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GREEN FUNCTIONS IN STOCHASTIC FIELD THEORY

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Functional representations are reviewed for the generating function of Green functions of stochastic problems stated either with the use of the Fokker–Planck equation or the master equation. Both cases are treated in a unified manner based on the operator approach similar to quantum mechanics. Solution of a second-order stochastic differential equation in the framework of stochastic field theory is constructed. Ambiguities in the mathematical formulation of stochastic field theory are discussed. The Schwinger–Keldysh representation is constructed for the Green functions of the stochastic field theory which yields a functional-integral representation with local action but without the explicit functional Jacobi determinant or ghost fields.

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1. INTRODUCTION

Diverse models describing evolution of physical, chemical, biological, social and financial processes are presented in the form of differential equations with the implicit understanding that they are mean-field equations for averages of quantities describing intrinsically random processes. There are two mainstream approaches to take into account the intrinsic randomness of such processes. The most straightforward way is to include some random function (source) in the original differential equation to obtain a stochastic differential equation (SDE). The alternative approach is to construct balance equations for the probability density functions of suitable random quantities in the system, which gives rise to master equations (ME).

1.1. Stochastic Differential Equation. To take fluctuations into account, a traditional and straightforward way to proceed is to introduce a source of randomness directly in the mean-field equation. This approach leads to description of fluctuations by the Langevin equation (stochastic differential equation)

$$\frac{\partial\varphi}{\partial t} = -K\varphi + U(\varphi) + fb(\varphi), \quad \langle f(t)f(t')\rangle = \delta(t-t')\,D. \tag{1}$$

The white-noise stochastic differential equation (1) is mathematically ill-defined. The straightforward way to use a δ sequence with finite correlation times to obtain

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a mathematically consistent equation with subsequent passing to the δ-function limit

$$\langle f(t,\mathbf{x})f(t',\mathbf{x}')\rangle = \overline{D}(t,\mathbf{x};t',\mathbf{x}') \to \delta(t-t')\,D(\mathbf{x},\mathbf{x}'), \quad t'\to t \tag{2}$$

leads to the Stratonovich interpretation of the SDE (1) (see, e.g., [1]). However, in theoretical analysis, as well as in perturbation theory, the Ito interpretation is more convenient, but not so easily approached from the finite correlation time problem.

In the overwhelming majority of applications the SDE with the first-order time derivative is used. There are, however, situations in which the second-order time derivative is included. Here, the second-order stochastic differential equation in the form of the direct generalization of (1) shall be discussed:

$$m\frac{\partial^2\varphi}{\partial t^2} + \beta\frac{\partial\varphi}{\partial t} = -K\varphi + U(\varphi) + fb(\varphi). \tag{3}$$

The use of second-order SDE seems to be growingly popular in financial applications.

Let us start from solution of the first-order equation (1) for simplicity. The standard tree-graph solution for the function φ

$$\varphi[\chi,f] = (\partial_t + K)^{-1}\chi + \text{ tree graphs}$$

yields the generating function of correlation functions of the stochastic process $\varphi[\chi,f]$ with the aid of Wick's theorem

$$G(J) = \left\langle \mathrm{e}^{\varphi[\chi,f]J} \right\rangle = \int \mathcal{D}f\,\exp\left(-\frac{1}{2}f\overline{D}^{-1}f\right)\mathrm{e}^{\varphi[\chi,f]J}. \tag{4}$$

It should be noted that in this representation there are closed loops of the free-field Green function $\Delta = (\partial_t + K)^{-1}$. Representation (4) is inconvenient for practical calculations; therefore, it is a widespread trick to change variables to arrive at the Martin–Siggia–Rose (MSR) field theory [2] with the generating function

$$\begin{aligned} G(J) = \int \mathcal{D}\varphi\,\langle\delta(\varphi - \varphi[\chi,f])\rangle\,\mathrm{e}^{\varphi J} = \\ = \iiint \mathcal{D}f\,\mathcal{D}\varphi\,\mathcal{D}\tilde\varphi\,\left|\det(-\partial_t - K + U' + fb')\right| \times \\ \times \exp\left\{-\frac{1}{2}f\overline{D}^{-1}f + \tilde\varphi[-\partial_t\varphi - K\varphi + U(\varphi) + fb(\varphi)] + \varphi J\right\}. \end{aligned} \tag{5}$$

According to the standard Feynman rules, the exponential expression in the right-hand side of (5) gives rise to graphs containing closed loops of propagators Δ.

In case of the first-order SDE only closed loops with a single propagator survive, but in case of second-order SDE graphs with loops of any number of propagators may appear. Loop expansion of the determinant $|\det(-\partial_t - K + U' + fb')|$ serves to remove Δ loops. The loop expansion of the determinant itself is somewhat ambiguous in a manner similar to the ambiguity of the SDE itself.

The point of introducing of the SDE with white noise is to avoid dealing with the limit (2) explicitly. To this end, instead of the mathematically problematic SDE (1), the stochastic problem may be equivalently stated in terms of the Fokker–Planck equation (FPE) for both the conditional probability density $p(\varphi, t|\varphi_0, t_0)$ and the probability density $p(\varphi, t)$ of the variable φ. Other joint probability densities follow from the assumption of a Markov process. The main advantage of the Fokker–Planck equation is that the equation itself is a completely well-defined partial differential (or functional-differential for field variables) equation. The ambiguity of the Langevin problem shows in the fact that the FPE is different for different interpretations of the SDE.

With the use of the standard rules [1], it is readily seen that the SDE (1) in the Stratonovich sense yields the FPE

$$\frac{\partial}{\partial t} p(\varphi, t|\varphi_0, t_0) = -\frac{\partial}{\partial \varphi}\left\{[-K\varphi + U(\varphi)]\, p(\varphi, t|\varphi_0, t_0)\right\} + \\ + \frac{1}{2}\frac{\partial}{\partial \varphi}\left\{b(\varphi)\frac{\partial}{\partial \varphi}\left[Db(\varphi)p(\varphi, t|\varphi_0, t_0)\right]\right\}, \quad (6)$$

whereas the Ito interpretation of the same SDE yields the FPE

$$\frac{\partial}{\partial t} p(\varphi, t|\varphi_0, t_0) = -\frac{\partial}{\partial \varphi}\left\{[-K\varphi + U(\varphi)]p(\varphi, t|\varphi_0, t_0)\right\} + \\ + \frac{1}{2}\frac{\partial^2}{\partial \varphi^2}\left[b(\varphi)Db(\varphi)p(\varphi, t|\varphi_0, t_0)\right]. \quad (7)$$

These equations coincide when $b(\varphi)$ is a constant (additive noise).

In case of the second-order SDE (3), it is convenient to cast the problem into the form of a set of two first-order equations for a doubled set of variables:

$$\frac{\partial \varphi}{\partial t} = v, \quad m\frac{\partial v}{\partial t} + \beta v = -K\varphi + U(\varphi) + fb(\varphi).$$

Standard rules yield for the conditional PDF of this problem $p(t, \varphi, v|t_0, \varphi_0, v_0)$ the Fokker–Planck equation

$$\frac{\partial}{\partial t} p = -\frac{\partial}{\partial \varphi}(vp) + \frac{\partial}{\partial v}\left[\left(\frac{\beta}{m}v + \frac{K\varphi}{m} - \frac{U}{m}\right)p\right] + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left[bDbp\right], \quad (8)$$

where the Ito interpretation of the SDE has been adopted. As will be seen further, this is the more convenient choice for the construction of the functional representation.

