ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
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DIFFEOMORPHISM-INVARIANT LATTICE ACTIONS

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We present a lattice-discretization procedure which is based on the simplicial lattice and preserves diffeomorphism invariance. The presented procedure is the straightforward generalization for the procedure used for discretization of the spinor gravity [7]. As a stable way to guarantee the removing of the lattice regularization, i.e., the continuum limit, for lattice diffeomorphism-invariant theories, we propose to tune the system to the point of phase transition. We expect that the Einstein gravitation is achieved at this point.

Представлена процедура решеточной дискретизации, сохраняющая диффеоморфную инвариантность действия, основанная на использовании симплектических решеток. Данная процедура является прямым обобщением дискретизации, используемой в спинорной гравитации [7]. В качестве гарантии снятия решеточной регуляризации, т. е. существования непрерывного предела, мы предлагаем тонкую настройку системы в точку фазового перехода. Мы также предполагаем, что в этой точке будет реализована гравитация Эйнштейна.

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INTRODUCTION

Diffeomorphism invariance, or reparametrization invariance, is the key feature of general relativity. However, perturbative expansion about a background metric usually breaks diffeomorphism invariance: one has to «fix the gauge». In order to quantize gravity, one needs nonperturbative approaches that preserve diffeomorphism invariance explicitly.

The most straightforward nonperturbative approaches preserving diffeomorphism invariance are based on lattice regularization of the path integrals over quantum fluctuations; in addition, they usually allow direct numerical simulations. Many well-known quantum gravity theories, such as the Ponzano–Regge model [1] and the Barrett–Crane model [2], are based on the lattice discretization of the classical diffeomorphism-invariant action. Most part of those models have the BF-type action, or can be related to a model with the BF action (see, e.g., (24)). This helps to discretize the action without loosing the diffeomorphism invariance. However, the BF-like models are sensitive to the change of the action, and it is difficult to consider coupling of matter within such models.

Lattice versions of the diffeomorphism invariance require the independence of the lattice action on the particular realization of the lattice grid in curved space, i.e., the independence of the positioning of the lattice vertices in space. There are several ways to construct such a lattice, see, e.g., [3–5]. In this paper we present a lattice-discretization procedure that preserves the diffeomorphism invariance of *any* action.

In most lattice approaches the continuum action is replaced by the sum over lattice vertices, where the field derivatives are replaced by the finite differences of the fields between neighboring lattice points. In this way, the construction of the diffeomorphism-invariant lattice action is hardly possible. We propose to replace the action over a manifold by a sum over the lattice *cells*, and to adjust the Lagrangian to a cell, rather than to a vertex. We show that in this way any diffeomorphism-invariant action involving only covariant fields can be discretized in the way that preserves the diffeomorphism invariance in the continuum limit. In this approach, it is necessary to use simplicial lattices rather than hypercubic ones.

The discretization procedures similar to the present one were described in several recent papers [5, 7] for particular models. Here, we formalize the procedure in a more general way. The essential point is that the lattice action should not only represent the continuum Lagrangian locally but also should correctly restore the integration over the whole volume. By using simplicial lattices and by summing up actions over all lattice cells, we get confident that the full action contains neither holes nor overlaps.

The structure of the paper is as follows: in Sec. 1 we formulate the procedure of lattice discretization supporting diffeomorphism invariance and give examples of the discretization for a simple bosonic theory and a theory with gauge symmetry. We also compare our method with that of dual lattice discretization. In Sec. 2 we discuss the continuum limit of the lattice models, the problem of the restoration of the low-energy Einstein–Hilbert action, the appearance of scales, and the possible solutions to those problems.

1. DIFFEOMORPHISM-INVARIANT ACTIONS ON A LATTICE

The diffeomorphism invariance means the independence of the action under general coordinate transformation,

$$\int d^d x\, \mathcal{L}(x) \xrightarrow{\text{diffeomorphism}} \int d^d x'\, \mathcal{L}(x'), \tag{1}$$

where $x'^{\mu}(x)$ are differentiable bijective functions. The lattice-regularized version of the action is defined on a graph which we shall call the *lattice*, with fields assigned either to the vertices or to the edges of the graph. The lattice version of the diffeomorphism invariance is the invariance of the action under the arbitrary continuous vertex displacement. It means that the lattice action should depend only on the topology, i.e., on the neighborhood structure of the lattice. We call the lattice the «number space».

1.1. General Construction. It is possible to cover the whole d-dimensional space by $(d+1)$-cells or simplices, although the number of edges entering one vertex may not be the same for all vertices. Alternatively, the number of edges coming from all vertices is the same but then the edges lengths may vary, if one attempts to embed the lattice into flat space. Since only the set of the nearest neighbors matters and the abstract «number space» does not need to be flat, this is also acceptable. The important thing is that the chosen set of cells should fill in the space without holes and without overlapping.

