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MODIFIED FREQUENTIST DETERMINATION OF CONFIDENCE INTERVALS FOR POISSON DISTRIBUTION

S. I. Bitioukov, N. V. Krasnikov

Istitute for Nuclear Research of the Russian Academy of Sciences, Moscow

We propose the modified frequentist definition for the determination of confidence intervals for the case of Poisson statistics. Namely, we require that $1-\beta' \geqslant \sum\limits_{n=0}^{n=n_{\rm obs}+k} P(n|\lambda) \geqslant \alpha'$. We show that this definition is equivalent to the Bayesian method with prior $\pi(\lambda) \sim \lambda^k$. We also propose the modified frequentist definition for the case of nonzero background.

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In high-energy physics one of the standard problems [1] is the determination of the confidence intervals for the parameter λ in the Poisson distribution

$$P(n|\lambda) = \frac{\lambda^n}{n!}\exp(-\lambda). \tag{1}$$

There are two methods to solve this problem — the frequentist and the Bayesian.

In this paper, we propose the modified frequentist definition of the confidence interval for the case of the Poisson distribution. We show that the modified frequentist distribution is equivalent to the Bayesian approach.

In the Bayesian method [1, 2], due to the Bayes theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)},$$

the probability density for the λ parameter is determined as

$$p(\lambda|n_{\rm obs}) = \frac{P(n_{\rm obs}|\lambda)\pi(\lambda)}{\int\limits_0^\infty P(n_{\rm obs}|\lambda')\pi(\lambda')\,d\lambda'}. \tag{2}$$

Here $\pi(\lambda)$ is the prior function and, in general, it is not known what is the main problem of the Bayesian method. Formula (2) reduces the statistics problem to the probability problem. At the $(1-\alpha)$ probability level the parameters $\lambda_{\rm up}$ and $\lambda_{\rm down}$ are determined from the equation*

$$\int\limits_{\lambda_{\rm down}}^{\lambda_{\rm up}} p(\lambda|n_{\rm obs})\, d\lambda = 1-\alpha, \tag{3}$$

and the unknown parameter λ lies between $\lambda_{\rm down}$ and $\lambda_{\rm up}$ with the probability $1-\alpha$. The solution of Eq. (3) is not unique. One can define

$$\int\limits_{\lambda_{\rm up}}^{\infty} p(\lambda|n_{\rm obs})\, d\lambda = \alpha', \tag{4}$$

$$\int\limits_{0}^{\lambda_{\rm down}} p(\lambda|n_{\rm obs})\, d\lambda = \beta'. \tag{5}$$

In general, the parameters α' and β' are arbitrary except the evident equality

$$\alpha' + \beta' = \alpha. \tag{6}$$

The most popular are the following options [1]:

1. $\lambda_{\rm down} = 0$ — upper limit.
2. $\lambda_{\rm up} = \infty$ — lower limit.
3. $\int\limits_{0}^{\lambda_{\rm up}} p(\lambda|n_{\rm obs})\, d\lambda = \int\limits_{\lambda_{\rm down}}^{\infty} p(\lambda|n_{\rm obs})\, d\lambda = \alpha/2$ — symmetric interval.
4. The shortest interval — $p(\lambda|n_{\rm obs})$ inside the interval is bigger or equal to $p(\lambda|n_{\rm obs})$ outside the interval.

In frequentist approach, the Neyman belt construction [3] (see Fig. 1) is used for the determination of the confidence intervals.

For the continuous observable $-\infty < x < \infty$ with the probability density $f(x,\lambda)$**, we require that

$$\int\limits_{x_{\rm down}}^{x_{\rm up}} f(x,\lambda)\, dx = 1-\alpha, \tag{7}$$

*Usually α is taken equal to 0.05.

