UNDERSTANDING PT RESULTS FOR DECAYS OF τ LEPTONS INTO HADRONS

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We review some of our results obtained in the field of the structure of PT series in high orders and its resummation in some particular circumstances.

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INTRODUCTION

A mathematical framework of QFT is a powerful working tool for description of elementary particle interactions. The concept of renormalization and the practical technique of R-operation generate the renormalizable models of QFT as self-consistent tools for phenomenological analysis [1]. The SM of «all interactions» is formulated entirely as a local quantum field theoretical model, and the proof of its renormalizability has required some extension of the standard techniques [2, 3]. Despite the pronounced success in explaining the properties of interactions at high energies, the complete solution of the model is still absent. The main route of quantitative investigation is still the use of PT in the small coupling constant based on the computation of integrals represented by Feynman diagrams.

The classical example of the PT approach has been developed in QED that is a part of SM. After renormalization of QED there are two parameters — the electron mass and the fine structure constant α — that are to be fixed to observables which are computed as PT series in α. With $\alpha = 1/137$ being small, the precision of the results is impressive. However, the question has been raised about the behavior of the PT series in high orders. The arguments have been put forward that the series is merely asymptotic one [4, 5]. This has caused much interest in summation of asymptotic series in QFT [6]. Different theoretical aspects of the problem have been discussed in [7]. Direct quantitative investigation of PT series in QFT and the question of convergence has been presented in [8].

Within the problem of PT convergence, it has been realized that one can identify some «important» terms in the entire PT series and resum them. This first leads to the technique of resummation of «large logs» [9]. The first results even provoked extreme statements about the fate of local QFT [10]. The correct interpretation of the «resummed» results emerged within the technique of renormalization group [11] (as a review, see [12]). Different types of resummation have been later suggested: analysis of classical solutions [13], the famous example of $1/N_c$ expansion in QCD [14], large β_0 limit in QED [15], and naive nonabelianization in QCD [16].

The SM requires a hard use of RG for precision analysis. The energy scales of SM are quite different and coupling constants (especially α_s) are quite large. Indeed, EW physics scale is quite high as given by $M_{Z,W} \sim 90$ GeV and $m_t = 175$ GeV, while the hadronic scales are rather low as $m_b = 5$ GeV and light hadrons at 1 GeV. The process of DIS allows for wide range of scale by varying Q. The modern experiments include hadrons (LHC) which requires the analysis of hadronization and the use of QCD with $\alpha_s(1 \text{ GeV}) = 0.45$. Thus, the use of PT in $\alpha_s(\mu)$ at low energies $\mu = 1$ GeV requires resummation as the convergence is slow.

Actual topics of the present phenomenology in SM include: i) account for higher-order PT corrections with NNLO being almost a standard; ii) precise definition of the expansion parameter or the choice of the renormalization scheme, $\overline{\text{MS}}$ scheme is a standard for technical reasons as dimensional regularization comprises a main computational framework, however, more physical schemes are also in use like α_V from the Coloumb part of the potential of heavy quarks for the dynamics in NR QCD for production near the threshold; iii) resummation of some infinite subsets of the series in α_s. The most popular ways of improving PT is the use of RG that sums powers of logs $(\alpha_s \ln (Q/\mu))^n$, $\alpha_s(\alpha_s \ln (Q/\mu))^n$. But some other ways are also used: β_0 dominance (naive nonabelianization) that sums terms of the form $(\beta_0 \alpha_s)^n$, or effects of analytic continuation between Euclidean and Minkowskian regions that basically deal with terms $(\pi \beta_0 \alpha_s)^{2n}$ [17–19].

Using infinite subsets of PT allows for nonpolynomial terms in α_s. The PT series are asymptotic and resummation may provide terms that interfere with non-PT expansions for description of some processes that account for the terms of the form $\exp[-(1/\alpha_s(Q))]$: higher twists in light-cone type expansions, or condensate type terms for correlation functions at short distance expansions. This means that numerical values of such non-PT parameters depend on how PT series are treated.

This is very important for analysis of hadronic τ decays. Theoretical description is simple and related to e^+e^- annihilation. The record number of PT terms are available. Therefore, the structure of the series is important. Also, it is not an academic exercise as it is important for hadronic contributions to $\alpha_{\text{em}}(M_Z)$ [20,21] and to the muon $g - 2$ [22,23] which are important for Higgs

mass determination and are also key players in constraining new physics search beyond SM [24]. And, it is also a field of interest of D. Kazakov.

1. HIGH-ORDER PT IN QCD

A classic example of application of PT series in QCD is analysis of e^+e^- cross section [25–27]. The relevant expansion parameter is large, such that a proper accuracy requires a long PT expansion. With such a large number of terms, one may already encounter the asymptotic nature of the perturbation series in which case no further increase of precision is possible. The main problem for the theory is convergence and the interpretation of the numerical values given by the series. An additional freedom and also complication is that the expansion parameter is not uniquely determined and the series should be analyzed in a scheme invariant way [28]. Because of the scheme redefinition freedom, it is difficult to judge the quality of convergence of the series. In this section, we present a way to bypass this complication by establishing the relation between observables.

1.1. Comparison of Observables in e^+e^- and τ Decays. Within massless pQCD the same Green's function determines the hadronic contribution to the τ-decay width and the moments of the e^+e^- cross section. This allows one to obtain relations between physical observables in the two processes up to an unprecedented high order of perturbative QCD [29]. A precision measurement of the τ-decay width allows one then to predict the first few moments of the spectral density in e^+e^- annihilations integrated up to $s \sim m_\tau^2$ with high accuracy.

The question of numerical convergence is influenced to a large extent by the freedom of the choice of the renormalization scheme for the truncated perturbation series [30–32]. Therefore, it is desirable to obtain predictions for observables which are renormalization scheme independent.