All vertices in a simplicial lattice can be characterized by a set of d integers. For brevity, we label these d numbers by a single integer i. Each vertex has its unique integer label i, supplemented with a rule what labels are ascribed to the neighbor vertices forming elementary cells. We shall denote the $d+1$ labels belonging to one cell by $i = 0, 1, \ldots, d$.

Each vertex in the abstract number space corresponds to the real world coordinate by a certain map x_i^μ. The goal is to write possible action terms in such a way that, if the fields vary slowly from one vertex (or link) to the topologically neighbor one, the action reduces to desired continuum action.

To perform this task, we introduce the set of vectors $\Delta x_{ij}^\mu = x_i^\mu - x_j^\mu$, $i \neq j$. A vector Δx_{ij}^μ is adjusted to a particular cell, which is in our case simplex and has $d+1$ vertices and $d(d+1)/2$ edges. The upper index μ is a Lorentz index of «usual» space and runs from 1 to d. The lower indices are runs over the vertices of the cell $0, 1, \ldots, d$ to which Δx_{ij}^μ belongs. The meaning of the vector Δx_{ij}^μ is the following: it is a vector (in «usual» space) pointing from vertex i to vertex j; in other words, it is a vector representing the link (ij). Fixing the label i on some vertex, say 0th vertex, the set of vectors Δx_{j0}^μ forms the matrix $d \times d$ which mixes the Lorentz index with the index enumerating the rest vertices $(1, \ldots, d)$. The determinant of this matrix gives the volume of a simplex:

$$V_{d\text{-simplex}} = \frac{1}{d!} \epsilon_{\mu_1 \mu_2 \ldots \mu_d} \Delta x_{10}^{\mu_1} \Delta x_{20}^{\mu_2} \cdots \Delta x_{d0}^{\mu_d}. \tag{2}$$

It does not matter which vertex is taken as a base: for any base the result would be the same up to a general sign. Only the numeration of the vertices plays a role; one can fix the vertices enumeration such that $V_{d\text{-simplex}} > 0$ for each cell. Therefore, introducing additional $(d+1)$-indexed totally antisymmetric symbol, we remove the preferential position of 0th vertex:

$$V_{d\text{-simplex}} = \frac{1}{(d+1)!d!} \epsilon^{i_0 i_1 i_2 \ldots i_d} \epsilon_{\mu_1 \mu_2 \ldots \mu_d} \Delta x_{i_1 i_0}^{\mu_1} \Delta x_{i_2 i_0}^{\mu_2} \cdots \Delta x_{i_d i_0}^{\mu_d}. \tag{3}$$

The volume element in this form will be our main ingredient for lattice regularization.

The next ingredient is the lattice derivative which, for some scalar field φ, is defined on a lattice edge as follows:

$$\Delta_{ij}(\varphi) = \phi(x_i) - \phi(x_j). \tag{4}$$

In the case of slowly varying field from x_i to x_j, one has $\Delta_{ij}(\varphi) \simeq \Delta x^{\mu}_{ij}\, \partial_\mu \varphi(x_j)$. In this limit the inverse relation reads

$$\partial_\mu \varphi(x_i) \simeq \left(\Delta x_i^{-1}\right)^j_\mu \Delta_{ji}(\varphi), \tag{5}$$

where $\left(\Delta x_i^{-1}\right)^j_\mu$ is the inverse matrix for Δx^{μ}_{ji} with fixed base vertex i. Note that this matrix is always invertible, except in the case of zero-volume simplices, and, therefore, unambiguously defined. At the same time, the expression (5) is not attached to specific lattice vertices.

The vector and tensor fields can be ascribed to the edges and to sides of the simplices, respectively. First of all, we pick out some vertex j of the simplex and construct the $d \times d$ matrix with indices μ and i: Δx^{μ}_{ij}, where j is the chosen vertex. The matrix Δx^{μ}_{ij} can be viewed as a field that connects the coordinate frame with the local frame built from the vectors formed by the edges of a simplex, pointing out of a given vertex x_i. With the help of this matrix we obtain the arbitrary tensor field:

$$T_{\mu\nu..}(x_i) = \left(\Delta x_i^{-1}\right)^j_\mu \left(\Delta x_i^{-1}\right)^k_\nu \cdots T_{jk\ldots,i}. \tag{6}$$

All coordinate-dependent properties of the field, such as gauge transformations, are related to the space point x_i. The contravariant vectors and tensors are constructed in the same way but with the help of the direct matrix:

$$T^{\mu\nu..}(x_i) = \Delta x^{\mu}_{ji} \Delta x^{\nu}_{ki} \cdots T^{jk\ldots,i}. \tag{7}$$

Any diffeomorphism-invariant combinations discretized in such a way preserve the lattice diffeomorphism invariance since all «curved» (Greek) indices arising from the matrix Δx are combined into some tensor in vertex numbering, which is independent of the coordinate x. The quantities that transform as the inverse Jacobian, such as the Lagrangian, obtain the inverse volume factor.