**Here λ is some unknown parameter, and $\int\limits_{-\infty}^{\infty} f(x,\lambda)\, dx = 1$.

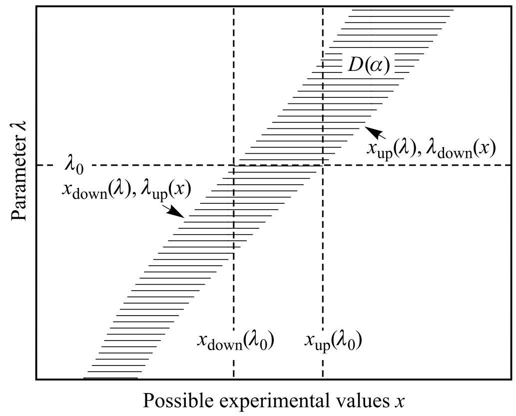

Fig. 1. The Neyman belt construction

or

$$\int\limits_{x_{\rm up}}^{\infty} f(x,\lambda)\,dx = \beta', \tag{8}$$

$$\int\limits_{-\infty}^{x_{\rm down}} f(x,\lambda)\,dx = \alpha', \tag{9}$$

$$\alpha' + \beta' = \alpha. \tag{10}$$

The equations*

$$\int\limits_{x_{\rm obs}}^{\infty} f(x,\lambda_{\rm down})\,dx = \beta', \tag{11}$$

$$\int\limits_{-\infty}^{x_{\rm obs}} f(x,\lambda_{\rm up})\,dx = \alpha' \tag{12}$$

determine the interval of possible values $\lambda_{\rm down} \leqslant \lambda \leqslant \lambda_{\rm up}$ of the parameter λ at the $(1-\alpha)$ confidence level. Equations (11), (12) are equivalent to the equations

$$\int\limits_{x_{\rm obs}}^{\infty} f(x,\lambda_{\rm down})\,dx = \beta', \tag{13}$$

*Here $x_{\rm obs}$ is the observed value of random variable x.

$$\int_{x_{\text{obs}}}^{\infty} f(x, \lambda_{\text{up}})\, dx = 1 - \alpha', \tag{14}$$

or to the equations

$$\int_{-\infty}^{x_{\text{obs}}} f(x, \lambda_{\text{down}})\, dx = \alpha', \tag{15}$$

$$\int_{-\infty}^{x_{\text{obs}}} f(x, \lambda_{\text{down}})\, dx = 1 - \beta'. \tag{16}$$

For the Poisson distribution $P(n|\lambda)$, the analog of Eq. (7) has the form

$$\sum_{n_{\text{down}}(\lambda)}^{n_{\text{up}}(\lambda)} P(n|\lambda) \geqslant 1 - \alpha. \tag{17}$$

The equations for the determination of λ_{down} and λ_{up} (analogs of Eqs. (11), 12)) have the form [4–6]

$$\sum_{n=n_{\text{obs}}}^{\infty} P(n|\lambda_{\text{down}}) = \beta', \tag{18}$$

$$\sum_{n=0}^{n_{\text{obs}}} P(n|\lambda_{\text{up}}) = \alpha'. \tag{19}$$

The analogs of the equations (13), (14) and (15), (16) for the Poisson distribution are

$$\sum_{n=n_{\text{obs}}}^{\infty} P(n|\lambda_{\text{down}}) = \beta', \tag{20}$$

$$\sum_{n=n_{\text{obs}}}^{\infty} P(n|\lambda_{\text{up}}) = 1 - \alpha', \tag{21}$$

and

$$\sum_{n=0}^{n_{\text{obs}}} P(n|\lambda_{\text{down}}) = 1 - \beta', \tag{22}$$

$$\sum_{n=0}^{n_{\text{obs}}} P(n|\lambda_{\text{up}}) = \alpha', \tag{23}$$

correspondingly. Unlike to the case of continuous variable, the equations (18)–(21) and (22), (23) are not equivalent for the discrete variable n and they differ in the presence or absence of $P(n_{\rm obs}|\lambda_{\rm up,\,down})$ in some equations. For instance, for $\beta' = 0$, $\alpha' = \alpha$ (upper limit case) the equations (19), (23) coincide and read as