We compare moments of the spectral density in e^+e^- annihilation and the hadronic contributions to $\Gamma(\tau \rightarrow \nu_\tau + \text{hadrons})$ [33–36]. The reduced decay width r_τ

$$R_\tau = \frac{\Gamma(\tau \rightarrow \nu_\tau + \text{hadrons})}{\Gamma(\tau \rightarrow \nu_\tau + \mu + \bar{\nu}_\mu)} = 3(1 + r_\tau) \tag{1}$$

is determined by massless pQCD, for which the axial and vector contributions are identical. The expansion for r_τ starts directly with $a(\mu^2)$, where $a = \alpha_s/\pi$. In e^+e^- annihilation the cross section is determined by the imaginary part of the vacuum polarization,

$$R_{e^+e^-}(s) = 12\pi \operatorname{Im} \Pi(s) = N_c \sum Q_i^2(1 + r(s)) = 2(1 + r(s)). \tag{2}$$

In pQCD, one has

$$r(s) = a(\mu^2) + (k_1 + \beta_0 L)a^2(\mu^2) + \left(k_2 - \frac{1}{3}\pi^2\beta_0^2 + (2\beta_0 k_1 + \beta_1)L + \beta_0^2 L^2\right)a^3(\mu^2) +$$

$$+ \left(k_3 - \pi^2\beta_0^2 k_1 - \frac{5}{6}\pi^2\beta_0\beta_1 + (3\beta_0 k_2 + 2\beta_1 k_1 + \beta_2 - \pi^2\beta_0^3)L +$$

$$+ \left(3\beta_0 k_1 + \frac{5}{2}\beta_1\right)L^2 + \beta_0^3 L^3\right)a^4(\mu^2) + \ldots \quad (3)$$

with $L = \ln(\mu^2/s)$. We define moments of $r(s)$,

$$r_n(s_0) = (n+1)\int\limits_0^{s_0} \frac{ds}{s_0}\left(\frac{s}{s_0}\right)^n r(s) \quad (4)$$

and

$$r_\tau = 2r_0(m_\tau^2) - 2r_2(m_\tau^2) + r_3(m_\tau^2). \quad (5)$$

Equation (5) is inverted within perturbation theory. One can then express the perturbative representation of one observable, i.e., any given e^+e^- moment $r_n(m_\tau^2)$, in powers of r_τ using the perturbative expansion of the τ-decay observable. The strong coupling constant α_s in any given scheme serves only as an intermediate agent to obtain relations between physical observables. The reexpression of one perturbative observable through another is a perfectly legitimate procedure in perturbation theory, and the result is independent of the choice of the renormalization scheme. One finds

$$r_n(m_\tau^2) = f_{0n}r_\tau + f_{1n}r_\tau^2 + f_{2n}r_\tau^3 + f_{3n}r_\tau^4 + f_{4n}r_\tau^5 + O(r_\tau^6), \quad (6)$$

where coefficients f_{in} are given in the Appendix.

For $r_\tau^{\exp} = 0.216 \pm 0.005$, one can investigate the convergence properties of the series for the first few moments that go

$$r_0/0.216 = 1 - 0.284 - 0.069 + 0.110 + \ldots, \quad (7)$$
$$r_1/0.216 = 1 - 0.527 - 0.143 + 0.177 + \ldots, \quad (8)$$
$$r_2 = 0.216(1 - 0.608 - 0.115 + 0.269 + \ldots, \quad (9)$$
$$r_3 = 0.216(1 - 0.648 - 0.091 + 0.317 + \ldots \quad (10)$$

One clearly see the divergence of the perturbation series for the moments $n = 0, 1, 2, 3$. Since these are scheme-independent perturbative relations between different sets of observables, there is no freedom in redefinition of expansion parameter. One sees very slow(?) convergence. Has asymptotics been already reached?

1.2. Sign of Asymptotics in τ Moments. How can the asymptotic nature of the series reveal itself? With redefinition of the charge, one can create any type of the series that hides the true rate of convergence. Therefore, one should work in a scheme-invariant way. We concentrate on the analysis of the tau system only.

The spectral density has been calculated with a very high degree of accuracy within perturbation theory (see, e.g., [37–40]) and been confronted with experiment to a very high precision [33–36]. In the paper [41], arguments have been given that in some sense, within the finite-order perturbation theory analysis, the ultimate theoretical precision has been reached already now. The limit of precision exists due to the asymptotic nature of the perturbation theory series. The actual magnitude of this limiting precision depends on the numerical value of the coupling constant which is the expansion parameter. We do not touch on power corrections here [42, 43].

The central quantity of interest in the τ system is the hadronic spectral density which can be measured in the finite energy interval $(0, M_\tau = 1.777 \text{ GeV})$. The appropriate quantities to be analyzed are the moments. We define moments of the spectral density by (with M_τ chosen to be the unity of mass)

$$M_n = (n + 1) \int_0^1 \rho(s) s^n \, ds \equiv 1 + m_n. \tag{11}$$

The invariant content of the investigation of the spectrum, i.e., independent of any definition of the charge, is the simultaneous analysis of all the moments.

In order to get rid of artificial scheme-dependent constants in the perturbation theory expressions for the moments, we define an effective coupling $a(s)$ directly on the physical cut through the relation

$$\rho(s) = 1 + a(s). \tag{12}$$

All the constants that may appear due to a particular choice of the renormalization scheme are absorbed into the definition of the effective charge (e.g., [30, 44–46]). When defining the effective charge directly through $\rho(s)$ itself, we get theoretical perturbative corrections to the moments only because of running. Without running, one would have

$$M_n = 1 + a(M_\tau) \equiv 1 + a \quad \text{or} \quad m_n \equiv a. \tag{13}$$

In any given order of PT, the running of the coupling $a(s)$ contains only logarithms of s:

$$a(s) = a + \beta_0 L a^2 + (\beta_1 L + \beta_0^2 L^2) a^3 + \left(\beta_2 L + \frac{5}{2} \beta_1 \beta_0 L^2 + \beta_0^3 L^3 \right) a^4 + \dots, \tag{14}$$

where $a = a(M_\tau^2)$, $L = \ln(M_\tau^2/s)$. At fixed order of PT the effects of running die out for large-n moments, improving the convergence of the series

$$m_0 = a + 2.25a^2 + 14.13a^3 + 87.66a^4 + 654.16a^5,$$
$$m_1 = a + 1.125a^2 + 4.531a^3 + 6.949a^4 - 64.77a^5, \tag{15}$$
$$m_2 = a + 0.75a^2 + 2.458a^3 - 1.032a^4 - 68.98a^5.$$

For large n the moments behave better because the infrared region of integration is suppressed, but in high orders they start to diverge. The coefficients of the series in Eq. (15) are saturated with the lowest power of logarithm for large n.

To suppress experimental errors from the high energy end of the spectrum, the modified system of moments

$$\tilde{M}_{kl} = \frac{(k+1)!}{k!} \int\limits_0^1 \rho(s)(1-s)^k \, ds \equiv 1 + \tilde{m}_k \tag{16}$$

can be used. The integral in Eq. (16) is dominated by contributions from around low scale. A disadvantage of choosing such moments is that the $(1-s)^k$ factor enhances the infrared region strongly and ruins the perturbation theory convergence. As an example, one has

$$\tilde{m}_0 = a + 2.25a^2 + 14.13a^3 + 87.66a^4 + 654.2a^5,$$
$$\tilde{m}_1 = a + 3.375a^2 + 23.72a^3 + 168.4a^4 + 1373.29a^5, \tag{17}$$

which shows bad convergence. The reason is the contribution of the log-term

$$(k+1) \int\limits_0^1 (1-s)^k \ln(1/s) \, ds = \sum_{j=1}^{k+1} \frac{1}{j} \tag{18}$$

and

$$(k+1) \int\limits_0^1 (1-s)^k \ln^2(1/s) \, ds = \left(\sum_{j=1}^{k+1} \frac{1}{j} \right)^2 + \sum_{j=1}^{k+1} \frac{1}{j^2} \tag{19}$$

that grow as $\ln(k)$ and $\ln^2(k)$ for large k.