Strictly speaking, the objects (6) and (7) are not tensors, because matrices Δx^{μ}_{ji} and $\left(\Delta x_i^{-1}\right)^j_\mu$ do not transform as vectors. These matrices remain vectors only in the limit then sizes of all links are infinitesimal. This conjecture is irrelevant for a diffeomorphism-invariant theory. Indeed, applying diffeomorphism transformation, one can make any infinitesimal distance finite. Nevertheless, our construction would reproduce correct properties of any indexless composition of fields and volumes; i.e., it would transform with a Jacobian in a proper power under diffeomorphism transformations. Only these objects are interesting in a quantum diffeomorphism-invariant theory: they are terms of action and operators for which one can calculate expectation values. An average for non-diffeomorphism-invariant object is presumably zero.

Let us illustrate the scheme in a simple example. We consider the simplest diffeomorphism-invariant action, namely, the cosmological term in four dimensions. In terms of the frame field or the tetrad e_μ^A, the cosmological term is

$$S_{\rm cosm} = \int d^4x \det(e) = \int d^4x \frac{1}{4!}\epsilon^{ABCD}\epsilon^{\mu\nu\kappa\lambda} e_\mu^A(x)\, e_\nu^B(x)\, e_\kappa^C(x)\, e_\lambda^D(x), \tag{8}$$

where the indices run from 1 to 4. Under the diffeomorphism transformation the frame field transforms as a covariant vector:

$$e_\mu^A(x) \xrightarrow{\text{diffeomorphism}} e_{\mu'}^A(x')\frac{\partial {x'}^{\mu'}}{\partial x^\mu}. \tag{9}$$

The action (8) is also invariant under the local $SO(4)$ or Lorentz transformation:

$$e_\mu^A(x) \xrightarrow{\text{Lorentz}} O^{AB}(x)\, e_\mu^B(x). \tag{10}$$

Since $A, B, \ldots = 1, \ldots, d$ are flat group indices in Euclidean signature, we can equivalently write them either as subscripts or superscripts. The lattice version of the tetrad field is ascribed to the edges: $e_\mu^A(x_i) = (\Delta x_i^{-1})_\mu^j\, e_{ji}^A$. Substituting this expression into (8), we obtain

$$\begin{aligned}\det\, e(x_i) = \frac{1}{4!}\epsilon^{ABCD}\epsilon^{\mu\nu\alpha\beta}(\Delta x_i^{-1})_\mu^j\, (\Delta x_i^{-1})_\nu^k\, (\Delta x_i^{-1})_\alpha^l\, (\Delta x_i^{-1})_\beta^m \times \\ \times\, e_{ji}^A\, e_{ki}^B\, e_{li}^C\, e_{mi}^D = \frac{V_{\rm cell}^{-1}}{4!}\epsilon^{ABCD}\epsilon^{i,jklm}\, e_{ji}^A\, e_{ki}^B\, e_{li}^C\, e_{mi}^D\Big|_{\text{no sum over } i}.\end{aligned} \tag{11}$$

The integral over space is the sum of cell volumes. Therefore, we have

$$S_\Lambda \xrightarrow{\text{lattice}} \sum_{\text{cells}} \frac{1}{5!}\epsilon^{ABCD}\epsilon^{i,jklm}\, e_{ji}^A\, e_{ki}^B\, e_{li}^C\, e_{mi}^D, \tag{12}$$

where we have also performed summation over i. This expression is gauge-invariant, since all group transformations «live» in the same vertex x_i.

In such a way any continuum diffeomorphism-invariant action can be put on a lattice. The diffeomorphism-equivalent degrees of freedom are removed from the action. The fields involved into the lattice action, e.g., the set of e_{ij}^A, are not related to each other by a continuous transformation.

The procedure described above can be generalized to lattice constructions involving other polyhedra whose volume can be expressed through the ϵ-tensor. For example, one can consider elementary cells in the form of octahedra and its higher-dimensional analogs (dual cubes). Such a construction is presented in [5]. However, for dimensions higher than two one cannot cover the space only by one kind of geometrical figures, except the simplices. Instead, one can use a combined lattice built from different polyhedra, but then the resulting action must contain nonuniform constructions. In this sense, our version of the action discretization is the most compact, and it can be applied to any number of space dimensions and to any topology of the background manifold.