$$\sum_{n=0}^{n_{\rm obs}} P(n|\lambda_{\rm up}) = \alpha, \tag{24}$$

while Eq. (21) is equivalent to

$$\sum_{n=0}^{n_{\rm obs}-1} P(n|\lambda_{\rm up}) = \alpha. \tag{25}$$

For $n_{\rm obs} = 3$ and $\alpha = 0.05$, we find that

$$\lambda \leqslant 7.75 \quad \text{(Eq. (24))}, \tag{26}$$

$$\lambda \leqslant 6.30 \quad \text{(Eq. (25))}. \tag{27}$$

Consider the probability to observe the number of events $n \leqslant n_{\rm obs}$

$$P_{-}(n_{\rm obs}|\lambda) = \sum_{n=0}^{n_{\rm obs}} P(n|\lambda). \tag{28}$$

To determine possible values $\lambda_{\rm down}$ and $\lambda_{\rm up}$ of the confidence interval, we require that

$$1 - \beta' \geqslant P_{-}(n_{\rm obs}|\lambda) \geqslant \alpha', \tag{29}$$

where $\alpha' + \beta' = \alpha$. The equations for the determination of $\lambda_{\rm up}$ and $\lambda_{\rm down}$ have the form

$$P_{-}(n_{\rm obs}|\lambda_{\rm up}) = \alpha', \tag{30}$$

$$P_{-}(n_{\rm obs}|\lambda_{\rm down}) = 1 - \beta'. \tag{31}$$

Note that as in the case of the Bayesian approach, the choice of α' and β' is not unique. Due to the identity [6]

$$P_{-}(n_{\rm obs}|\lambda) = \int\limits_{\lambda}^{\infty} P(n_{\rm obs}|\lambda')\, d\lambda', \tag{32}$$

the confidence interval $[\lambda_{\rm down}, \lambda_{\rm up}]$ is determined from the equations

$$\alpha' = \int\limits_{\lambda_{\rm up}}^{\infty} P(n_{\rm obs}|\lambda')\, d\lambda', \tag{33}$$

$$\beta' = \int\limits_{0}^{\lambda_{\rm down}} P(n_{\rm obs}|\lambda')\, d\lambda'. \tag{34}$$

The parameter λ lies in the interval

$$\lambda_{\text{down}} \leqslant \lambda \leqslant \lambda_{\text{up}} \tag{35}$$

with the probability $(1 - \alpha' - \beta')$. Due to the equations (33), (34), our modified frequentist definition (29) is equivalent to the Bayes definitions (3)–(5) with flat prior $\pi(\lambda) = 1$, namely:

$$\int\limits_{\lambda_{\text{down}}}^{\lambda_{\text{up}}} P(n_{\text{obs}}|\lambda')\, d\lambda' = 1 - \alpha' - \beta'. \tag{36}$$

As an alternative to the definition (29) we can require that

$$1 - \alpha' \geqslant \sum_{n=n_{\text{obs}}}^{\infty} P(n, \lambda) = -P_{-}(n_{\text{obs}}, \lambda) + 1 + P(n_{\text{obs}}, \lambda) \geqslant \beta'. \tag{37}$$

The definition (37) leads to the equations (20), (21) for the determination of λ_{down} and λ_{up}. The equations (20), (21) are equivalent to the Bayes equations with the prior function $\pi(\lambda) \sim 1/\lambda$.