The large difference in accuracy between m_0 and m_1 is a general feature of the moment observables at fifth order of perturbation theory: one cannot get a uniform smallness at this order for several moments at the same time. For any single moment, one can always redefine the charge and make the series converge well at any desired rate, but then other moments become bad in terms of this charge. The invariant statement about the asymptotic growth is that the system

of moments m_n, with $n = 0$ included, cannot be treated perturbatively at the fifth order of perturbation theory.

To demonstrate this in a scheme-invariant way, we choose the second moment (which is already well convergent) as a definition of our experimental charge and find

$$m_0 = m_2 + 1.5m_2^2 + 9.417m_2^3 + 59.28m_2^4 + 457.54m_2^5,$$
$$m_1 = m_2 + 0.375m_2^2 + 1.51m_2^3 + 2.527m_2^4 - 17.64m_2^5,$$
$$m_2 = m_2, \tag{20}$$
$$m_3 = m_2 - 0.19m_2^2 - 0.544m_2^3 + 0.742m_2^4 + 16.8m_2^5,$$
$$m_4 = m_2 - 0.3m_2^2 - 0.803m_2^3 + 1.69m_2^4 + 27.2m_2^5.$$

The convergence is absent.

1.3. α_s from τ Width in an RG-Invariant Way. Having in mind that the series expansion has reached the ultimate accuracy, we try to avoid expansions and to analyze the system in a concise way [47]. The observation is that any perturbation theory observable generates a scale due to dimensional transmutation and this is its internal scale. It is natural for a numerical analysis (and is our suggestion) to determine this scale fisrt and then to transform the result into an $\overline{\text{MS}}$-scheme parameter using the renormalization group invariance.

We use the explicit renormalization scheme invariance of the theory to bring the result of the perturbation theory calculation into a special scheme first, then we perform a numerical analysis in this particular scheme. Only after that, we transform the obtained numbers into the reference $\overline{\text{MS}}$ scheme.

A dimensional scale in QCD emerges as a boundary value parameterizing the evolution trajectoty of the coupling constant. The renormalization group equation

$$\mu^2 \frac{d}{d\mu^2} a(\mu^2) = \beta(a(\mu^2)), \quad a = \frac{\alpha}{\pi} \tag{21}$$

is solved by the integral

$$\ln\left(\frac{\mu^2}{\Lambda^2}\right) = \Phi(a(\mu^2)) + \int_0^{a(\mu^2)} \left(\frac{1}{\beta(\xi)} - \frac{1}{\beta_2(\xi)}\right) d\xi \tag{22}$$

with

$$\Phi(a) = \frac{1}{a\beta_0} + \frac{\beta_1}{\beta_0^2} \ln\left(\frac{a\beta_0^2}{\beta_0 + a\beta_1}\right), \quad \beta_2(a) = -a^2(\beta_0 + a\beta_1). \tag{23}$$

The $\overline{\text{MS}}$-scheme parameter Λ is defined through expansion

$$a(Q^2) = \frac{1}{\beta_0 L}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\ln(L)}{L^2}\right) + O\left(\frac{1}{L^3}\right), \quad L = \ln\left(\frac{Q^2}{\Lambda^2}\right). \tag{24}$$

The evolution trajectory of the coupling constant is parameterized by the scale parameter Λ and the coefficients of the β function β_i with $i > 2$ (see, e.g., [31]). The evolution is invariant under the renormalization group transformation

$$a \to a(1 + \kappa_1 a + \kappa_2 a^2 + \kappa_3 a^3 + \ldots) \tag{25}$$

with the simultaneous change

$$\Lambda^2 \to \Lambda^2 \, e^{-\kappa_1/\beta_0}, \tag{26}$$

$\beta_{0,1}$ left invariant and

$$\beta_2 \to \beta_2 - \kappa_1^2 \beta_0 + \kappa_2 \beta_0 - \kappa_1 \beta_1,$$
$$\beta_3 \to \beta_3 + 4\kappa_1^3 \beta_0 + 2\kappa_3 \beta_0 + \kappa_1^2 \beta_1 - 2\kappa_1(3\kappa_2 \beta_0 + \beta_2).$$

This invariance is violated in higher orders of the coupling constant because of omitting higher orders for the β functions. This is the source for different numerical outputs of analyses in different schemes.

We introduce an effective charge $a_\tau = \delta_P^{\text{th}}$ [30,46] and extract the parameter Λ_τ which is associated with a_τ through Eq. (22) with an effective β function

$$\beta_\tau(a_\tau) = -a_\tau^2(2.25 + 4a_\tau - 12.3a_\tau^2 + 38.1a_\tau^3). \tag{27}$$

In this procedure the only perturbative objects present are the β functions. We treat both as concise expressions, and at every order of the analysis we use the whole information of the perturbation theory calculation. For the coupling constant in the $\overline{\text{MS}}$ scheme, we finally find

$$\alpha_s = 0.3184 \pm 0.0159. \tag{28}$$

For the reference value for the coupling constant at the scale $M_Z = 91.187$ GeV, we run to this reference scale with the four-loop β function in the $\overline{\text{MS}}$ scheme [48] and three-loop matching conditions at the heavy quark (charm and bottom) thresholds [49] to get [47]

$$\alpha_s(M_Z) = 0.1184 \pm 0.0007(\text{exp.}) \pm 0.0006(\text{cb.}). \tag{29}$$

Can one do better than finite order of PT? Yes, but then one has to resum!

2. RESUMMATION ON THE CUT: $q^2 > 0$

With new high-order corrections of perturbation theory hardly available anymore in cases like e^+e^- annihilation or τ-lepton width, it is tempting to speculate on the general structure of series within perturbation theory (PT) [50–52].

Much attention has recently been paid to possible factorial divergences in PT series [53–57] generated through the integration over an infrared region in momentum space [58].