1.2. Lattice Spinor Gravity. The spinor gravity model has only the fermion spinor fields and the gauge fields. The action of the spinor gravity is the same as the usual action of the first-order gravity, but with the tetrad field being a bilinear fermion combination. One can take two distinct bilinear combinations of the fermion fields, transforming as the frame field

$$e_\mu^A = i(\psi^\dagger \gamma^A \nabla_\mu \psi + \psi^\dagger \overleftarrow{\nabla}_\mu \gamma^A \psi), \tag{13}$$

$$f_\mu^A = \psi^\dagger \gamma^A \nabla_\mu \psi - \psi^\dagger \overleftarrow{\nabla}_\mu \gamma^A \psi. \tag{14}$$

Here ∇_μ is the covariant derivative in the spinor representation,

$$\nabla_\mu = \partial_\mu - \frac{i}{2}\omega_\mu^{AB}\Sigma_{AB}, \qquad \overleftarrow{\nabla}_\mu = \overleftarrow{\partial}_\mu + \frac{i}{2}\omega_\mu^{AB}\Sigma_{AB}, \tag{15}$$

where ω_μ^{AB} is the spin connection in the adjoint representation of the $SO(d)$ group, and Σ_{AB} are its generators: $\Sigma_{AB} = (i/4)[\gamma_A \gamma_B]$. Under local Lorentz transformations and under the diffeomorphisms, the tetrad fields (13) and (14) transform as (9) and (10).

One can now construct a sequence of many-fermion actions that are invariant under local Lorentz transformations and also are diffeomorphism-invariant, using either e_μ^A or f_μ^A (or both) as building blocks:

$$S_k = \int d^d x \, \frac{1}{d!} \, \epsilon^{\mu_1 \mu_2 \ldots \mu_d} \, \epsilon^{A_1 A_2 .. A_d} \left(\mathcal{F}_{\mu_1 \mu_2}^{A_1 A_2} \cdots \mathcal{F}_{\mu_{2k-1}\mu_{2k}}^{A_{2k-1} A_{2k}} \right) \left(e_{\mu_{2k+1}}^{A_{2k+1}} \cdots e_{\mu_d}^{A_d} \right), \quad k = 0, 1, \ldots, [d/2], \tag{16}$$

where the Yang–Mills curvature tensor is

$$\mathcal{F}_{\mu\nu}^{AB} = \partial_\mu \omega_\nu^{AB} + \omega_\mu^{AC}\omega_\nu^{CB} - (\mu \leftrightarrow \nu). \tag{17}$$

Note that S_0 is the analog of the cosmological term, but there are many of them since one can replace any number of e_μ^A's by f_μ^A's; S_1 is the analog of the

Einstein–Hilbert–Cartan action linear in curvature, and the last action term $S_{[d/2]}$ for even d is a full derivative. Apart from full derivatives, there are 3 possible action terms in $2d$, 6 terms in $3d$, 8 terms in $4d$, 12 terms in $5d$, etc. One can also add the number of P- and T-odd term, which we do not consider here.

In the spinor gravity we have an additional gauge field. On a lattice, the gauge field is represented by the parallel transporter U_{ij}. As in any lattice gauge theory, we replace the connection ω_μ by a unitary matrix «living» on lattice links [6],

$$U_{ij} = P \exp\left(-\frac{i}{2} \int_{x_i}^{x_j} \omega_\mu^{AB} \Sigma^{AB} dx^\mu \right), \quad U_{ji} = U_{ij}^\dagger. \tag{18}$$

By this link variable we connect the γ matrix at point x_i to the next spinor field. Applying the derivative rule (5) to the composite tetrad, we obtain its descretized version:

$$\tilde{e}_{i,j}^A = i(\psi_j^\dagger U_{ji} \gamma^A U_{ij} \psi_j - \psi_i^\dagger \gamma^A \psi_i), \tag{19}$$

$$\tilde{f}_{i,j}^A = \psi_i^\dagger \gamma^A U_{ij} \psi_j - \psi_j^\dagger U_{ji} \gamma^A \psi_i. \tag{20}$$

The difference between $\tilde{e}$ and $\tilde{f}$ is that the first has both fermions in the same vertex, whereas in the second fermions are residing in the neighbor vertices.

We also need the discretized version of the curvature tensor $\mathcal{F}_{\mu\nu}^{\mathcal{AB}}$: it is a plaquette. For the simplicial lattice the plaquettes are triangles, and we define the parallel transporter along a closed triangle spanning the i, j, k vertices:

$$P_{ijk} = U_{ij} U_{jk} U_{ki}, \quad P_{ijk}^{AB} = \frac{1}{d_f} \mathrm{Tr}\,(\Sigma^{AB} P_{ijk}). \tag{21}$$

For the slowly varying gauge fields, we have

$$P_{ijk} = 1 - \frac{i}{4} \Delta x_{ji}^\mu \Delta x_{ki}^\nu \mathcal{F}_{\mu\nu}(x) + \ldots,$$

and

$$P_{ijk}^{AB} = -\frac{i}{4} \Delta x_{ji}^\mu \Delta x_{ki}^\nu \mathcal{F}_{\mu\nu}^{AB}(x).$$

Using the above ingredients, one can easily construct the lattice-regularized version of the action terms (16). For example, the action terms S_k of Eq. (16) reads

$$\tilde{S}_k = (4i)^k \sum_{\text{cells}} \frac{\epsilon^{i_0 i_1 \ldots i_d}}{(d+1)!} \frac{\epsilon^{A_1 A_2 \ldots A_d}}{d!} \times \\ \times \left(P_{i_0 i_1 i_2}^{A_1 A_2} P_{i_0 i_3 i_4}^{A_3 A_4} \cdots P_{i_0 i_{2k-1} i_{2k}}^{A_{2k-1} A_{2k}} \right) \left(\tilde{e}_{i_0 i_{2k+1}}^{A_{2k+1}} \cdots \tilde{e}_{i_0 i_d}^{A_d} \right), \tag{22}$$

where the total number of plaquette factors P (21) is k, $k = 0, 1, \ldots, [d/2]$.