The coverage of the definition (29) means the following. For a hypothetical ensemble of similar experiments, the probability of observing the number of events $n \leqslant n_{\text{obs}}$ satisfies the inequalities (29). As we noted before, the choice of λ_{down} and λ_{up} is not unique. Probably, the most natural choice is the use of the ordering principle. According to this principle, we require that the probability density $P(n_{\text{obs}}|\lambda)$ inside the confidence interval $[\lambda_{\text{down}}, \lambda_{\text{up}}]$ is bigger or equal to the probability density outside this interval. For the Poisson distribution this requirement leads to the formula

$$P(n_{\text{obs}}|\lambda_{\text{down}}) = P(n_{\text{obs}}|\lambda_{\text{up}}), \tag{38}$$

for the determination of λ_{up} and λ_{down}. For such ordering principle α' and β' are not independent quantities. It is natural to use $\alpha = \alpha' + \beta'$ as a single free parameter. Note that the equations (19) and (23) for the determination of an upper limit λ_{up} in frequentist and modified frequentist approach coincide, whereas the equations (19) and (21) are different. Namely, the equation (21) is equivalent to the equation

$$\sum_{n=0}^{n_{\text{obs}}-1} P(n|\lambda_{\text{up}}) = \alpha'. \tag{39}$$

Classical frequentist equation (18) is equivalent to the Bayes equation (5) with prior $\pi(\lambda) \sim 1/\lambda$.

It is possible to generalize our modified frequentist definition (29), namely:

$$1 - \beta' \geqslant P_-(n_{\rm obs}|\lambda; k) \geqslant \alpha', \tag{40}$$

where

$$P_-(n_{\rm obs}|\lambda; k) \equiv \sum_{n=0}^{n_{\rm obs}+k} P(n|\lambda) \tag{41}$$

and $k = 0, \pm1, \pm2, \ldots$

One can find that the definitions (40), (41) lead to the Bayes equations (4), (5) with the prior function $\pi(\lambda) \sim \lambda^k$. Upper limits for three values of k are shown in Table 1 ($\alpha = 0.1$), in Table 2 ($\alpha = 0.05$) and, correspondingly, in Figs. 2 and 3.

Table 1. **Upper limits ($\lambda_{\rm up}$) for the confidence level 90%** ($\alpha = 0.1$)

$n_{\rm obs}$	$k = -1$	$k = 0$	$k = +1$
0	—	2.30	3.89
1	2.30	3.89	5.32
2	3.89	5.32	6.68
3	5.32	6.68	7.99
4	6.68	7.99	9.27
5	7.99	9.27	10.53
6	9.27	10.53	11.77
7	10.53	11.77	12.99
8	11.77	12.99	14.21
9	12.99	14.21	15.41
10	14.21	15.41	16.60

Table 2. **Upper limits ($\lambda_{\rm up}$) for the confidence level 95%** ($\alpha = 0.05$)

$n_{\rm obs}$	$k = -1$	$k = 0$	$k = +1$
0	—	3.00	4.74
1	3.00	4.74	6.30
2	4.74	6.30	7.75
3	6.30	7.75	9.15
4	7.75	9.15	10.51
5	9.15	10.51	11.84
6	10.51	11.84	13.15
7	11.84	13.15	14.43
8	13.15	14.43	15.71
9	14.43	15.71	16.96
10	15.71	16.96	18.21

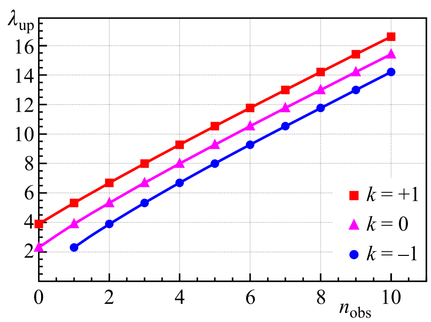

Fig. 2. Upper limits ($\lambda_{\rm up}$) for the confidence level 90% ($\alpha = 0.1$), $k = -1,\ 0, +1$