Direct going beyond FOPT by account of RG logs in $\rho(s)$ does not work. For a simple approximation $\rho(t) = \alpha_s(t)$ and

$$\alpha_s(t) = \frac{\alpha_s(s)}{1 - \beta_0 \alpha_s(s) \ln (s/t)} = \alpha_s(s) \sum_{n=0}^{\infty} (\beta_0 \alpha_s(s) \ln (s/t))^n,$$

one finds

$$F(s) = \frac{1}{s} \int_0^s \alpha_s(t) \, dt = \alpha_s(s) \sum_{n=0}^{\infty} (\beta_0 \alpha_s(s))^n n!$$

with factorial growth that is not Borel summable. The reason is Landau pole in the expression for $\alpha_s(t)$ or divergence of the integrand outside the convergence circle $|a(s) \ln (s/t)| < 1$. Higher-order terms in $\rho(t)$ are important [59].

For estimation of uncertainties of theoretical predictions for the τ-lepton width, different approaches for defining the integration over infrared region are used. This problem has been widely discussed in the literature (see, e.g., [60]). We propose a set of schemes that regularize the infrared behavior of the coupling constant in general and allow one to use any reference scheme for high-energy domain [59]. All these schemes are perturbatively equivalent at high energies. The uncertainties that come from low-energy region are quite essential as our study shows.

2.1. Example with Explicit Solution. Consider first an example with explicit solution for $\alpha_s(\mu^2)$ [59]. Consider a β function

$$\beta(a) = -\frac{a^2}{1 + \kappa a^2}, \quad \kappa > 0$$

with $\beta^{\text{as}}(a) = -a^2 + \ldots$ at $a \to 0$. The RG equation has a solution

$$a(\mu^2) = \frac{-\ln \frac{\mu^2}{\Lambda^2} + \sqrt{\ln^2 \frac{\mu^2}{\Lambda^2} + 4\kappa}}{2\kappa},$$

and the pole at $\mu^2 = \Lambda^2$ of the asymptotic solution $a^{\text{as}}(\mu^2) = (\ln (\mu^2/\Lambda^2))^{-1}$ disappears.

Thus, the particular way of summing an infinite number of specific perturbative terms for the β function can cure the Landau pole problem. There are no nonperturbative terms added, but the freedom of choosing a renormalization scheme for an infinite series was used. It can be considered either as a pure PT result in some particular RG after an infinite resummation or as a sort of Pade

approximation of some real β function that might include nonperturbative terms as well. The only important point for us here is that the running coupling obeying the RG equation with such a β function has a smooth continuation to the infrared region. Because the expansion parameter becomes large in the infrared region the polynomial approximation is invalid in this domain. Here we encounter a particular case of the general situation that the expansion in unphysical parameter α_s is incorrect, and the proper way of action is to expand one physical quantity through another.

2.2. Example with Explicit Expression for the Integral. Consider now the example with

$$\beta(a) = \frac{-a^2}{1 + 2a}.$$

In this case the integral

$$F(s) = \frac{1}{s} \int\limits_0^s a(t)\, dt$$

can be found explicitly [59]. The RG equation for the effective charge $a(s)$ is given by

$$\ln\left(s/\Lambda^2\right) = \frac{1}{a(s)} - 2\ln a(s),$$

and $F(s)$ reads

$$F(s) = \frac{1}{s} \int\limits_0^s a(t)\, dt = a(s) + a(s)^2 - a(s)^2 \exp\left(-\frac{1}{a(s)}\right).$$

The last term gives the «condensate» contribution. Up to logarithmic corrections $F^{\mathrm{cond}}(s) \sim \Lambda^2/s$. This example shows that a change of the evolution in the IR region resums factorials. The last term cannot be detected if integrated by series near $a = 0$.

To study higher orders of PT, the Borel transformation is often used. The Borel analysis for this example goes as follows:

$$F(a) = \int\limits_0^\infty \mathrm{e}^{-\xi/a} B(\xi)\, d\xi, \tag{30}$$

and the Borel image $B(x)$ is $B(x) = 1 + x + (1-x)\,\theta(x-1)$. The PT series with all coefficients known still does not allow one to restore the exact answer through Borel summation. A naive Borel image for polynomials from definition (30) behaves as $B(x) = \sum\limits_{k=1} f_k x^k/(k-1)!$ which is correct at small x but not at large x. The explicit result shows that the Borel image is singular at $x = 1$.

There is still another possibility to treat the PT series: resummation in the complex q^2 plane [19].

3. RESUMMATION ON THE CONTOUR

Integrating the function $\Pi^{\text{had}}(z)$ over a contour in the complex q^2 plane beyond the physical cut $s > 0$, one finds

$$\oint_C \Pi(z)\, dz = \int_{\text{cut}} \rho(s)\, ds,$$

with

$$\rho(s) = \frac{1}{2\pi i} \left(\Pi(s + i0) - \Pi(s - i0) \right).$$

Using the approximation $\Pi^{\text{had}}(z)|_{z \in C} \approx \Pi^{\text{PT}}(z)|_{z \in C}$, which is well justified sufficiently far from the physical cut, one obtains

$$\oint_C \Pi^{\text{had}}(z)\, dz = \int_{\text{cut}} \rho(s)\, ds = \oint_C \Pi^{\text{PT}}(z)\, dz,$$

with the integral over the hadronic spectrum computable in pQCD. The total decay rate of the τ and its moments are the quantities that can be computed this way. The use of RG improved $\Pi^{\text{PT}}(z)$ on the contour has first been considered in [19]. The technique is now known as CIPT [19, 61]. Parameterizing the contour by $Q^2 = M_\tau^2\, e^{i\varphi}$, one obtains for M_{kl}

$$M_{kl} = 1 + m_{kl} = \frac{(-1)^l}{2\pi} \frac{(k+l+1)!}{k!l!} \int_{-\pi}^{\pi} \Pi(M_\tau^2\, e^{i\varphi})(1 + e^{i\varphi})^k\, e^{i(l+1)\varphi}\, d\varphi.$$

This program has been realized for τ decays and extraction of $\alpha_s(M_\tau^2)$ and $m_s(M_\tau^2)$ in a series of papers [62–64].