1.2. Master Equation. Markov processes described in terms of the Fokker–Planck equation have continuous sample paths; i.e., $\varphi = \varphi(t)$ is a continuous function of time. Not all interesting stochastic processes belong to this category. A wide class of such processes describe changes in occupation numbers (e.g., individuals of some population, molecules in chemical reaction) which cannot be naturally described by continuous paths. This kind of processes are described by *master equations*.

The generic form of a master equation written for the conditional probability density $p(\varphi, t|\varphi_0, t_0)$ of a Markov process is

$$\frac{\partial}{\partial t} p(\varphi, t\,|\,\varphi_0, t_0) = \int d\chi [W(\varphi\,|\,\chi, t) p(\chi, t\,|\,\varphi_0, t_0) - W(\chi\,|\,\varphi, t) p(\varphi, t\,|\,\varphi_0, t_0)], \tag{9}$$

where $W(\varphi|\chi, t)$ is the *transition probability* per unit time, whose formal definition from the differential Kolmogorov equation is (for all $\varepsilon > 0$)

$$W(\varphi\,|\,\chi, t) = \lim_{\Delta t \to 0} \frac{p(\varphi, t + \Delta t\,|\,\chi, t)}{\Delta t},$$

uniformly in φ, χ and t for all $|\varphi - \chi| \geqslant \varepsilon$ [1].

Here, the master equation shall be used for discrete variables (occupation numbers). In this case the discontinuous character of the paths of the *jump processes* described by the master equation is especially transparent. The transition probabilities are usually simple functions of the occupation number n. As an example, consider the generic master equation for the *stochastic Verhulst model* [3]

$$\frac{dP(t,n)}{dt} = [\beta(n+1) + \gamma(n+1)^2] P(t, n+1) + \lambda(n-1) P(t, n-1) - \\ - (\beta n + \lambda n + \gamma n^2) P(t, n), \tag{10}$$

where β is the death rate, λ the birth rate and γ the damping coefficient necessary to bring about a saturation for the population. The choice $\lambda = \beta = 0$ leads to the master equation for the annihilation reaction $A + A \to A$. The set of master equations may also be cast in the form of an evolution equation of the type of Schrödinger equation in a Fock space of many-particle quantum mechanics. This representation is due to Doi [4]. Contrary to the widely known Martin–Siggia–Rose approach, the method of Doi has gained due attention only recently.

2. STOCHASTIC FIELD THEORY

Formulation of stochastic problems in terms of quantum field theory gives rise to stochastic field theory. The two main approaches are the analysis of the solution of the SDE with the use of field operators initiated by Martin, Siggia and

Rose and the somewhat analogous representation of stochastic problems stated in terms of master equations due to Doi. In case of stochastic processes with continuous sample paths, the two approaches are equivalent, but there is no consistent SDE representation in the case of discontinuous sample paths.

2.1. Field Theory for Fokker–Planck Equation. The Fokker–Planck equation is similar to the Schrödinger equation (with imaginary time). Using this analogy, the solution of the FPE, as well as calculation of expectation values, may be represented in a way analogous to quantum field theory [5]. Construction with the FPE as the starting point gives rise to the famous Martin–Siggia–Rose solution of the SDE [2], but avoids ambiguities inherent in the SDE (they have been fixed by the choice of the FPE).

Consider, for definiteness, the Fokker–Planck equation (7) corresponding to the Ito interpretation of the Langevin equation (1). Introduce — in analogy with Dirac's notation in quantum mechanics — the state vector $|p_t\rangle$ according to the following representation of the PDF:

$$p(\varphi,t)=\langle\varphi|p_t\rangle,$$

which is the solution of the FPE (7) with the initial condition $p(\varphi,0)=p_0(\varphi)$. To construct the evolution operator for the state vector, introduce momentum and coordinate operators in the manner of quantum mechanics by relations

$$\hat\pi f(\varphi)=-\frac{\partial}{\partial\varphi}f(\varphi),\quad \hat\varphi f(\varphi)=\varphi f(\varphi),\quad [\hat\varphi,\hat\pi]=1.$$

In these terms, the FPE for the PDF gives rise to the evolution equation for the state vector in the form

$$\frac{\partial}{\partial t}|p_t\rangle=\hat L|p_t\rangle,$$

where the Liouville operator for the FPE corresponding to the Ito interpretation of the SDE assumes, according to (7), the form

$$\hat L=\hat\pi[-K\hat\varphi+U(\hat\varphi)]+\frac{1}{2}\hat\pi^2 b(\hat\varphi)\,Db(\hat\varphi). \tag{11}$$

Note that, contrary to quantum mechanics, there is no ordering ambiguity in the construction of the Liouville operator here.

In this notation, the conditional PDF may be expressed as the matrix element

$$p\,(\varphi,t\,|\,\varphi_0,t_0)=\langle\varphi|\,\mathrm{e}^{\hat L(t-t_0)}\,|\varphi_0\rangle. \tag{12}$$

The Fokker–Planck equation (8) of the second-order SDE (3) gives rise to the set of operators

$$\hat\varphi,\ \hat\pi=-\frac{\partial}{\partial\varphi},\ [\hat\varphi,\hat\pi]=1;\quad \hat v,\ \hat\eta=-\frac{\partial}{\partial v},\ [\hat v,\hat\eta]=1.$$

The Liouville operator of the Fokker–Planck equation (8) for the single-time PDF $p(t,\varphi,v) = \langle \varphi, v | p_t \rangle$ is

$$\hat{L} = \hat{\pi}\hat{v} - \hat{\eta}\left[\frac{\beta}{m}\hat{v} + \frac{K}{m}\hat{\varphi} - \frac{U(\hat{\varphi})}{m}\right] + \frac{1}{2}\hat{\eta}^2 b(\hat{\varphi}) D b(\hat{\varphi}). \tag{13}$$

The conditional PDF $p(t,\phi,v|t_0,\phi_0,v_0)$ is, as usual, the Green function of this equation.

Introduce time-dependent operators $\hat{\varphi}(t)$ in the Heisenberg picture of imaginary time quantum mechanics (i.e., Euclidean quantum mechanics):

$$\hat{\varphi}_H(t) = \mathrm{e}^{-\hat{L}(t-t_0)}\hat{\varphi}\,\mathrm{e}^{\hat{L}(t-t_0)}, \tag{14}$$

and define the time-ordered product (chronological product, T product) of time-dependent operators

$$T[\hat{A}_1(t_1)\cdots\hat{A}_n(t_n)] = \sum_{P(1,\ldots,n)} P[\theta\,(t_1\cdots t_n)\,\hat{A}_1(t_1)\cdots\hat{A}_n(t_n)], \tag{15}$$

where

$$\theta(t_1\cdots t_n) \equiv \theta(t_1-t_2)\theta(t_2-t_3)\cdots\theta(t_{n-1}-t_n).$$

In definition (15) the sum is taken over all permutations of the labels of the time arguments $\{t_i\}_{i=1}^n$ and the operators in each term are put in the order of growing time arguments from the right to the left. Thus, under the T-product sign operators commute. It should be noted that the definition of the time-ordered product should be amended for coinciding time arguments. Here, it is convenient to define the T product at coinciding time arguments as the normal-ordered product (N product) (see, e.g., [6]).