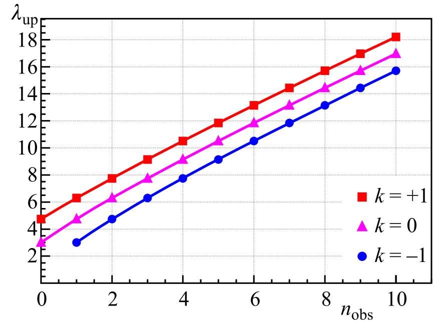

Fig. 3. Upper limits ($\lambda_{\rm up}$) for the confidence level 95% ($\alpha = 0.05$), $k = -1,\ 0, +1$

We can further generalize the definitions (40), (41) by the introduction of

$$P_-(n_{\rm obs}|\lambda; c_k) \equiv \sum_k c_k^2 P_-(n_{\rm obs}|\lambda; k), \tag{42}$$

where $\sum\limits_k c_k^2 = 1$. Again we require that

$$1 - \beta' \geqslant P_-(n_{\rm obs}|\lambda; c_k) \geqslant \alpha'. \tag{43}$$

One can find that our definition (43) is equivalent to the Bayes approach with prior function

$$\pi(\lambda) = \sum_k c_k^2 l_k \lambda^k, \tag{44}$$

where

$$l_k = \frac{n!}{(n+k)!}. \tag{45}$$

For the case when we have nonzero background, the parameter λ is represented in the form

$$\lambda = b + s. \tag{46}$$

Here $b \geqslant 0$ is the known background, and s is the unknown signal. In the Bayes approach, the generalization of the formula (2) reads

$$p(s|n_{\rm obs}, b) = \frac{P(n_{\rm obs}|b+s)\pi(b,s)}{\int\limits_0^\infty P(n_{\rm obs}|b+s')\pi(b,s')\,ds'}. \tag{47}$$

For flat prior, we have

$$p(s|n_{\rm obs}, b) = \frac{P(n_{\rm obs}|b+s)}{\int\limits_b^\infty P(n_{\rm obs}|\lambda')\, d\lambda'}. \qquad (48)$$

So, we see that the main effect of nonzero background is the appearance of the factor

$$K(n_{\rm obs}, b) = \int\limits_b^\infty P(n_{\rm obs}|\lambda')\, d\lambda' \qquad (49)$$

in the denominator of the formula (48). For zero background $K(n_{\rm obs}, b=0)=1$. One can interpret the appearance of additional factor $K(n_{\rm obs}, b)$ in terms of conditional probability. Really, for flat prior the $P(n_{\rm obs}, \lambda)\, d\lambda$ is the probability that parameter λ lies in the interval $[\lambda, \lambda+d\lambda]$. For the case of nonzero background b parameter $\lambda = b + s \geqslant b$. The probability that $\lambda \geqslant b$ is equal to $p(\lambda \geqslant b|n_{\rm obs}) = K(n_{\rm obs}, b)$. The conditional probability that λ lies in the interval $[\lambda, \lambda+d\lambda]$ provided $\lambda \geqslant b$ is determined by the standard formula of the conditional probability

$$p(\lambda, n_{\rm obs}|\lambda \geqslant b)\, d\lambda = \frac{p(\lambda, n_{\rm obs})}{p(\lambda \geqslant b)}\, d\lambda = \frac{p(\lambda, n_{\rm obs})}{K(n_{\rm obs} s)}\, d\lambda, \qquad (50)$$

and it coincides with the Bayes formula (48).

In the frequentist approach, the naive generalization of the inequality (29) is

$$1 - \beta' \geqslant P_-(n_{\rm obs}|s+b) \geqslant \alpha'. \qquad (51)$$

One can show that

$$1 - \alpha' - \beta' = \int\limits_{b+s_{\rm down}}^{b+s_{\rm up}} P(n_{\rm obs}|\lambda')\, d\lambda' \leqslant \int\limits_b^\infty P(n_{\rm obs}|\lambda')\, d\lambda'. \qquad (52)$$