The lattice-regularized partition function for the spinor quantum gravity is quite similar to that of the common lattice gauge theory. One integrates with the Haar measure over link variables U_{ij} living on lattice edges, and over anticommuting fermion variables $\psi_i, \psi_i^\dagger$ living on lattice sites. The partition function is

$$\mathcal{Z} = \prod_{\text{vertices } i} \int d\psi_i^\dagger \, d\psi_i \prod_{\text{links } ij} \int dU_{ij} \, \exp\left(\sum_{\text{cells}} \lambda_k^{(m)} \tilde{S}_k^{(m)}(\psi^\dagger, \psi, U)\right), \quad (23)$$

where $\tilde{S}_k^{(m)}$ are lattice actions of the type (22) with any number of composite frame fields $\tilde{e}$ (19) replaced by the other composite frame fields $\tilde{f}$ (20). Path integrals for the spinor gravity are well defined, and that overcomes the problem of a bottomless action, which is one of the principal problems in the quantization of the diffeomorphism-invariant actions.

Because of the requirement of the diffeomorphism invariance, the lattice action has certain similarities with but in fact is different from those used in common lattice gauge theory. Typically, one has many-fermion terms in the action. There are no action terms without fermions. One can write 3 action terms in $2d$ (all of them are 4-fermion), 6 terms in $3d$ (four are 6-fermion and two are 2-fermion), 8 terms in $4d$ (five are 8-fermion and three are 4-fermion), etc. Therefore, it is difficult to simulate this theory numerically; however, one can successfully apply various mean-field procedures [7].

1.3. Diffeomorphism-Invariant Lattice vs. Spin-Foam Models. There are several quantum gravity theories based on the so-called spin foams. Roughly speaking, the spin foam is a lattice field theory with only gauge fields (links), for a review see [8] and references therein. Originally the spin-foam models were obtained as a solution of the canonical quantization approach to the three-dimensional gravity [9]. Later on, the same construction was applied to many diffeomorphism-invariant gravity-like actions. It is well known that the same constructions can be obtained by lattice regularization on a dual lattice for a corresponding action. Here, we want to review briefly the dual lattice regularization and compare it with the regularization presented above.

The main idea of obtaining a spin-foam model from the action is well illustrated by the Ponzano–Regge model [1]. The Ponzano–Regge model is a topological BF model in three-dimension with the action

$$S_{\text{PR}} = \int d^3x \, \epsilon^{\mu\nu\rho} \, \epsilon^{ABC} \, B_\mu^A \mathcal{F}_{\nu\rho}^{\mathcal{BC}}, \quad (24)$$

where B is a vector field, and $\mathcal{F}$ is the curvature tensor (17) in $3d$. To regularize the theory, one covers the space with simplices (tetrahedra in $3d$) with the vector field B living on the simplices edges. The field strength tensor $\mathcal{F}$ is presented by plaquettes corresponding to the faces of the dual lattice. Each dual face is in one-to-one correspondence with an edge of the simplex. Therefore, discretization of the action (24) is

$$\epsilon^{\mu\nu\rho}\epsilon^{ABC}B_\mu^A\mathcal{F}_{\nu\rho}^{BC} \xrightarrow{\text{lattice}} \epsilon^{\mu\nu\rho}\epsilon^{ABC}\Delta x_{ij}^\mu B_{ij}^A P_{\text{face}}^{BC} S_{\text{face}}^{\nu\rho},$$

where Δx_{ij}^μ is the difference of the coordinates of the edge endpoints, and $S_{\text{face}}^{\nu\rho}$ is the area of the dual face. One can see that the convolution of the vector indices gives the volume of the «tops» stretched on the edge and its dual face. These «tops» fill the space without holes, and the complete lattice action has the form

$$S_{\text{PR}} = \sum_{\text{edges}} \epsilon^{ABC} B_{\text{edge}}^A P_{\text{face}}^{BC}. \tag{25}$$

Integrating this expression over the B field, one obtains the system of links on the dual lattice, which forms the classical spin-foam model of Ponzano–Regge [10]. Other spin-foam models are constructed in a similar way.