Introduce then the n-point Green function as the quantum-mechanical expectation value of the T product of n operators (14)

$$G_n(t_1,t_2,\ldots,t_n) = \mathrm{Tr}\,\{\hat{p}_0 T[\hat{\varphi}_H(t_1)\hat{\varphi}_H(t_2)\cdots\hat{\varphi}_H(t_n)]\} \tag{16}$$

determined by the trace Tr and the density operator

$$\hat{p}_0 = \int d\varphi |p_0\rangle\langle\varphi|. \tag{17}$$

Choosing, for definiteness, the time sequence $t_1 > t_2 > t_3 > \ldots > t_{n-1} > t_n > t_0$, it is readily seen by direct substitution of relations (12), (14) and (17) in (16) with the aid of the normalization conditions of the PDF and insertions of the resolution of the identity $\int d\varphi\,|\varphi\rangle\langle\varphi| = 1$ that

$$\int d\varphi_1\cdots\int d\varphi_n\,\varphi_1\cdots\varphi_n p\,(\varphi_1,t_1;\varphi_2,t_2;\ldots;\varphi_n,t_n) = G_n(t_1,t_2,\ldots,t_n); \tag{18}$$

i.e., the Green function (16) is equal to the moment function (18). This relation connects the operator approach to evaluation of moments of the random process.

2.2. Field Theory for Master Equation. The set of master equations for $P(t,n)$ may be cast into a single kinetic equation by the «second quantization» of Doi [4]. Let us first construct a suitable Fock space spanned by the annihilation and creation operators $\hat{a}$, $\hat{a}^+$ and the basis vectors $|n\rangle$:

$$\hat{a}\,|0\rangle = 0, \quad \hat{a}^+\,|n\rangle = |n+1\rangle, \quad [\,\hat{a}, \hat{a}^+] = 1, \quad \langle n|m\rangle = n!\delta_{nm}. \tag{19}$$

The set of master equations yields for the state vector

$$|p_t\rangle = \sum_{n=0}^{\infty} P(t,n)|n\rangle \tag{20}$$

a kinetic equation in the form of a single evolution equation for the state vector (20) without any explicit dependence on the occupation number:

$$\frac{d|P_t\rangle}{dt} = \hat{L}(\hat{a}^+,\, \hat{a})|P_t\rangle, \tag{21}$$

where the Liouville operator $\hat{L}(\hat{a}^+,\, \hat{a})$ is constructed from the set of master equations according to rules:

$$\begin{aligned} nP(t,n)|n\rangle &= \hat{a}^+\hat{a}P(t,n)|n\rangle, \\ nP(t,n)|n-1\rangle &= \hat{a}P(t,n)|n\rangle, \\ nP(t,n)|n+1\rangle &= \hat{a}^+\hat{a}^+\hat{a}P(t,n)|n\rangle. \end{aligned}$$

For instance, the Liouville operator for the stochastic Verhulst model (10) is

$$\hat{L}(\hat{a}^+, \hat{a}) = \beta(I - \hat{a}^+)\hat{a} + \gamma(I - \hat{a}^+)\,\hat{a}\hat{a}^+\hat{a} + \lambda(\hat{a}^+ - I)\,\hat{a}^+\hat{a}. \tag{22}$$

Equation (21) gives rise to the Heisenberg evolution of operators in analogy with (14). To represent expectation values of occupation-number dependent quantities in the operator formalism use the projection vector $\langle P|$:

$$\langle P| = \sum_{n=0}^{\infty} \frac{1}{n!}\,\langle n| = \sum_{n=0}^{\infty} \frac{1}{n!}\,\langle 0|\,\hat{a}^n = \langle 0|\,\mathrm{e}^{\hat{a}}. \tag{23}$$

Consider the Green function of occupation-number operators $\hat{n}_H(t) = \hat{a}_H^+(t)\hat{a}_H(t)$:

$$G_m(t_1, t_2, \ldots, t_m) = \mathrm{Tr}\left\{\hat{P}_0 T[\hat{n}_H(t_1)\hat{n}_H(t_2)\cdots\hat{n}_H(t_m)]\right\}, \tag{24}$$

with the density operator

$$\hat{P}_0 = |p_0\rangle\langle P|. \tag{25}$$

From definitions it follows that the conditional probability density function for the master equation may be written as (the factorial in front of the matrix element is due to the normalization (19) of the basis states)

$$P(n,t|n_0,t_0) = \frac{1}{n!}\,\langle n|\,\mathrm{e}^{\hat{L}(t-t_0)}\,|n_0\rangle. \qquad (26)$$

Choosing, for definiteness, the time sequence $t_1 > t_2 > t_3 > \ldots > t_{n-1} > t_n > t_0$, it is readily seen by direct substitution of relations (26) and (25) in (24) with the aid of the normalization conditions of the PDF and insertions of the resolution of the identity

$$\sum_n \frac{1}{n!}\,|n\rangle\langle n| = 1$$

that

$$\sum_{n_1}\cdots\sum_{n_m} n_1\cdots n_m P\left(n_1,t_1;n_2,t_2;\ldots;n_m,t_m\right) = G_m(t_1,t_2,\ldots,t_m); \qquad (27)$$

i.e., the Green function (24) is equal to the moment function (27). This relation connects the operator approach to evaluation of moments of the random process described by a master equation.

3. GENERATING FUNCTIONALS FOR GREEN FUNCTIONS

The generic representation for the generating function of Green functions introduced in the previous section may be written in the form

$$G_{if}(J) = \mathrm{Tr}\left[\hat{\rho}_0 T\,\mathrm{e}^{\hat{S}_J}\right], \qquad (28)$$

where the source term and the density operator are

$$\hat{S}_J = \int_{t_i}^{t_f} dt\,\hat{\varphi}_H(t)J(t), \quad \hat{\rho}_0 = \int d\varphi|p_0\rangle\langle\varphi|$$

for the Fokker–Planck equation, or, for the master equation,

$$\hat{S}_J = \int_{t_i}^{t_f} dt\,\hat{a}_H^+(t)\hat{a}_H(t)J(t), \quad \hat{\rho}_0 = |P_0\rangle\langle P|.$$

Perturbation theory is constructed in the Dirac (interaction) picture of quantum mechanics. To this end, the Liouville operator is decomposed to the free and interaction parts $\hat{L}_0$ and $\hat{L}_I$:

$$\hat{L} = \hat{L}_0 + \hat{L}_I.$$

In our cases, the convenient choices are $\hat{L}_0 = -K\hat{\pi}\hat{\varphi}$ for (11), $\hat{L}_0 = \hat{\pi}\hat{v} - \dfrac{\beta}{m}\hat{\eta}\hat{v}$ for (13) and $\hat{L}_0 = -\beta(\hat{a}^+ - I)\hat{a}$ for (22), where $K > 0$, $\beta > 0$. Time-dependent operators in the interaction picture are defined according to

$$\hat{\varphi}(t) = \mathrm{e}^{-(t-t_0)\hat{L}_0}\,\hat{\varphi}\,\mathrm{e}^{(t-t_0)\hat{L}_0}.$$

The corresponding evolution operator may be expressed in terms of the T product (here, $t > t' > t_0$):

$$\hat{U}(t-t_0, t'-t_0) = \mathrm{e}^{-(t-t_0)\hat{L}_0}\,\mathrm{e}^{(t-t')\hat{L}}\,\mathrm{e}^{(t'-t_0)\hat{L}_0} = \\ = \mathrm{e}^{\hat{L}_0 t_0}\,T\exp\left[\int_{t'}^{t}\hat{L}_I(u)\,du\right]\mathrm{e}^{-\hat{L}_0 t_0}. \quad (29)$$