3.1. Resummation with 4-Loop β Function. Resummation of PT series for τ-decays related observables with 4-loop β function in MS scheme has been studied in [62]. The integrals over the contour give the resummed functions that are analytic at the origin with a finite radius of convergence. The convergence radius of a series given explicitly by integration over the contour can be analyzed through the singularity structure in the complex a_τ plane [62]. In the lowest order example, the radius of convergence is determined by the solution of the equation $1 - i\pi\beta_0 a = 0$, leading to the region of convergence $|a| < 1/\pi\beta_0$. In higher orders of β-function accuracy, the investigation of the convergence properties of

the resummed functions $M_{i,n}(a_\tau, \beta)$ is quite involved. The evolution of a along the contour $Q^2 = m_\tau^2\, e^{i\phi}$, $\phi \in [-\pi, \pi]$ is governed by the renormalization group equation

$$-i\frac{\partial a}{\partial \phi} = \beta(a) = -a^2(1 + ca + c_2 a^2 + c_3 a^3 + \ldots), \qquad (31)$$

and the closest singularity in the complex a_τ plane then determines the convergence radius of the resummed functions $M_{i,n}(a_\tau, \beta)$. The results for critical values of $\alpha_s = \alpha_s(m_\tau^2)$ in increasing orders of β function read

$$\bar{\alpha}_s^{(1)} = 0.444, \quad \bar{\alpha}_s^{(2)} = 0.331, \quad \bar{\alpha}_s^{(3)} = 0.310, \quad \bar{\alpha}_s^{(4)} = 0.299. \qquad (32)$$

The convergence radii become smaller as the order of the β function increases. It is interesting to speculate that the shrinking of the convergence radius continues as one goes to ever higher orders of β-function accuracy, including the possibility that the convergence radius shrinks to zero when the order of the perturbative β-function expansion goes to infinity.

The value of a_τ is outside the convergence region. This means that the perturbative approximation for moments diverges at the scale determined by the experimental data for the semileptonic τ-decay width. The resummed values are not accessible by using higher and higher order approximations of PT in polynomial form.

3.2. Relations between Observables with Resummation. Here we discuss a model of how the technique of using the direct PT relations between observables can give results that are also obtained in more sophisticated resummation approach [63]. Consider two observables given by perturbative series in some given scheme, namely,

$$f(a) = a(1 - a + a^2 - \ldots) = \frac{a}{1 + a} \qquad (33)$$

and

$$g(a) = a(1 - 2a + 4a^2 - \ldots) = \frac{a}{1 + 2a}. \qquad (34)$$

The functions $f(a)$ and $g(a)$ can be seen to be related by

$$g(f) = \frac{f}{1 + f} = f(1 - f + f^2 - \ldots). \qquad (35)$$

If we fit the right-hand side of Eq. (33) to an experimental value of about $f = 0.6$, we get $a = 1.5$. But for this value of the coupling, the series in Eq. (33) diverges. So we cannot get a from it without a proper resummation procedure that in this case is trivially given by the appended exact formula. Consequently, we cannot get a prediction for g, using the series in Eq. (34) in terms of a. On the other hand, the direct relation in terms of the series in Eq. (35) converges perfectly and

gives an unambiguous result for g in terms of measured f. Of course, for such an improvement to occur, one has to analyze in detail the underlying theory and the origin of the series. The analysis of τ system with account for m_s corrections has been performed in finite-order PT in [65, 66], and with resummation in [67]. Resummation on the contour along the above lines in effective scheme has been done in [64]. It happens that numerical results of the used techniques are different. It is important to understand the difference, or answer the question of to what extent the resummation recipe restores the same correlation function.

4. COMPARING RESUMMING TECHNIQUES

Because of arbitrariness of resummation, it is important to understand the relation between different techniques. Clearly, the different resummation of asymptotic series gives functions that have the same asymptotic expansion but differ as the total functions. General lore is that the difference behaves as $\exp{(-1/\alpha_s)}$. This form emerges from the Borel resummation recipe and is confirmed by the explicit example with resummation on the cut [59], although other forms are also possible [68].

Here we discuss the relation between CIPT and resummtion on the cut known as analytical PT [69, 70] following the lines of the paper [71]. At LO the moments can be expanded in a convergent series in α_τ for $\beta_0 \alpha_\tau < 1$. The finite radius of convergence within contour technique of resummation is a general feature which persists in higher orders of the β function [62, 63]. The convergence radius decreases when higher orders of the β function are included. In practice, for $\alpha_\tau^{\mathrm{exp}}/\pi = 0.14$ the relation $\alpha_\tau^{\mathrm{exp}} < 1/\beta_0$ is still marginally valid. However, the exact expression provides an analytic continuation beyond the convergence radius even when α_τ lies outside the convergence radius.

We consider the moment m_{00} and proceed with the analysis by constructing just an efficient computational scheme. Integrating n times by parts, one obtains

$$
m_{00} = \frac{1}{\pi\beta_0}\left\{ \phi + \sum_{j=1}^{n-1}\Gamma(j)\left(\frac{\beta_0\alpha_\tau}{\pi r}\right)^j \sin{(j\phi)}+ \right.
$$

$$
\left. + \frac{\Gamma(n)}{2}\left(\frac{\beta_0\alpha_\tau}{\pi}\right)^n \int_{-\pi}^{\pi} \frac{\mathrm{e}^{i\varphi}\,d\varphi}{(1+i\beta_0\alpha_\tau\varphi/\pi)^n} \right\}, \quad (36)
$$

with the polar coordinate functions r and ϕ defined by

$$
1 \pm i\beta_0\alpha_\tau = r\,\mathrm{e}^{\pm i\phi}, \quad r = \sqrt{1+\beta_0^2\alpha_\tau^2}, \quad \phi = \arctan{(\beta_0\alpha_\tau)}. \quad (37)
$$

The n-fold integration by parts removes a polynomial of order n from the expansion of the logarithm.

One gets an asymptotic expansion with the residual term, i.e., the last term in Eq. (36) being of the formal order α_τ^n. However, the obtained result is not a series expansion in the original coupling determined in the Euclidean domain but a more complicated system of functions related to it, as it is important for APT [70]. The system of functions is ordered, and the asymptotic expansion is valid in the sense of Poincaré. The system of functions is obtained by using the expression for the running coupling in the Euclidean domain and continuing it into the complex plane and onto the cut. When the analytic structure of the initial function is known, asymptotic expansions which converge fast for the first few terms (as a representation in the form of Eq. (36)) are useful for practical calculations. The expansion in Eq. (36) can give a better accuracy (for some n and α_τ) than a direct expansion in α_s. Indeed, this expansion includes a partial resummation of the π^2 terms, which is a consequence of the analytic continuation [17]. Therefore, the expansion can be understood as being done in terms of quantities defined on the cut. Because the region near the real axis is important, the continuation causes a change of the effective expansion parameter $\alpha_\tau \to \alpha_\tau/\sqrt{1 + \beta_0^2 \alpha_\tau^2}$. The first term in the expansion shown in Eq. (36) is just the value for the spectral density expressed through the coupling in the Euclidean domain.