As one can see, in this approach it is necessary to split the space into two lattices: the regular and the dual one. One keeps the dynamical fields on the dual lattice, whereas the auxiliary fields live on the regular lattice. It is impossible to avoid such a splitting: the dual cells are polyhedra with an arbitrary number of faces and one cannot construct the volumes of these cells in the closed form. Therefore, although the dual lattice simplifies many calculations, this approach is not universal. It seems that the most general action requesting the dual lattice is the BF action with a polynomial in B [11].

In contrast, the approach presented in this paper can be applied to any action with any set of fields, which makes the method more powerful. The application of our approach, to the action (24) leads to the Penrose spin network, which is known to be equivalent to the Ponzano–Regge spin-foam model. Additionally, in our approach, fermion or other matter fields can be easily added to the action. For the spin-foam models the implementation of fermions is a big problem. So far it has been solved only perturbatively, with the help of the hopping-parameter expansion.

2. CONTINUUM LIMIT AND THE EINSTEIN–HILBERT ACTION

A successful construction of a lattice model does not guarantee the proper continuum limit. Lattice diffeomorphism invariance implies the diffeomorphism invariance in the continuum limit only for slowly varying fields. How can one guarantee the continuum limit for such lattice models? Or at least, how can one guarantee the restoration of the rotational invariance in large lattices? The key property of the diffeomorphism-invariant models is the absence of any fundamental length parameters — this is one of the necessary requirements of the diffeomorphism invariance which includes invariance under dilatations. In the lattice gauge theory there are also no explicit dimensional parameters; however, taking the dimensionless lattice parameter $\beta \to \infty$ (the inverse gauge coupling constant) guarantees the continuum limit. In the diffeomorphism-invariant models there is no such obvious handle. The standard lattice gauge theory tools, such as the lattice renormalization group, are not applicable.

A typical situation in lattice models of gravity is that correlation functions decay exponentially over a few lattice cells. It means that the fields vary strongly from one lattice cell to another, which prevents the gradient expansion of the fields on the way to obtaining the continuum limit. Lattice models have a continuum limit when and if field correlations are long-ranged in lattice units.

A standard way to guarantee long-range correlations and hence the continuum limit is to show that there is a second-order phase transition. Second-order phase transitions occur in theories where there is an order parameter usually related to the spontaneous breaking of a continuous symmetry. The corresponding Goldstone particles propagate to long distances. However, this is not enough: in order for the system to totally loose memory about the original lattice, *all* degrees of freedom have to propagate to long distances in lattice units. This happens only exactly at the phase transition point when all correlation functions are long-range.

In [7] the presence of phase transitions of the Berezinsky–Kosterlitz–Thouless type has been demonstrated in the $2d$ spinor gravity. In particular, it was shown that the space of the (dimensionless) coupling constants has two distinct regions with and without the spontaneous breaking of chiral symmetry. This was shown by a lattice mean-field method. The method allows one to calculate local quantities such as the chiral condensate, with a possibility to systematically improve the accuracy. However, it is still unclear if there can be other phase transitions in the model, for example, the spontaneous breaking of Lorentz symmetry, which is difficult to search for on lattice models. Evidence of a possibility to have spontaneous breaking of the Lorentz symmetry in diffeomorphism-invariant lattice models has been recently shown in [12].

If long-range correlations in a lattice model of quantum gravity are guaranteed in this way or another, the classical metric tensor $g^{\rm cl}_{\mu\nu}$ and the effective action functional $\Gamma[g^{\rm cl}_{\mu\nu}]$ can be introduced by means of the Legendre transform [7] (proposed in this context also by Wetterich [5]). One introduces first the generating functional for the stress-energy tensor $\Theta^{\mu\nu}$ as an external source,

$$\mathrm{e}^{W[\Theta]} = \int d\psi^\dagger\, d\psi\, d\omega_\mu\, \exp\left(S + \frac{1}{2}\int \hat{g}_{\mu\nu}\, \Theta^{\mu\nu}\right), \tag{26}$$

where $\hat{g}_{\mu\nu}$ is a metric operator of the theory, possibly a composite field. The classic metric field is by definition

$$g^{\rm cl}_{\mu\nu} = \langle \hat{g}_{\mu\nu} \rangle = 2\frac{\delta W[\Theta]}{\delta\Theta^{\mu\nu}}. \tag{27}$$

This equation can be solved back to give the functional $\Theta^{\mu\nu}[g^{\rm cl}]$. Using it, one can construct the effective action as the Legendre transform,

$$\Gamma[g^{\rm cl}] = W[\Theta] - g^{\rm cl}_{\mu\nu}\,\Theta^{\mu\nu}. \tag{28}$$

At the phase transition all fluctuations are long-ranged. For long-range fluctuations, it is legal to take the continuum limit of the lattice, which is diffeomorphism-invariant. The low-energy limit of diffeomorphism-invariant actions for a quantity transforming as a metric tensor is uniquely given by the expansion

$$S_{\rm low} = \int dx\,\sqrt{g^{\rm cl}}(-c_1 + c_2 R(g^{\rm cl}) + \ldots), \tag{29}$$

where the constants $c_{1,2}$ are certain dimensionless constants expressed through the dimensionless couplings $\lambda_k^{(m)}$ of the original microscopic lattice theory, e.g., given by Eq. (23).