Note that the Dirac operators in the T product do not carry any t_0-dependence in (29). In the interaction picture the T product in (28) may be written as [6]

$$T\,\mathrm{e}^{\hat{S}_J} = \mathrm{e}^{\hat{L}_0 t_0}\,\hat{U}(t_0, t_f)T\left[\mathrm{e}^{\hat{S}_J+\hat{S}_I}\right]\hat{U}(t_i, t_0)\,\mathrm{e}^{-\hat{L}_0 t_0}, \quad (30)$$

where $\hat{S}_I = \int_{t_i}^{t_f}\hat{L}_I(t)\,dt$ and the time instants t_f and t_i are chosen such that $t_f > t_l > t_i > t_0$ for all t_l, $l = 1, 2, \ldots, n$. The evolution operator with the reversed order of time arguments may be cast into the form of the *antichronologically ordered* exponential ($t > t_0$)

$$\hat{U}(t_0, t) = \tilde{T}\,\exp\left[-\int_{t_0}^{t}\hat{L}(t)\,dt\right]. \quad (31)$$

With the use of representations (29) and (31), the chronological product in the generating function (28) may be expressed as the product of three chronological products:

$$T\,\mathrm{e}^{\hat{S}_J} = \mathrm{e}^{\hat{L}_0 t_0}\tilde{T}\,\exp\left[-\int_{t_0}^{t_f}\hat{L}(t)\,dt\right]T\left[\mathrm{e}^{\hat{S}_J+\hat{S}_I}\right]\times \\ \times T\,\exp\left[\int_{t_0}^{t_i}\hat{L}(t)\,dt\right]\mathrm{e}^{-\hat{L}_0 t_0}. \quad (32)$$

The three T products in (32) fuse — due to Wick's theorems — in a *normal product* ($\hat{\pi}$ to the left of $\hat{\varphi}$ or $\hat{a}^+$ to the left of $\hat{a}$), giving rise to the representation [6]

$$
\begin{aligned}
G_{if}(J) = \mathrm{Tr}\, \mathrm{e}^{-\hat{L}_0 t_0}\, \hat{\rho}_0\, \mathrm{e}^{\hat{L}_0 t_0} \times \\
\times N \Bigg\{ \exp\left[\frac{1}{2} \frac{\delta}{\delta\phi_1} \tilde{\Delta} \frac{\delta}{\delta\phi_1} + \frac{1}{2} \frac{\delta}{\delta\phi_2} \Delta \frac{\delta}{\delta\phi_2} + \frac{\delta}{\delta\phi_1} n \frac{\delta}{\delta\phi_2} \right] \times \\
\times \exp\left[S_J(\phi_2) - \int_{t_0}^{t_f} L_I(\phi_1)\, du + \int_{t_0}^{t_f} L_I(\phi_2)\, du \right]\Bigg|_{\phi_1=\phi_2=\hat{\phi}} \Bigg\}, \qquad (33)
\end{aligned}
$$

where $\hat{\phi}$ is a two-component shorthand for all the operators appearing in $\hat{L}_I$, i.e., either $\hat{\phi} = (\hat{\varphi}, \hat{\pi})$ or $\hat{\phi} = (\hat{a}, \hat{a}^+)$. In (33), the auxiliary field variables ϕ_1 and ϕ_2 correspond to functional arguments prescribed to the antichronological product and the chronological products in (32), respectively. Originally, in the functional representation of Wick's theorem for (32) to each T product a separate field variable is prescribed, but those corresponding to the two consecutive rightmost T-product factors may be replaced by a single variable ϕ_2. The propagators and contractions in (33) are standard [6]:

$$
\begin{aligned}
\tilde{\Delta}(t,t') &= \tilde{T}[\hat{\phi}(t)\hat{\phi}(t')] - N[\hat{\phi}(t)\hat{\phi}(t')], \\
\Delta(t,t') &= T[\hat{\phi}(t)\hat{\phi}(t')] - N[\hat{\phi}(t)\hat{\phi}(t')], \\
n(t,t') &= \hat{\phi}(t)\hat{\phi}(t') - N[\hat{\phi}(t)\hat{\phi}(t')].
\end{aligned}
$$

The functional L_I in (33) (not an operator any more) is ambiguous, however, because the chronological products have not been defined at coinciding time arguments. The choice of the value of the T and $\tilde{T}$ products at coinciding time arguments fixes the form of the interaction functional L_I. It should be noted that this choice affects the propagators as well. The *normal form* obtained by replacing operators by functions in the representation obeying $\hat{L}_I = N[\hat{L}_I]$ (see [6] for details) is the most convenient here. For the T product this definition boils down to choosing the temporal step function equal to zero at the origin. Thus, the practical rule to resolve this ambiguity sounds similar to that in the Ito interpretation of the SDE [1], but it should be borne in mind that the latter ambiguity has already been fixed by different means and there is no obligation to make the same choice here. Henceforth, the normal form of L_I is implied together with the corresponding choice: $\Delta(t,t) = 0$, $\tilde{\Delta}(t,t) = 0$.

In quantum field theory we would have substitutions $\hat{L} \to -i(\hat{H} - \mu\hat{N})/\hbar$, $\hat{\rho}_0 \to \mathrm{e}^{-(\hat{H}-\mu\hat{N})/T}/Z_G$, and the Green functions defined above would then be replaced by Green functions at finite temperature and give rise to the Keldysh rules for the graphical representation of the perturbation expansion.

Calculation of the trace in (33) may be reduced to the calculation of the trace of a linear exponential, since

$$\mathrm{Tr}\left\{\mathrm{e}^{-\hat{L}_0 t_0}\,\hat{\rho}_0\,\mathrm{e}^{\hat{L}_0 t_0}\,N[F(\hat{\phi})]\right\} = \\ = F\left(\frac{\delta}{\delta B}\right)\mathrm{Tr}\left\{\mathrm{e}^{-\hat{L}_0 t_0}\,\hat{\rho}_0\,\mathrm{e}^{\hat{L}_0 t_0}\,N\,\mathrm{e}^{B\hat{\phi}}\right\}\Big|_{B=0}. \quad (34)$$

The rules of construction and the form of the functional in the normal product in (33) are similar in all cases of quantum field theory and stochastic problems. The differences show in the result of the calculation of the «expectation value» of the linear exponential operator in (34).

3.1. Expectation Value of the Linear Exponential in Coordinate Basis. It is convenient to write out the linear exponential in terms of the operators $\hat{\pi}$ and $\hat{\varphi}$ instead of the generic notation. Thus,

$$\mathrm{Tr}\left\{\mathrm{e}^{-\hat{L}_0 t_0}\,\hat{\rho}_0\,\mathrm{e}^{\hat{L}_0 t_0}\,N\,\mathrm{e}^{\hat{\varphi}A+\hat{\pi}B}\right\} = \\ = \int d\chi\,\langle\chi|\,\mathrm{e}^{-\hat{L}_0 t_0}\int d\varphi\,|p_0\rangle\langle\varphi|\,\mathrm{e}^{\hat{L}_0 t_0}\,N\,\mathrm{e}^{\hat{\varphi}A+\hat{\pi}B}|\chi\rangle = \\ = \int d\chi\int d\varphi\int d\zeta\,\langle\chi|\,\mathrm{e}^{-\hat{L}_0 t_0}\,|p_0\rangle\langle\varphi|\,\mathrm{e}^{\hat{L}_0 t_0}|\zeta\rangle\langle\zeta|\,N\,\mathrm{e}^{\hat{\varphi}A+\hat{\pi}B}\,|\chi\rangle. \quad (35)$$

In the normal product the operators $\hat{\pi}(t)$ go to the left and $\hat{\varphi}(t)$ to the right as suggested by the form of the Liouvillean. The convenient choice for the time evolution of the free theory is given by the free Liouvillean in the form ($K > 0$)

$$\hat{L}_0 = -\hat{\pi}K\hat{\varphi} \quad (36)$$

with the time-dependent operators

$$\hat{\pi}(t) = \hat{\pi}\,\mathrm{e}^{Kt}, \quad \hat{\varphi}(t) = \hat{\varphi}\,\mathrm{e}^{-Kt}. \quad (37)$$

