{e}^{\pm i\phi} \, \langle \tilde{\rho}, \, n | \, \tilde{k} \rangle \, . \tag{9}$$

Also the operators exist corresponding to the eigenvalue $\cosh \chi/2$:

$$\tilde{c}_{\pm} = -\frac{1}{2i\rho} \, \mathrm{e}^{\mp 1/2 \, \partial_n} \{ (-i\rho \mp n) \, \mathrm{e}^{i/2 \, \partial_\rho} + (-i\rho \pm n) \, \mathrm{e}^{-i/2 \, \partial_\rho} \pm i \partial_\psi \, \mathrm{e}^{i/2 \, \partial_\rho} \} = \\ = \left(\frac{1}{2} \, \mathrm{e}^{-i\psi} \, \hat{k}^{\pm} + \mathrm{e}^{i/2 \partial_\rho} \, \mathrm{e}^{\mp 1/2 \partial_n} \right), \quad \hat{c}_{\pm} \, \langle \tilde{\rho}, \, n | \, \tilde{k} \rangle = \cosh \frac{\chi}{2} \, \langle \tilde{\rho}, \, n | \, \tilde{k} \rangle \, . \tag{10}$$

Now we define the differential calculus in RCS. The differential calculus over the associative algebra A is a $\mathbb{Z}$ graded associative algebra over $\mathbb{C}$:

$$\Omega(A) = \sum_{r=0} \oplus \, \Omega^r(A). \tag{11}$$

In our case, A is the commutative algebra generated by the coordinate functions $f(x), \quad x_i = (\rho, \psi, n)$. The elements of $\Omega^r(A)$ are called r-forms. There exists an exterior derivative $\hat{d}$ which satisfies the following conditions:

$$\hat{d}^2 = 0, \quad \hat{d}(\omega\omega') = (\hat{d}\omega)\,\omega' + (-1)^r \omega \hat{d}\omega', \tag{12}$$

where ω and ω' are r and r' forms, respectively. We define $\hat{d}$ as an operator valued 1-form which satisfies the relation

$$\hat{d}\omega = [\,\hat{d}, \,\omega]_\wedge = \hat{d}\,\omega - (-1)^r \omega\,\hat{d}. \tag{13}$$

Important difference exists between the standard differential calculus and the relativistic one. In the first one, the differential and the coordinate function commute $[dx_k, x_i] = 0$, because in the standard case x_i and dx_k are the independent numerical variables. But the relativistic differential calculus is noncommutative: $[dx_k, x_i] \neq 0$.

To determine $\hat{d}$, we recall first the simple connection existing between the nonrelativistic $\hat{d}$ and momentum operators: $\hat{d} = dx_1 \partial_{x_1} + dx_2 \partial_{x_2} = i(dx_+ \hat{k}_- + dx_- \hat{k}_+) = d\rho\,\partial_\rho + d\psi\,\partial_\psi$, where $dx_\pm = dx_1 \pm dx_2 = \mathrm{e}^{\pm i\psi}(d\rho \pm i\rho\, d\psi)$. In the relativistic case, the differentials $dx_\pm$ are modified and gain the operator valued form:

$$\hat{d}x_\pm = \mathrm{e}^{\pm i\psi} \left(d\rho \pm \mathrm{e}^{-i/2\,\partial_\rho} (i\rho \mp n) d\psi \right) \mathrm{e}^{-1/2\,\partial_n}. \tag{14}$$

The identity is satisfied

$$\frac{i}{2}(\hat{d}x_+\, \hat{k}^- + \hat{d}x_-\, \hat{k}^+) = d\rho \left(-2i \sinh \frac{i}{2}\partial_\rho \right) + d\psi\,\partial_\psi \longrightarrow d\rho\,\partial_\rho + d\psi\,\partial_\psi. \tag{15}$$