With the concise expression for the moments at hand, one can change the form of the residual term. The relation

$$(n-1)! \left(\frac{\beta_0 \alpha_\tau}{\pi}\right)^n \int\limits_{-\pi}^{\pi} \frac{e^{i\varphi}\, d\varphi}{(1 + i\beta_0 \alpha_\tau \varphi/\pi)^n} =$$

$$= 2\pi\, e^{-\pi/\beta_0 \alpha_\tau} - (n-1)! \left(\frac{\beta_0 \alpha_\tau}{\pi}\right)^n \left(\int\limits_{-\infty}^{-\pi} + \int\limits_{\pi}^{\infty}\right) \frac{e^{i\varphi}\, d\varphi}{(1 + i\beta_0 \alpha_\tau \varphi/\pi)^n}, \quad (38)$$

valid for any n, leads to a representation of the zeroth order moment in the form

$$m_{00} = \frac{1}{\pi\beta_0} \left\{ \pi\, e^{-\pi/\beta_0 \alpha_\tau} + \phi + \sum_{j=1}^{n-1} (j-1)! \left(\frac{\beta_0 \alpha_\tau}{\pi r}\right)^j \sin(j\phi) - \right.$$

$$\left. - \frac{(n-1)!}{2} \left(\frac{\beta_0 \alpha_\tau}{\pi}\right)^n \left(\int\limits_{-\infty}^{-\pi} + \int\limits_{\pi}^{\infty}\right) \frac{e^{i\varphi}\, d\varphi}{(1 + i\beta_0 \alpha_\tau \varphi/\pi)^n} \right\}. \quad (39)$$

Here a «nonperturbative» term $e^{-\pi/\beta_0 a_\tau}$ has appeared.

The moments are analytic functions of α_τ for small values of the coupling α_τ. This means that the nonanalytic piece in Eq. (38) cancels the corres-

ponding piece in the residual term. If the residual term is dropped, the analytic structure drastically changes depending on which representation, either Eq. (36) or (39), is used.

One can recover the form of the moments as integrals over a spectral density by going to the complex plane in φ (see Fig. 1)

$$m_{00} = \frac{\alpha_\tau}{2\pi^2} \int_{-\pi}^{\pi} \frac{(1 + e^{i\varphi})\, d\varphi}{1 + i\beta_0 \alpha_\tau \varphi/\pi}. \qquad (40)$$

This representation is different from Eq. (36) for $n = 1$. They differ by an integral which can be explicitly computed,

Fig. 1. Integration contour in the complex φ plane

$$\frac{\alpha_\tau}{2\pi^2} \int_{-\pi}^{\pi} \frac{d\varphi}{1 + i\beta_0 \alpha_\tau \varphi/\pi} = \frac{1}{\pi\beta_0} \arctan\left(\beta_0 \alpha_\tau\right). \qquad (41)$$

Now we consider the integration over a rectangular contour in the complex φ plane. The part of the contour on the real axis from $-\pi$ to π leads to the moments. The integral over the contour is given by the residue at the pole $\varphi = i\pi/\beta_0 \alpha_\tau$. We thus have

$$m_{00} = \frac{1}{\beta_0}(1 + e^{-\pi/\beta_0\alpha_\tau}) - \frac{1}{\beta_0} \int_0^\infty \frac{(1 - e^{-\xi})\, d\xi}{\pi^2 + (\xi - \pi/\beta_0\alpha_\tau)^2}. \qquad (42)$$

With substitutions $-\pi/\beta_0\alpha_\tau = \ln(\Lambda^2/M_\tau^2)$, $-\xi = \ln(s/M_\tau^2)$, one obtains

$$m_{00} = \frac{1}{\beta_0}\left(1 + \frac{\Lambda^2}{M_\tau^2}\right) - \frac{1}{\beta_0} \int_0^{M_\tau^2} \frac{(1 - s/M_\tau^2)\, ds}{(\pi^2 + \ln^2(s/\Lambda^2))s}. \qquad (43)$$

Finally,

$$m_{00} = \frac{1}{\beta_0}\left(\frac{\Lambda^2}{M_\tau^2}\right) + \frac{1}{\pi\beta_0} \int_0^{M_\tau^2} \arccos\left(\frac{\ln(s/\Lambda^2)}{\sqrt{\pi^2 + \ln^2(s/\Lambda^2)}}\right) \frac{ds}{M_\tau^2}. \qquad (44)$$

One recognizes this representation as an integration over the singularities of $\Pi(q^2)$. In addition to a cut along the positive semi-axis, there also appears a part
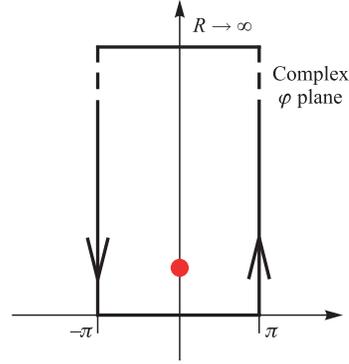
of the singularity on the negative real s axis. This part is a pure mathematical feature of the concrete approximation chosen for $\Pi(q^2)$. The result reads

$$m_{00} = \int\limits_{-\Lambda^2}^{M_\tau^2} \frac{\sigma(s)\, ds}{M_\tau^2} \tag{45}$$

with

$$\sigma(s) = \frac{1}{\beta_0}\theta(\Lambda^2 + s)\theta(-s) + \frac{1}{\pi\beta_0}\theta(s)\arccos\left(\frac{\ln(s/\Lambda^2)}{\sqrt{\pi^2 + \ln^2(s/\Lambda^2)}}\right). \tag{46}$$

This formal result can be reformulated as integration over the spectrum $\sigma(s)$ using Cauchy's theorem. Indeed,

$$\text{Disc}\,\Pi(s) = \frac{2\pi i}{\beta_0}\left\{\theta(\Lambda^2 + s)\theta(-s) + \right.$$

$$\left. + \frac{1}{\pi}\theta(s)\arccos\left(\frac{\ln(s/\Lambda^2)}{\sqrt{\pi^2 + \ln^2(s/\Lambda^2)}}\right)\right\}, \tag{47}$$

which coincides with $\sigma(s)$ in Eq. (46). The part of the spectrum on the positive real axis is an analytic continuation of function $\Pi(Q^2)$ to the cut [17–19,69]. It can be written in the form

$$\sigma_c(s) = \frac{1}{\pi\beta_0}\arctan(\beta_0\alpha(s)). \tag{48}$$

The differential equation determining the continuum part $\sigma_c(s)$ through its initial value $\sigma(M_\tau^2)$ can be constructed by differentiating Eq. (48) with respect to s,

$$s\frac{d\sigma_c(s)}{ds} = -\beta_0\left(\frac{\alpha(s)}{\pi}\right)^2\frac{1}{1 + \beta_0^2\alpha(s)^2}. \tag{49}$$