Calculate first the matrix element of the normal product of the linear exponential in the coordinate basis:

$$\langle\zeta|\,N\left[\mathrm{e}^{\hat{\varphi}\,A+\hat{\pi}B}\right]|\chi\rangle = \langle\zeta|\,\mathrm{e}^{\hat{\pi}\tilde{B}}\mathrm{e}^{\hat{\varphi}\tilde{A}}|\chi\rangle, \quad (38)$$

where $\tilde{A} = \int \mathrm{e}^{-Kt}A(t)\,dt$ and $\tilde{B} = \int \mathrm{e}^{Kt}\,B(t)\,dt$. In the coordinate basis the operator $\hat{\varphi}$ is the multiplication operator, whereas the exponential of the momentum operator is the translation operator. Therefore, (38) immediately yields

$$\langle\zeta|\,N[\mathrm{e}^{\hat{\varphi}\,A+\hat{\pi}B}]\,|\chi\rangle = \delta(\zeta-\chi-\tilde{B})\,\mathrm{e}^{\chi\tilde{A}}. \quad (39)$$

The matrix element $\langle\chi|\,\mathrm{e}^{-\hat{L}_0 t_0}\,|p_0\rangle = p_1(t_0)$ in (35) is the solution of the first-order partial differential equation

$$\left[\frac{\partial}{\partial t}+\hat{L}_0\right]p_1(t,\chi)=\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial\chi}K\chi\right]p_1(t,\chi)=0$$

with the initial condition $p_1(0,\chi)=\langle chi|p_0\rangle = p_0(\chi)$. Thus,

$$\langle\chi|\,\mathrm{e}^{-\hat{L}_0 t_0}\,|p_0\rangle=\mathrm{e}^{-Kt_0}\,p_0(\mathrm{e}^{-Kt_0}\,\chi). \tag{40}$$

The matrix element of the other evolution operator in (35) yields the unity due to probability conservation:

$$\int d\varphi\,\langle\varphi|\,\mathrm{e}^{\hat{L}_0 t_0}\,|\zeta\rangle=1.$$

Thus, the average of the linear exponential (35) is reduced to

$$\begin{aligned}&\mathrm{Tr}\left\{\mathrm{e}^{-\hat{L}_0 t_0}\,\hat{\rho}_0\,\mathrm{e}^{\hat{L}_0 t_0}\,N\,\mathrm{e}^{\hat{\varphi}A+\hat{\pi}B}\right\}=\\=&\int d\chi\int d\zeta\,\delta(\zeta-\chi-\tilde{B})\,\mathrm{e}^{\chi\tilde{A}}\,\mathrm{e}^{-Kt_0}\,p_0(\mathrm{e}^{-Kt_0}\,\chi)=\int d\chi\,\mathrm{e}^{\mathrm{e}^{Kt_0}\chi\tilde{A}}\,p_0(\chi),\end{aligned}\tag{41}$$

and for any operator functional $F[\hat{\pi},\hat{\varphi}]$ we obtain

$$\mathrm{Tr}\,\mathrm{e}^{-\hat{L}_0 t_0}\,\hat{\rho}_0\,\mathrm{e}^{\hat{L}_0 t_0}\,N\left\{F[\hat{\pi},\hat{\varphi}]\right\}=\int d\varphi\,p_0(\varphi)F[0,n\varphi], \tag{42}$$

where $n\varphi = n(t,t_0)\varphi = \mathrm{e}^{-K(t-t_0)}\,\varphi$. Therefore,

$$\begin{aligned}G_{if}(J)=&\int d\varphi\,p_0(\varphi)\exp\left[\frac{\delta}{\delta\varphi_1}\tilde{\Delta}\frac{\delta}{\delta\pi_1}+\frac{\delta}{\delta\varphi_2}\Delta\frac{\delta}{\delta\pi_2}+\frac{\delta}{\delta\varphi_1}n\frac{\delta}{\delta\pi_2}\right]\times\\&\times\exp\left[\int_{t_i}^{t_f}dt\,\varphi_2(t)J(t)-\int_{t_0}^{t_f}L_I(\pi_1,\varphi_1)\,dt+\int_{t_0}^{t_f}L_I(\pi_2,\varphi_2)\,dt\right]\Bigg|_{\substack{\pi_i=0\\ \varphi_i=n\varphi}}.\end{aligned}\tag{43}$$

It should be noted that here we have two sets of variables evolving in opposite directions of time. This is usually referred to as the result of introduction of path-ordered products of operators in the construction of the generating function (43). The generic expression (43) in a finite time interval is somewhat complicated due to the lack of translation invariance in time. Therefore, it is convenient to pass to the the limit $t_f\to\infty$, $t_i, t_0\to-\infty$ and arrive at the Keldysh rules of the graphical representation. Moreover, the dependence on the initial field in the exponential disappears — provided the condition $K > 0$ holds — and the normalization condition of the PDF leads to

$$G(J) = \exp\left[\frac{\delta}{\delta\varphi_1}\tilde{\Delta}\frac{\delta}{\delta\pi_1} + \frac{\delta}{\delta\varphi_2}\Delta\frac{\delta}{\delta\pi_2} + \frac{\delta}{\delta\varphi_1} n \frac{\delta}{\delta\pi_2}\right] \times$$
$$\times \exp\left[\int_{-\infty}^{\infty} dt\, \varphi_2(t) J(t) - \int_{-\infty}^{\infty} L_I(\pi_1, \varphi_1)\, dt + \int_{-\infty}^{\infty} L_I(\pi_2, \varphi_2)\, dt\right]\Bigg|_{\substack{\pi_i=0\\ \varphi_i=0}} . \quad (44)$$

Inspection of diagrams of the model allows us to conclude that closed propagator loops of the physical set of fields π_2, φ_2 are cancelled due to the contribution produced by the auxiliary set of fields π_1, φ_1.

In first-order models, closed loops of the propagators Δ and $\tilde{\Delta}$ vanish. The contribution of fields π_1, φ_1 is then reduced to a constant, and we obtain the expression

$$G(J) = \left[\exp\left(\frac{\delta}{\delta\varphi_2}\Delta\frac{\delta}{\delta\pi_2}\right)\exp\left(S_I(\pi_2, \varphi_2) + \varphi_2 J\right)\right]\Bigg|_{\substack{\pi_2=0\\ \varphi_2=0}} , \quad (45)$$

which is the functional representation of the generating function of the MSR field theory.

In a similar fashion, the generating function for problem corresponding to the second-order SDE may be cast in the form

$$G_{2\,if}(J) = \iint d\varphi\, dv\, p_0(\varphi, v) \exp\left[\frac{\delta}{\delta\varphi_1}\tilde{\Delta}\frac{\delta}{\delta\eta_1} + \frac{\delta}{\delta\varphi_2}\Delta\frac{\delta}{\delta\eta_2} + \frac{\delta}{\delta\varphi_1} n_2 \frac{\delta}{\delta\eta_2}\right] \times$$
$$\times \exp\left[\int_{t_i}^{t_f} dt\, \varphi_2(t) J(t) - \int_{t_0}^{t_f} L_I(\eta_1, \varphi_1)\, dt + \int_{t_0}^{t_f} L_I(\eta_2, \varphi_2)\, dt\right]\Bigg|_{\substack{\eta_i=0\\ \varphi_i=n_1\varphi+n_2 v}} , \quad (46)$$

where the functions n_i are commutators of the Dirac fields

$$n_1(t, t') = [\hat{\varphi}(t), \hat{\pi}(t')] , \quad n_2(t, t') = [\hat{\varphi}(t), \hat{\eta}(t')] , \quad n_3(t, t') = [\hat{v}(t), \hat{\eta}(t')]$$

and, e.g., $n_3 v = n_3(t, t_0) v$, where v is the initial value of the field.