It is easily seen that this expression is incomplete. It must be extended to involve the «cosh $\chi/2$» terms or operators $\hat{c}_\pm$ in order to satisfy the generalized Leibnitz rule of the noncommutative derivation:

$$\hat{d} = \frac{i}{2}(\hat{d}x_+ \hat{k}^- + \hat{d}x_- \hat{k}^+ + d\rho_+ (\mathrm{e}^{1/2\,\partial_n} \hat{c}_+ + \mathrm{e}^{-1/2\,\partial_n} \hat{c}_- - 2) + \\ + d\rho_- (\mathrm{e}^{1/2\,\partial_n} \hat{c}_+ - \mathrm{e}^{-1/2\,\partial_n} \hat{c}_-)). \tag{16}$$

We omit the intermediate calculations and deliver here in an explicit form only the operators of the left and right interior derivatives

$$\overrightarrow{\partial} = \frac{\mathrm{e}^{i/2\,\partial_\rho} - 1}{i/2} \quad \text{and} \quad \overleftarrow{\partial} = \frac{\mathrm{e}^{-i(i/2)\partial_\rho} - 1}{-i/2}, \tag{17}$$

which can be easily derived in the framework of the noncommutative differential calculus above. It is sufficient to keep only one of the $\overleftrightarrow{\partial}$ derivatives and we put $d\rho_{\leftarrow} = 0$ (and omit the subscript $_{\rightarrow}$: $d\rho_{\rightarrow} = d\rho$)

$$\hat{d} = d\rho \overrightarrow{\partial} + d\psi \partial_{\psi}. \tag{18}$$

In the relativistic differential calculus the differential $d\rho$ does not commute with the coordinate: $[d\rho,\, \rho] \neq 0$. We recall that in the standard calculus ρ and $d\rho$, of course, commute . In the relativistic calculus

$$\hat{d}\rho = [\hat{d},\, \rho] = [\hat{d}\rho\, \partial_{\rho}, \rho] = d\rho\, \frac{\mathrm{e}^{i/2\, \partial_{\rho}} - 1}{i/2}\, \rho - \rho\, d\rho\, \frac{\mathrm{e}^{i/2\, \partial_{\rho}} - 1}{i/2} = d\rho\, \mathrm{e}^{i/2\, \partial_{\rho}}, \tag{19}$$

and we obtain

$$[\hat{d}\rho,\, \rho] = \frac{i}{2}\, \hat{d}\rho. \tag{20}$$

It follows directly from (20)

$$[\hat{d}\rho,\, f(\rho)] = \frac{i}{2}\, (\overrightarrow{\partial} f(\rho))\, \hat{d}\rho = \frac{i}{2}\, \hat{d}\rho\, (\overleftarrow{\partial} f(\rho)). \tag{21}$$

Free Hamiltonian has a form (cf. (3))

$$\hat{H}_0 - mc^2 = \frac{2\, \hat{k}^{+} \hat{k}^{-}}{2m}. \tag{22}$$

Potential $V(\rho)$ is introduced (see [1–5]) as addition to the free Hamiltonian and we come to the relativistic Schrödinger equation which does not differ by form from the nonrelativistic one

$$(\hat{H}_0 + V(\rho))\psi(\tilde{\rho}) = E\psi(\tilde{\rho}). \tag{23}$$

At last, we write down the k-momentum operators corresponding to the half of the distance in the Lobachevsky momentum space in terms of the relativistic noncommutative differential calculus

$$\hat{k}^{\pm} = \mathrm{e}^{\pm i\psi} \left(\frac{-i\rho - 1/2\, \pm\, n}{2\rho} (\overrightarrow{*} + \overleftarrow{*})\, \hat{d}\rho - \frac{1}{\rho} \left(1 + \frac{i}{2}\, \overrightarrow{*}\, \hat{d}\rho \right) * \hat{d}_{\psi} \right), \tag{24}$$

where $\overrightarrow{*}$ and $\overleftarrow{*}$ are left and right noncommutative Hodge symbols introduced in [5], $*$ is the standard Hodge symbol corresponding to the commutative differentiation in ψ.

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