By inverting Eq. (48) we have $\beta_0\alpha(s) = \tan(\pi\beta_0\sigma_c(s))$ for $s > 0$ and, therefore, obtain

$$s\frac{d}{ds}\sigma_c(s) = -\frac{1}{\pi^2\beta_0}\sin^2(\pi\beta_0\sigma_c(s)) \quad \text{for} \quad s > 0. \tag{50}$$

This equation can indeed be considered as an evolution equation for the spectral density $\sigma_c(s)$ determining $\sigma_c(s)$ through its initial value $\sigma(M_\tau^2)$. Therefore, one can introduce an effective charge $\alpha_M(s) = \pi\sigma_c(s)$ with an evolution equation

$$s\frac{da_M(s)}{ds} = -\frac{1}{\pi^2\beta_0}\sin^2(\pi\beta_0 a_M(s)). \tag{51}$$

Thus, one defines the coupling as the value of the spectral density on the cut far from the IR region. The evolution of this coupling, however, is calculated by taking into account the analytic continuation. Then it has an IR fixed point with the value $a_M(0) = 1/\beta_0$. If Adler's function starts with other power of the coupling constant as it is the case for gluonic observables, for instance, this picture will change. For

$$D(Q^2) = \left(\frac{\alpha_E(Q^2)}{\pi} \right)^2 \tag{52}$$

the spectral density in leading-order β-function approximation reads

$$\rho(s) = \frac{1}{\beta_0^2} \frac{1}{\ln^2(s/\Lambda^2) + \pi^2} = \frac{\alpha^2(s)}{\pi^2(1 + \beta_0^2 \alpha^2(s))}, \tag{53}$$

and an effective coupling is

$$\bar{a}_M(s) = \frac{\alpha(s)}{\pi \sqrt{1 + \beta_0^2 \alpha^2(s)}}. \tag{54}$$

The β function for the effective coupling obtained from Eq. (54)

$$\bar{\beta}_M(\bar{a}_M) = -\beta_0 \bar{a}_M^2 \sqrt{1 - (\pi \beta_0 \bar{a}_M)^2} =$$
$$= -\beta_0 \bar{a}_M^2 \left(1 - \frac{1}{2}(\pi \beta_0 \bar{a}_M)^2 + O(\bar{a}_M^4) \right) \tag{55}$$

differs from the β function of the effective coupling a_M (c.f. Eq. (51))

$$\beta_M(a_M) = -\frac{1}{\pi^2 \beta_0} \sin^2(\pi \beta_0 a_M) =$$
$$= -\beta_0 a_M^2 \left(1 - \frac{1}{3}(\pi \beta_0 a_M)^2 + O(a_M^4) \right) \tag{56}$$

at the next-to-leading order. Thus, the resummation on the contour and on the positive semi-axis for s differs by the integral over the negative real semi-axis for s.

In the contour formulation it is not essential what particular point-by-point behavior in the IR region exists. For the analytically continued correlator, this is not important unless the contour crosses a nonanalytic region. Whatever singularities exist in the IR region (regions B, B', or B'' in Fig. 2), the contour includes them. The resummation on the contour is explicitly perturbative. For resummation on the cut the extrapolation of the running of the coupling constant to the IR region is crucial since one has to interpret integration over IR region.
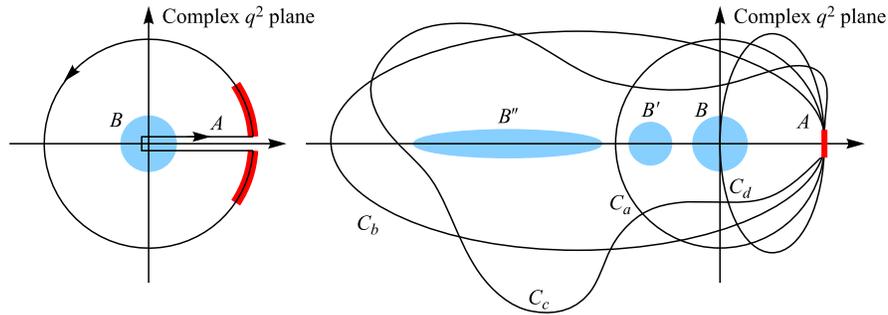
Fig. 2. Contours in the complex plane with a fixation in the region A, taking into account possible occurrences of singularities. The figure part in the left-hand side shows the standard circle path which circumvents the singular region B, while the occurrence of other singularity regions as discussed in the text (regions B' and B'') may lead to different possibilities for choosing a path (C_a, C_b, C_c). The path C_d crosses the singularity region and, therefore, cannot be used from the perturbation theory point of view

Formal manipulations with $t = \beta_0 \alpha(M_\tau^2) \ln (M_\tau^2/s)$ give

$$\int\limits_0^{M_\tau^2} \alpha(s)\, ds = \int\limits_0^{M_\tau^2} \frac{\alpha(M_\tau^2)\, ds}{1 + \beta_0 \alpha(M_\tau^2) \ln (s/M_\tau^2)} = \frac{M_\tau^2}{\beta_0} \int\limits_0^\infty \frac{e^{-t/\beta_0 \alpha(M_\tau^2)}\, dt}{1 - t}. \qquad (57)$$

All expressions are not well-defined from the beginning, the third form being a Borel representation. The problem can be reformulated as a divergence of the asymptotic series. Indeed, by expanding the expression for the running coupling under the integration sign in a PT series, one has

$$\int\limits_0^{M_\tau^2} \alpha(s)\, ds = \sum_n n! \left(\frac{\beta_0 \alpha(M_\tau^2)}{\pi} \right)^n. \qquad (58)$$

The summation of the series in Eq. (58) is related to the interpretation of the integral. Therefore, an integrable behavior of the coupling constant at small s offers a recipe of the summation of the asymptotic series. This solution is strongly model-dependent because the extrapolation of the evolution into the IR region is essentially arbitrary. The explicit form of the extrapolation in Eq. (46) gives an extrapolation motivated by analytic continuation. It can also be considered as a special change of the renormalization scheme [59]. Indeed, for the coupling a_M with the evolution given in Eq. (51), one obtains an IR fixed point. If the analytic

moments are defined simply by

$$m_{00}^{\text{anal}} = \int\limits_{0}^{M_\tau^2} \frac{\sigma_c(s)\,ds}{M_\tau^2},\tag{59}$$

without the negative part of the spectrum that emerges in the exact treatment of quantities originally defined in the Euclidean domain, then one relates these moments to the moments on the contour by the relation

$$m_{00}^{\text{anal}} = -\frac{1}{\beta_0}\,\mathrm{e}^{-\pi/\beta_0\alpha_\tau} + m_{00},\tag{60}$$

which contains explicitly «nonperturbative» term.