3.2. Expectation Value of the Linear Exponential in Occupation Number Basis. In case of master equation the form of the evolution operator is just the same as in the case of Fokker–Planck equation, but the density operator is different. Operators $\hat{a}^+$ and $\hat{a}$, specific for this problem, shall be used in what follows

$$\operatorname{Tr}\left\{ \mathrm{e}^{-\hat{L}_0 t_0}\,\hat{\rho}_0\,\mathrm{e}^{\hat{L}_0 t_0}\,N\,\mathrm{e}^{\hat{a}B^+ + \hat{a}^+ B}\right\} = \\ = \sum_n \frac{1}{n!}\langle n|P_0\langle P|\,\mathrm{e}^{\hat{L}_0 t_0}\,\mathrm{e}^{\hat{a}^+ B}\,\mathrm{e}^{\hat{a}B^+}\,\mathrm{e}^{-\hat{L}_0 t_0}\,|n\rangle. \quad (47)$$

Choose the free Liouville operator in the form

$$\hat{L}_0 = -\left(\hat{a}^+ - I\right) K\hat{a}, \quad (48)$$

for which the «conservation of probability» holds:

$$\langle P|\,\mathrm{e}^{\hat{L}_0 t_0} = \langle P|,$$

because the projection vector is the left eigenstate of the creation operator

$$\langle P|\hat{a}^+ = \langle P|.$$

The free Liouvillean (48) gives rise to time-dependent operators

$$\hat{a}^+(t) = \hat{a}^+\,\mathrm{e}^{Kt} + (1 - \mathrm{e}^{Kt}), \quad \hat{a}(t) = \hat{a}\,\mathrm{e}^{-Kt}. \quad (49)$$

Therefore,

$$\langle P|\,\mathrm{e}^{\hat{L}_0 t_0}\,\mathrm{e}^{\hat{a}^+ B}\,\mathrm{e}^{\hat{a}B^+}\,\mathrm{e}^{-\hat{L}_0 t_0}\,|n\rangle = \\ = \exp\left[\int (1 - \mathrm{e}^{Kt})B(t)\,dt\right]\langle P|\,\mathrm{e}^{\hat{a}^+\tilde{B}}\,\mathrm{e}^{\hat{a}\tilde{B^+}}\,\mathrm{e}^{-\hat{L}_0 t_0}\,|n\rangle, \quad (50)$$

where $\tilde{B} = \int \mathrm{e}^{Kt}B(t)\,dt$ and $\tilde{B^+} = \int \mathrm{e}^{-Kt}B^+(t)\,dt$. Pull now the operator exponential $\mathrm{e}^{\hat{a}}$ from $\langle P|$ to the right by the rule

$$\mathrm{e}^{\hat{a}}\hat{a}^+ = (\hat{a}^+ + I)\,\mathrm{e}^{\hat{a}},$$

which boils down to the shift $\hat{a}^+ \to \hat{a}^+ + I$ in operators, through which the coherent-state exponential is pulled:

$$\langle P|\,\mathrm{e}^{\hat{a}^+\tilde{B}}\,\mathrm{e}^{\hat{a}\tilde{B^+}}\,\mathrm{e}^{-\hat{L}_0 t_0}\,|n\rangle = \langle 0|\,\mathrm{e}^{(\hat{a}^+ + I)\,\tilde{B}}\,\mathrm{e}^{\hat{a}\tilde{B^+}}\,\mathrm{e}^{-\hat{L}'_0 t_0}\,(\hat{a}^+ + I)^n|0\rangle. \quad (51)$$

The basis state $|n\rangle$ is an eigenstate of the shifted free Liouville operator (48):

$$\hat{L}'_0\,|n\rangle = -\hat{a}^+ K\hat{a}\,|n\rangle = -Kn\,|n\rangle;$$

therefore,

$$\mathrm{e}^{-\hat{L}'_0 t_0}\left(\hat{a}^+ + I\right)^n|0\rangle = \left(\hat{a}^+\mathrm{e}^{Kt_0} + I\right)^n|0\rangle. \quad (52)$$

Combining (50), (51) and (52), we obtain

$$\frac{1}{n!}\langle n|P_0\langle P|\,e^{\hat{L}_0 t_0}\,e^{\hat{a}^+ B}\,e^{\hat{a}B^+}\,e^{-\hat{L}_0 t_0}\,|n\rangle = \\ = P(0,n)\,\exp\left[\int B(t)\,dt\right]\left[\int e^{-K(t-t_0)}\,B^+(t)\,dt+1\right]^n. \quad (53)$$

Thus, the expectation value of the linear exponential (47) is

$$\mathrm{Tr}\,\left\{e^{-\hat{L}_0 t_0}\,\hat{\rho}_0\,e^{\hat{L}_0 t_0}\,N\,e^{\hat{a}B^+ + \hat{a}^+ B}\right\} = \\ = \sum_n P(0,n)\exp\left[\int B(t)\,dt\right]\left[\int e^{-K(t-t_0)}\,B^+(t)\,dt+1\right]^n. \quad (54)$$

The result for the expectation value of the normal form of an operator functional $F[\hat{a}^+(t),\hat{a}(t)]$ may be written in a closed form with the use of the identity

$$x^n = \frac{n!}{2\pi i}\oint\limits_C \frac{e^{xz}}{z^{n+1}}\,dz,$$

where C is a closed Jordan contour encircling the origin of the complex z plane. We obtain

$$\mathrm{Tr}\,\left(e^{-\hat{L}_0 t_0}\,\hat{\rho}_0\,e^{\hat{L}_0 t_0}\,N\left\{F\left[\hat{a}^+(t),\hat{a}(t)\right]\right\}\right) = \\ = F\left[\frac{\delta}{\delta B(t)},\frac{\delta}{\delta B^+(t)}\right]\,\mathrm{Tr}\,\left(e^{-\hat{L}_0 t_0}\,\hat{\rho}_0\,e^{\hat{L}_0 t_0}\,N\,e^{\hat{a}B^+ + \hat{a}^+ B}\right)_{B=B^+=0} = \\ = F\left[\frac{\delta}{\delta B(t)},\frac{\delta}{\delta B^+(t)}\right]\exp\left[\int B(t)\,dt\right]\times \\ \times\sum_n P(0,n)\frac{n!}{2\pi i}\oint\limits_C \frac{\exp\left[z\left(\int e^{-K(t-t_0)}\,B^+(t)\,dt+1\right)\right]}{z^{n+1}}\,dz\Bigg|_{B=B^+=0} = \\ = \sum_n P(0,n)\frac{n!}{2\pi i}\oint\limits_C \frac{e^z}{z^{n+1}}\,F[1,nz]\,dz = \frac{1}{2\pi i}\oint\limits_C e^z\widetilde{G}(z)\,F[1,nz]\,dz, \quad (55)$$

where $nz = n(t,t_0)z = e^{-K(t-t_0)}z$. In (55) the shorthand notation

$$\widetilde{G}(z) = \sum_n \frac{P(0,n)n!}{z^{n+1}}$$

has been used on the right side. For the generating function we thus obtain the representation

$$G_{if}(J) = \frac{1}{2\pi i} \oint_C \mathrm{e}^z\, \widetilde{G}(z)\, \exp\left[\frac{\delta}{\delta a_1}\tilde{\Delta}\frac{\delta}{\delta a_1^+} + \frac{\delta}{\delta a_2}\Delta\frac{\delta}{\delta a_2^+} + \frac{\delta}{\delta a_1}n\frac{\delta}{\delta a_2^+}\right] \times$$
$$\times \exp\left[\int_{t_i}^{t_f} dt\, a_2^+(t)\, a_2(t) J(t) - \int_{t_0}^{t_f} L_I(a_1^+, a_1)\, dt + \int_{t_0}^{t_f} L_I(a_2^+, a_2)\, dt\right]\Bigg|_{\substack{a_i^+=1\\ a_i=nz}} . \quad (56)$$