However, the main drawback of the definition (51) is that the probability, that the signal s lies in the interval $0 \leqslant s \leqslant \infty$, is equal to $\int\limits_b^\infty P(n_{\rm obs}|\lambda')\, d\lambda'$ and it is less than unity for nonzero background $s > 0$ that contradicts the intuition that the full probability, that the signal s lies between zero and infinity, must be equal to unity. To cure this drawback, let us require that

$$1 - \beta' \geqslant \frac{P_-(n_{\rm obs}|s+b)}{P_-(n_{\rm obs}|b)} \geqslant \alpha'. \qquad (53)$$

The inequality (53) leads to the equations for the determination of $s_{\rm down}$ and $s_{\rm up}$ which coincide with the corresponding Bayes equations. The generalization of the inequalities (53) is straightforward, for instance, the inequality (43) reads

$$1-\beta' \geqslant \frac{P_-(n_{\rm obs}|b+s;c_k)}{P_-(n_{\rm obs}|b;c_k)} \geqslant \alpha'. \tag{54}$$

Upper limit on the signal s derived from the inequality (53) coincides with the upper limit in CL_s method [7, 8].

Note that frequentist equations (18), (19) for $\lambda_{\rm up} = \lambda_{\rm down}$ do not satisfy the evident equality $\alpha' + \beta' = 1$. One of the possible generalizations of the equations (18), (19) looks as follows:

$$P_{-1}(n_{\rm obs}|\lambda_{\rm up}) = \alpha', \tag{55}$$

$$P_{+1}(n_{\rm obs}|\lambda_{\rm down}) = \beta', \tag{56}$$

where

$$P_{-1}(n_{\rm obs}|\lambda) = \sum_{n=0}^{n_{\rm obs}-1} P(n|\lambda) + \frac{1}{2}P(n_{\rm obs}|\lambda), \tag{57}$$

$$P_{+1}(n_{\rm obs}|\lambda) = \sum_{n_{\rm obs}+1}^{n=\infty} P(n|\lambda) + \frac{1}{2}P(n_{\rm obs}|\lambda). \tag{58}$$

Note that

$$P_{-1}(n_{\rm obs}|\lambda) + P_{+1}(n_{\rm obs}|\lambda) = 1 \tag{59}$$

and $\alpha' + \beta' = 1$ for $\lambda_{\rm up} = \lambda_{\rm down}$. The equations (55)–(58) are equivalent to the Bayes equations (4), (5) with prior

$$\pi(\lambda) = \frac{1}{2}\left(1 + \frac{n_{\rm obs}}{\lambda}\right). \tag{60}$$

The modified frequentist definition (29) takes the form

$$1-\beta' \geqslant P_{-1}(n_{\rm obs}|\lambda) \geqslant \alpha'. \tag{61}$$

To conclude, let us stress our main result. For the Poisson distribution we have the proposed modified frequentist definition of the confidence interval and have shown the equivalence of the modified frequentist approach and the Bayes approach. It means that the frequentist approach is in fact nonunique.

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REFERENCES

1. As a review, see, for example: *James F.* Statistical Methods in Experimental Physics. 2nd ed. World Sci., 2006.
2. See, for example: *Agostini G.D.* Bayesian Reasoning in Data Analysis, a Critical Introduction. Hackensack, NJ: World Sci., 2003.
3. *Neyman Y.* // Philos. Trans. Roy. Soc. London Sec. A. 1937. V. 236. P. 333.
4. *Garwood F.* // Biometrica. 1936. V. 28. P. 437.
5. *Ricker W. E.* // J. Am. Stat. Assoc. 1937. V. 32. P. 349.
6. *Cousins R. D.* // Am. J. Phys. 1995. V. 63. P. 398.
7. *Junk T.* // Nucl. Instr. Meth. A. 1999. V. 434. P. 435.
8. *Read A.L.* Modified Frequentist Analysis of Search Results (the CL_s Method). CERN Yellow Report CERN-2000-005. 2000. P. 81.