The next important question for the diffeomorphism-invariant lattice theories is the appearance of the dimensions. The historic tradition in General Relativity is that the space-time at infinity is supposed to be flat; therefore, one can safely choose the coordinate system such that $g_{\mu\nu}$ is a unity matrix there. This sets the traditional dimensions of the fields. In particular, the scalar curvature has the dimension 1/length^2, the fermion fields have the dimension 1/length$^{3/2}$, etc. However, in a diffeomorphism-invariant quantum theory where one can perform arbitrary change of coordinates $x^\mu \to x'^\mu(x)$, this convention is neither natural, nor convenient.

The natural dimensions of the fields are those that are in accordance with their transformation properties: any contravariant vector transforms as x^μ and has the dimension of length, a covariant vector, in particular, the frame field e_μ transforms as a derivative and has the dimension 1/length, $g_{\mu\nu}$ has the dimension 1/length^2, etc. World scalars like the scalar curvature and the fermion fields are dimensionless. In fact, it is a tautology: a quantity invariant under diffeomorphisms is in particular invariant under dilatations and hence has to be dimensionless.

In this convention, any diffeomorphism-invariant action term is by construction dimensionless and is accompanied by a dimensionless coupling constant, as in (23).

Let us suppose that we have a microscopic quantum gravity theory at hand that successfully generates the first terms in the derivative expansion of the effective action (29). The ground state of that action is the space with constant curvature $R = 2c_1/c_2$, represented, e.g., by a conformal-flat metric

$$g_{\mu\nu} = \frac{6c_2}{c_1}\left(\frac{2\rho}{((x-x_0)^2+\rho^2)}\right)^2 \delta_{\mu\nu}, \tag{30}$$

where x_0 and ρ are arbitrary. At the vicinity of some observation point x_0, it can be made a unity matrix by rescaling the metric tensor,

$$g^{\rm cl}_{\mu\nu} = m^2\,\bar g_{\mu\nu}, \quad \bar g_{\mu\nu} = \delta_{\mu\nu}, \quad m \sim \sqrt{\frac{c_2}{c_1}}\,\frac{1}{\rho}. \tag{31}$$

The rescaling factor m has the dimension of mass, that is 1/length, such that $\bar g_{\mu\nu}$ has the conventional zero dimension. At this point one can rescale other fields to conventional dimensions, in particular, introduce the new fermion field $\bar\psi$ of conventional dimension $m^{3/2}$:

$$\psi = m^{-3/2}\,\bar\psi, \quad \psi^\dagger = m^{-3/2}\,\bar\psi^\dagger. \tag{32}$$

One can now rewrite the action (29) together with the fermionic matter in terms of the new rescaled fields denoted by a bar,

$$S = -\underbrace{c_1\,m^4}_{2\Lambda=\lambda^4}\int d^4x\,\sqrt{\bar g} + \underbrace{c_2\,m^2}_{M_P^2=1/\sqrt{16\pi G_N}}\int d^4x\,\sqrt{\bar g}\bar R + \\ + m^0\int d^4x\,\sqrt{\bar g}\,\bar e^{A\mu}\left(\bar\psi^\dagger\gamma^A\nabla_\mu\bar\psi + \text{h.c.}\right). \tag{33}$$

Underbraced are the cosmological constant and the Planck mass squared, respectively; numerically, $\lambda = 2.39\cdot10^{-3}$ eV, $M_P = 1.72\cdot10^{18}$ GeV. The dimensionless ratio of these values,

$$\frac{\lambda}{M_P} = \left(\frac{c_1\,m^4}{c_2^2\,m^4}\right)^{1/4} = \left(\frac{c_1}{c_2^2}\right)^{1/4} = 1.39\cdot10^{-30}, \tag{34}$$

is the only meaningful quantity in pure gravity theory, independent of the arbitrary scale parameter m.

If a fermion obtains an effective mass, e.g., as a result of the spontaneous chiral symmetry breaking, leading to an additional term in the effective low-energy action

$$S_m = \int d^4x\,\sqrt{g}\,\psi^\dagger\,\mathcal{M}\,\psi = \underbrace{m\,\mathcal{M}}_{\text{fermion mass } m_f}\int d^4x\,\sqrt{\bar g}\,\bar\psi^\dagger\bar\psi, \tag{35}$$

then the theory has to predict also other dimensionless ratios. For example, taking the top quark mass $m_t = 172$ GeV, one has to be able to explain the ratio

$$\frac{m_t}{\sqrt{\lambda M_P}} = \frac{\mathcal{M}}{c_1^{1/4}c_2^{1/2}} = 0.0848. \tag{36}$$

In other words, one can measure the Newton constant (or the Planck mass) or the cosmological constant in units of the quark or lepton masses or the Bohr radius. Only dimensionless ratios make sense and can be, as a matter of principle, calculated from a microscopic theory. It is convenient and legitimate to use natural dimensions when $g_{\mu\nu}$ has the dimension 1/length^2 whereas all world scalars are dimensionless, be it the scalar curvature R, the interval ds, the fermion field ψ or any diffeomorphism-invariant action term.