Master equations have so far been applied to first-order models, in which closed loops of the propagators Δ and $\tilde{\Delta}$ vanish. The contribution of fields a_1^+, a_1 is then reduced to a constant and we obtain the expression

$$G_{if}(J) = \frac{1}{2\pi i} \oint_C \mathrm{e}^z \widetilde{G}(z)\, \exp\left[\frac{\delta}{\delta a_2}\Delta\frac{\delta}{\delta a_2^+}\right] \times$$
$$\times \exp\left[\int_{t_i}^{t_f} dt\, a_2^+(t) a_2(t) J(t) + \int_{t_0}^{t_f} L_I(a_2^+, a_2)\, dt\right]\Bigg|_{\substack{a_2^+=1\\ a_2=nz}} . \quad (57)$$

To restore translation invariance with respect to time, it is convenient to pass to the the limit $t_f \to \infty$, $t_i, t_0 \to -\infty$. If the condition $K > 0$ holds, then dependence on the initial condition in the exponential in (57) disappears. However, in master equation problems a uniform initial density is often assumed, and in diffusion-limited cases the wave-number dependent operator K then vanishes for the zero wave-number and $nz \to \delta(\mathbf{k})z$ in the limit $t_0 \to -\infty$.

4. FUNCTIONAL INTEGRALS FOR STOCHASTIC FIELD THEORY

If the cancellation of closed loops of propagators in the perturbation expansion of first-order models is taken as granted for the model, then we arrive at the functional representation of the type (45), which may be cast into a functional integral through the Gaussian representation of the differential operator:

$$\exp\left(\frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\pi}\right) =$$
$$= (\det \pi\Delta)^{-1} \iint \mathcal{D}\phi\, \mathcal{D}\tilde{\phi}\, \exp\left(-\tilde{\phi}\Delta^{-1}\phi + \tilde{\phi}\frac{\delta}{\delta\pi} + \phi\frac{\delta}{\delta\varphi}\right). \quad (58)$$

The result is the functional-integral representation for the generating function of the Martin–Siggia–Rose theory:

$$G(J) = \iiint \mathcal{D}\varphi\,\mathcal{D}\phi\,\mathcal{D}\tilde{\phi}\,p_0(\varphi)\times$$
$$\times\exp\bigl[-\tilde{\phi}(\partial_t + K)\phi + S_I(\tilde{\phi}, \phi + n\varphi) + (\phi + n\varphi)J\bigr]. \quad (59)$$

Note the explicit appearance of the initial condition. Vanishing of propagator loops, however, is not at all obvious in the functional integral, in which only the differential operator but not the propagator is present. Therefore, the functional representation with the two sets of fields might be needed even in the first-order case.

In representation of the generating function (43) with both sets of fields as a functional integral, care has to be exercised due to the fact that the normal contraction n is not a Green function, but a solution of the homogeneous differential equation related to the free part of the Liouville operator. In particular, representation (58) cannot be used separately for each differential operator in (43), because the normal contraction n does not have an inverse. In the matrix interpretation of (58) the correspondence between the propagator matrix and the differential operator matrix is highly ambiguous due to the presence of the normal contraction n. Of course, it should be possible to fix this ambiguity by a suitable choice of the measure of functional integration. However, at present there is no consistent way to do this in the generic functional integral. In the framework of perturbation theory, manipulations with the manner of introducing the integral representation may be useful in defining a reasonable functional integral for the the generating function (43).

To this end, it is proposed to use the following procedure, in which the integral representations are carried out with the use of operators possessing unique inverse operators. First, since the interaction functional L_I in (43) is quadratic in π:

$$L_I = \pi U(\varphi) + \frac{1}{2}\pi^2 b(\varphi)\,Db(\varphi),$$

we may write (time integrals on the right side are implied)

$$\exp\left[-\int_{t_0}^{t_f} L_I(\pi_1, \varphi_1)\,dt\right] =$$
$$= \bigl[\det 2\pi\,(\partial_t + K)^{-\top} b(\varphi_1)\,Db(\varphi_1)(\partial_t + K)^{-1}\bigr]^{-1/2}\times$$
$$\times\int \mathcal{D}\phi_1 \exp\biggl\{-\frac{1}{2}(\partial_t + K)\phi_1\,[b(\varphi_1)\,Db(\varphi_1)]^{-1}\,(\partial_t + K)\phi_1 -$$

$$- i(\partial_t + K)\phi_1 \left[\pi_1 + [b(\varphi_1)\, Db(\varphi_1)]^{-1}\, U(\varphi_1)\right] + \\ + \frac{1}{2} U(\varphi_1)\, [b(\varphi_1)\, Db(\varphi_1)]^{-1}\, U(\varphi_1)\Big\} \quad (60)$$

and

$$\exp\left[\int_{t_0}^{t_f} L_I(\pi_2, \varphi_2)\, dt\right] = \\ = \left[\det 2\pi\, (\partial_t + K)^{-\top} b(\varphi_2)\, Db(\varphi_2)(\partial_t + K)^{-1}\right]^{-1/2} \times \\ \times \int \mathcal{D}\phi_2\, \exp\Big\{-\frac{1}{2}(\partial_t + K)\phi_2\, [b(\varphi_2)\, Db(\varphi_2)]^{-1}\, (\partial_t + K)\phi_2 + \\ + (\partial_t + K)\phi_2 \left[\pi_2 + [b(\varphi_2)\, Db(\varphi_2)]^{-1}\, U(\varphi_2)\right] - \\ - \frac{1}{2} U(\varphi_2)\, [b(\varphi_2)\, Db(\varphi_2)]^{-1}\, U(\varphi_2)\Big\}. \quad (61)$$

The point of extracting the free-field differential operators $\partial_t + K$ from the integration variables in (60) and (61) will become clear shortly. The integration fields ϕ_1 and ϕ_2 are assumed to fall off rapidly in space and vanish at the initial time instant: $\phi_1(t_0) = \phi_2(t_0) = 0$.