The question of uniqueness of resummation is important. However, it is only relevant if many terms of PT expansion are known. It happens that the very uniqueness can be useful for multiloop computations [72,73]. Note that it is also important to have examples of high PT orders behavior in simple QFT models as unique record-breaking computations show [74,75] or in some particular cases as just a good laboratory for checking the techniques [76–78]. The methods are also applied in some exotic areas as nonlinear sigma model, topological theories, SUSY [79].

5. SUMMARY

We have reviewed some ways of interpreting the PT series for describing τ-decay observables. Experimental data are very precise and theory matches it by record numbers of PT terms. High orders of PT are available for many cases, which allows both description of data and extraction of the parameters of the theory with high accuracy. However, PT series converge slowly, requiring improvement that can be achieved through: i) manipulating with schemes for close observables avoiding artificial intermediaries as $\overline{\text{MS}}$ quantities; ii) resummation of different kinds; iii) not using unnecessary expansions — treating polynomials as compact expressions; and some others that you name. This richness is available due to RG properties, which allows one to control scaling behavior and invariance of the theory.

APPENDIX

Phenomenology of hadronic τ decays is contained in the correlator of weak currents $j_\mu^W(x) = \cos\left(\theta_C\right)\bar{u}\gamma_\mu(1 - \gamma_5)\,d + \sin\left(\theta_C\right)\bar{u}\gamma_\mu(1 - \gamma_5)s$

$$i\int \langle Tj_\mu^W(x)j_\nu^{W+}(0)\rangle\,\mathrm{e}^{iqx}\,dx = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi^{\text{had}}(q^2)$$

with $\rho(s) = \mathrm{Im}\,\Pi^{\mathrm{had}}(s+i0)/\pi$ and

$$\Pi^{\mathrm{had}}(q^2) = \int \frac{\rho(s)\,ds}{s-q^2}.$$

Total τ-lepton rate

$$R_{\tau S=0} = \frac{\Gamma(\tau \to H_{S=0}\nu)}{\Gamma(\tau \to l\bar{\nu}\nu)} \sim \int\limits_0^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + \frac{2s}{M_\tau^2}\right)\rho(s)\,ds$$

is a useful observable that is experimentally measured with high precision. One considers moments of the spectral density of the form

$$M_{kl} = \frac{(k+l+1)!}{k!l!} \int\limits_0^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^k \left(\frac{s}{M_\tau^2}\right)^l \frac{\rho(s)\,ds}{M_\tau^2} \equiv 1 + m_{kl},$$

which contain information about the hadronic spectral density $\rho^{\mathrm{had}}(s)$ since $\rho^{\mathrm{had}}(s)$ itself is a distribution. The function $\rho(s)$ is related to Adler's function $D(Q^2)$:

$$D(Q^2) = -Q^2 \frac{d}{dQ^2}\Pi(Q^2) = Q^2 \int \frac{\rho(s)\,ds}{(s+Q^2)^2},$$

where $Q^2 = -q^2$ and $D(Q^2)$ is computable in PT. In the massless limit, PT expression for Adler's function reads

$$D(Q^2) = 1 + a_s + k_1 a_s^2 + k_2 a_s^3 + k_3 a_s^4 + O(a_s^5)$$

with $a_s = \alpha_s(Q^2)/\pi$, and in the $\overline{\mathrm{MS}}$ scheme

$$k_1 = \frac{299}{24} - 9\zeta(3), \quad k_2 = \frac{58057}{288} - \frac{779}{4}\zeta(3) + \frac{75}{2}\zeta(5), \tag{61}$$

while $k_3 = 49.08$ [38,40]. The correction to the total width δ_P is

$$\delta_P^{\mathrm{th}} = a_s + 5.2023a_s^2 + 26.366a_s^3 + (78.003 + 49.08)a_s^4 + O(a_s^5), \tag{62}$$

which corresponds to the experimental value $\delta^{\mathrm{exp}} = 0.216 \pm 0.005$.

The renormalization group equation for $a(\mu^2)$ reads

$$\mu^2 \frac{da}{d\mu^2} = \beta(a) = -a^2(\beta_0 + \beta_1 a + \beta_2 a^2 + \beta_3 a^3 + \ldots) \tag{63}$$

with

$$\beta_0 = \frac{9}{4}, \quad \beta_1 = 4, \quad \beta_2 = \frac{3863}{384}, \quad \beta_3 = \frac{140599}{4608} + \frac{445}{32}\zeta(3) \qquad (64)$$

for $N_c = n_f = 3$ [48, 80].

The coefficients f_{in} are

$$f_{0n} = \tilde{I}(0,n), \quad f_{1n} = \beta_0 \tilde{I}(1,n), \quad f_{2n} = \beta_0^2(\tilde{I}(2,n) + \rho_1 \tilde{I}(1,n)),$$

$$f_{3n} = \beta_0^3\left(\tilde{I}(3,n) + \left(I_\tau(2) - I_\tau(1)^2 - \frac{1}{3}\pi^2\right)\tilde{I}(1,n) + \frac{5}{2}\rho_1\tilde{I}(2,n) + \rho_2\tilde{I}(1,n)\right),$$

$$f_{4n} = \beta_0^4\left(\tilde{I}(4,n) - 3\left(I_\tau(2) - I_\tau(1)^2 - \frac{1}{3}\pi^2\right)\tilde{I}(2,n) + \right. \qquad (65)$$

$$+ 2(I_\tau(3) - 3I_\tau(1)I_\tau(2) + 2I_\tau(1)^3)\tilde{I}(1,n)+$$

$$+ \rho_1\left(\frac{13}{3}\tilde{I}(3,n) + 5\left(I_\tau(2) - I_\tau(1)^2 - \frac{1}{3}\pi^2\right)\tilde{I}(1,n)\right)+$$

$$\left. + 3\rho_2\tilde{I}(2,n) + \rho_3\tilde{I}(1,n)\right)$$

with

$$I(m,n) = \frac{m!}{(n+1)^m}, \quad I_\tau(m) = 2I(m,0) - 2I(m,2) + I(m,3),$$

$$\qquad (66)$$

$$I(m,n) = I(m) + \sum_{p=0}^{m}\binom{m}{p}I_\tau(p)\,\tilde{I}(m-p,n).$$

The ρ_i are scheme-independent quantities given by

$$\rho_1 = \frac{\beta_1}{\beta_0^2}, \quad \rho_2 = \frac{1}{\beta_0^3}\left[\beta_2 - \beta_1 k_1 + \beta_0(k_2 - k_1^2)\right],$$

$$\qquad (67)$$

$$\rho_3 = \frac{1}{\beta_0^4}\left[\beta_3 - 2\beta_2 k_1 + \beta_1 k_1^2 + 2\beta_0(k_3 - 3k_1 k_2 + 2k_1^3)\right].$$

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