Finally, we would like to remark that, if the standard Einstein's gravity is obtained as a low-energy effective theory arising from a microscopic well-defined diffeomorphism-invariant lattice theory at its phase-transition point, the cosmological term, by construction, needs to have zero coefficient, $c_1 = 0$. Otherwise, the graviton would propagate to a finite distance $\sqrt{c_2/c_1}$, which contradicts the masslessness of the fluctuations at the phase transition. This is how one can recover Einstein's gravity from the lattice-regularized theory.

CONCLUSIONS

We have formulated the general method of lattice discretization for the diffeomorphism-invariant quantum field theories. The method is based on simplicial lattice and suitable for regularization of any diffeomorphism-invariant theory. As an example of application, we apply the presented method to the spinor gravity theory: the local Lorentz invariant and diffeomorphism-invariant theory based on fermions. The resulting action is very difficult for the direct numerical simulations, due to the dominant contribution of the multi-fermion interaction. The details on the lattice-regularized spinor gravity can be found in [6, 7].

Applying the lattice regularization to the quantum theory, one supposes that the lattice regularization can be removed in final result; i.e., one supposes that the continuum limit exists. In the diffeomorphism-invariant theories the continuum limit is not under control, due to the lack of natural scale, such as lattice spacing in lattice QCD. The continuum limit shows up if all degrees of freedom or at least some of them are slowly varying fields from one lattice cell to another. This is, generally, not fulfilled: generically, all correlation functions decay exponentially over a few lattice cells. For such «massive» degrees of freedom the theory is at the «strong coupling» regime where the continuum limit is not achieved and remains dormant.

There must be special physical reasons for massless excitations in the theory, for which the continuum limit makes sense and diffeomorphism invariance becomes manifest. One such reason is spontaneous breaking of continuous symmetry where the existence of massless fields is guaranteed by the Goldstone theorem. To obtain the low-energy Einstein limit, one has to stay at the second-order phase transition surface in the space of the coupling constants. We expect that at the phase transition surface the effective low-energy action for the classical metric tensor, derived through the Legendre transform, is just the Einstein–Hilbert action, with the zero cosmological term.

The described situation takes place in the spinor gravity [7], where one has the second-order phase transition, in plane of constants $\lambda_{1,2}$. This fact allows us to consider the spinor gravity as a serious candidate for the theory of microscopic general relativity.

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REFERENCES

1. *Ponzano G., Regge T.* Semiclassical Limits of Racach Coefficients // Spectroscopic and Theoretical Methods in Physics / Ed. F. Block. Amsterdam: North-Holland, 1968.
2. *De Pietri R., Freidel L.* $SO(4)$ Plebanski Action and Relativistic Spin Foam Model // Class. Quant. Grav. 1999. V. 16. P. 2187; gr-qc/9804071.
3. *Corichi A., Zapata J. A.* On Diffeomorphism Invariance for Lattice Theories // Nucl. Phys. B. 1997. V. 493. P. 475; gr-qc/9611034.
4. *Dittrich B.* How to Construct Diffeomorphism Symmetry on the Lattice. arXiv:1201.3840 [gr-qc].
5. *Wetterich C.* Lattice Diffeomorphism Invariance // Phys. Rev. D. 2012. V. 85. P. 104017; arXiv:1110.1539 [hep-lat].
6. *Diakonov D.* Towards Lattice-Regularized Quantum Gravity. arXiv:1109.0091 [hep-th].
7. *Vladimirov A. A., Diakonov D.* Phase Transitions in Spinor Quantum Gravity on a Lattice // Phys. Rev. D. 2012. V. 86. P. 104019; arXiv:1208.1254 [hep-th].
8. *Perez A.* The Spin Foam Approach to Quantum Gravity. arXiv:1205.2019 [gr-qc].
9. *Rovelli C., Smolin L.* Loop Space Representation of Quantum General Relativity // Nucl. Phys. B. 1990. V. 331. P. 80.
10. *Rovelli C.* The Basis of the Ponzano–Regge–Turaev–Viro–Ooguri Quantum Gravity Model in the Loop Representation Basis // Phys. Rev. D. 1993. V. 48. P. 2702; hep-th/9304164.
11. *Freidel L., Krasnov K.* Spin Foam Models and the Classical Action Principle // Adv. Theor. Math. Phys. 1999. V. 2. P. 1183; hep-th/9807092.
12. *Sexty D., Wetterich C.* Emergent Gravity in Two Dimensions. arXiv:1208.2168 [hep-th].