Second, the result of the action of the differential operator from (43) on the π field may be calculated explicitly in this representation:

$$\exp\left[\frac{\delta}{\delta\varphi_1}\tilde{\Delta}\frac{\delta}{\delta\pi_1} + \frac{\delta}{\delta\varphi_2}\Delta\frac{\delta}{\delta\pi_2} + \frac{\delta}{\delta\varphi_1}n\frac{\delta}{\delta\pi_2}\right] \times \\ \times \exp\left[\int_{t_i}^{t_f} dt\, \varphi_2(t) J(t) - \int_{t_0}^{t_f} L_I(\pi_1, \varphi_1)\, dt + \int_{t_0}^{t_f} L_I(\pi_2, \varphi_2)\, dt\right]\Bigg|_{\substack{\pi_i = 0 \\ \varphi_i = n\varphi}} = \\ = \int \mathcal{D}\phi_1 \int \mathcal{D}\phi_2 \exp\Big[-i(\partial_t + K)\, \phi_1 \tilde{\Delta}^\top \frac{\delta}{\delta\varphi_1} + \\ + (\partial_t + K)\phi_2 \Delta^\top \frac{\delta}{\delta\varphi_2} + (\partial_t + K)\phi_2 n^\top \frac{\delta}{\delta\varphi_1}\Big] \times \\ \times \left[\det 2\pi\, \Delta^\top b(\varphi_1) Db(\varphi_1)\Delta\right]^{-1/2} \left[\det 2\pi\, \Delta^\top b(\varphi_2) Db(\varphi_2)\Delta\right]^{-1/2} \times \\ \times \exp\Big\{J\varphi_2 - \frac{1}{2}(\partial_t + K)\phi_1\, [b(\varphi_1) Db(\varphi_1)]^{-1}\, (\partial_t + K)\phi_1 - \\ - i(\partial_t + K)\phi_1\, [b(\varphi_1) Db(\varphi_1)]^{-1}\, U(\varphi_1) + \frac{1}{2} U(\varphi_1)\, [b(\varphi_1) Db(\varphi_1)]^{-1}\, U(\varphi_1) -$$

$$
-\frac{1}{2}(\partial_t+K)\phi_2\left[b(\varphi_2)Db(\varphi_2)\right]^{-1}(\partial_t+K)\phi_2+(\partial_t+K)\phi_2\left[b(\varphi_2)Db(\varphi_2)\right]^{-1}\times
$$

$$
\times U(\varphi_2)-\frac{1}{2}U(\varphi_2)\left[b(\varphi_2)Db(\varphi_2)\right]^{-1}U(\varphi_2)\Bigg\}\Bigg|_{\varphi_i=n\varphi}, \quad (62)
$$

where the remaining derivatives with respect to φ_i give rise to shifts of the functional arguments. The result for the generating function is

$$
G_{if}(J)=\iiint\mathcal{D}\varphi\,\mathcal{D}\phi_1\mathcal{D}\phi_2\,p_0(\varphi)\left[\det 2\pi\,\Delta^\top b_1Db_1\Delta\right]^{-1/2}\times
$$

$$
\times\left[\det 2\pi\,\Delta^\top b_2Db_2\Delta\right]^{-1/2}\exp\Bigg\{J(\phi_2+n\varphi)-\frac{1}{2}(\partial_t+K)\phi_2\,(b_2Db_2)^{-1}\times
$$

$$
\times(\partial_t+K)\phi_2-\frac{1}{2}(\partial_t+K)\phi_1\,(b_1Db_1)^{-1}(\partial_t+K)\phi_1-i(\partial_t+K)\phi_1(b_1Db_1)^{-1}U_1+
$$

$$
+(\partial_t+K)\phi_2(b_2Db_2)^{-1}U_2+\frac{1}{2}U_1(b_1Db_1)^{-1}U_1-\frac{1}{2}U_2(b_2Db_2)^{-1}U_2\Bigg\}, \quad (63)
$$

where the shorthand notation

$$
\begin{aligned}
b_1&=b(i\phi_1+\phi_2+n\varphi), &\quad b_2&=b(\phi_2+n\varphi),\\
U_1&=U(i\phi_1+\phi_2+n\varphi), &\quad U_2&=U(\phi_2+n\varphi)
\end{aligned}
$$

has been introduced. Cancellations of contributions from the graphs with direct and inverse time flows are now explicit in the functional integral (63). No operator inversions have been carried out for this representation, and it is unambiguous in this sense. This representation with ordinary number fields only should be more convenient numerically than the use of ghost fields. Introduction of integral representations similar to (60) and (61) in (63) leads to a representation similar to the MSR theory:

$$
G_{if}(J)=\int\cdots\int\mathcal{D}\varphi\,\mathcal{D}\phi_1\,\mathcal{D}\phi_2\,\mathcal{D}\tilde\phi_1\,\mathcal{D}\tilde\phi_2\,p_0(\varphi)\left[\det(2\pi)^2\,\Delta^\top\Delta\right]^{-1}\times
$$

$$
\times\exp\Bigg\{J(\phi_2+n\varphi)-\frac{1}{2}\tilde\phi_2b_2Db_2\tilde\phi_2+i\tilde\phi_2\left[-(\partial_t+K)\,\phi_2+U_2\right]-
$$

$$
-\frac{1}{2}\tilde\phi_1b_1Db_1\tilde\phi_1-\tilde\phi_1\left[-(\partial_t+K)\,(i\phi_1+\phi_2)+U_1\right]\Bigg\}. \quad (64)
$$

The generating function of Green functions of the Fokker–Planck equation of the Ito interpretation of the second-order SDE (3) may be obtained in a similar fashion in the form

$$G_{2if}(J) = \int \cdots \int \mathcal{D}\varphi\, \mathcal{D}v\, \mathcal{D}\phi_1\, \mathcal{D}\tilde{\phi}_1\, \mathcal{D}\phi_2\, \mathcal{D}\tilde{\phi}_2\, p_0(\varphi, v) \times$$
$$\times \exp\big[J(\phi_2 + n\varphi) - i\tilde{\phi}_2(m\partial_t^2 + \beta\partial_t + K)\phi_2 + S_I(i\tilde{\phi}_2, \phi_2 + n\varphi) +$$
$$+ \tilde{\phi}_1(m\partial_t^2 + \beta\partial_t + K)(i\phi_1 + \phi_2) - S_I(\tilde{\phi}_1, i\phi_1 + \phi_2 + n\varphi)\big], \quad (65)$$

in which the propagator determinants are omitted for brevity. The only difference between (64) and (65) is the form of the free action. In representation (65) there is no functional determinant, there are no ghost fields, and the dynamic action is local! Moreover, all variables are usual numbers; therefore, numerical evaluation might be more straightforward and unambiguous than in other representations of the generating function.

5. CONCLUSION

Functional representations for the generating function of Green functions of stochastic problems stated either with the use of the Fokker–Planck equation or the master equation have been constructed with the use of the operator approach of quantum field theory. Initial conditions are explicitly included in this representation and the translation-invariant limit with respect to time discussed. A representation of the generating function of the master equation in the form of a functional integral is constructed with the generic initial condition for the probability density function.

Solution of first- and second-order stochastic differential equations in the framework of stochastic field theory has been analyzed. Ambiguities in the mathematical formulation of stochastic field theory arising from both the ambiguity of the stochastic differential equation and the ambiguity of the functional representation are discussed. To resolve the former ambiguity, the Fokker–Planck equation of a Markov process has been used as the starting point of the construction of the stochastic field theory. The Fokker–Planck equation corresponding to the Ito interpretation of the SDE is put forward as more convenient in practical calculations than that corresponding to the Stratonovich interpretation. The latter ambiguity gives rise to the determinant problem in the construction of the corresponding functional integral. The ambiguity in the functional formulation is related to the definition of the time-ordered product at coinciding time arguments and is not connected with the choice between Ito and Stratonovich interpretations. In the framework of the Ito interpretation, it is argued that the most convenient choice of the time-ordered product is to define it as the normal product at coinciding time arguments.

The functional representation for the generating function of Green functions allows one to generate the perturbation expansion without any reference to the functional determinant. A representation of the generating function in the form of a functional integral is constructed in a way in which the quadratic form of the dynamic action is determined unambiguously. In this approach, a new solution is proposed to the functional Jacobi determinant problem, in which by the introduction of an additional set of variables a functional-integral representation of the generating function is obtained with local action, but without any explicit determinant and without ghost fields.

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