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A QUEER REDUCTION OF DEGREES OF FREEDOM

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The classical dynamics of antisymmetric second-rank tensor matter fields is analyzed. The conformally invariant action for the tensor field leads to a positive-definite Hamiltonian on the class of the solutions that are bounded at the time infinity (plane waves). Only the longitudinal waves contribute to the energy and momentum. The helicity proves to be equal to zero.

Проанализирована классическая динамика антисимметричных тензорных полей материи второго ранга. Конформно-инвариантное действие для тензорного поля приводит к положительно-определенному гамильтониану на классе решений, которые ограничены на временной бесконечности (плоские волны). В энергию и импульс дают вклад только продольные волны. Показано, что спиральность равна нулю.

In the field theory an elementary particle with an integer (a half integer) spin s is usually described by a totally symmetric tensor field $\Phi_{\mu_1\ldots\mu_s}$ (a tensor spinor field $\psi_{\mu_1\ldots\mu_{s-1/2}}$) [1]. The tensors (spinor-tensors) symmetrized by other Young tableaux were also considered in the literature. Two forms of the action are known for the antisymmetric second-rank tensor fields. In the first case the action possesses the gauge symmetry [2]:

$$\mathcal{S}_A = -\int d^4x \left[\frac{1}{4}(\partial_\lambda A_{\mu\nu})\partial^\lambda A^{\mu\nu} - \frac{1}{2}(\partial_\mu A^{\mu\lambda})\partial^\nu A_{\nu\lambda}\right], \qquad \delta A_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu. \quad (1)$$

The gauge fields (1) appear in supergravity theories in diverse dimensions [3] and in the effective low-energy field theory derived from relativistic strings [4]. These fields have been well studied, and eventually, a unitary S-matrix was constructed even in the non-Abelian case [5]. The gauge invariance leaves only one (longitudinal) polarization state for the tensor gauge field, in contrast to the ordinary vector gauge fields.

Another possibility is the conformally invariant action for the antisymmetric tensor matter fields $T_{\mu\nu}$:

$$\mathcal{S}_T = \int d^4x \left[\frac{1}{4}(\partial_\lambda T_{\mu\nu})\partial^\lambda T^{\mu\nu} - (\partial_\mu T^{\mu\lambda})\partial^\nu T_{\nu\lambda}\right]. \quad (2)$$

Such an action naturally arises in the conformal field theory [6] and in conformal supergravity [7]. However, the dynamics of such fields has not been completely investigated. There is an assertion that, in the Euclidean space, $T_{\mu\nu}$ describe three physical and three ghost degrees of freedom [8]. We are going to study the classical dynamics of the antisymmetric tensor matter fields in the Minkowski space. If we limit ourselves to the solutions that are bounded

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at the time infinity, then the Hamiltonian proves to be positive-definite; just two degrees of freedom contribute to the energy and momentum.

Six independent components of the real tensor field $T_{\mu\nu}$ can be parameterized by a three-dimensional vector $A_i = T_{0i}$ and a pseudovector $B_i = \frac{1}{2}\epsilon_{ijk}\, T_{jk}$, where the Latin indices take on values 1, 2, 3, and are summed over when repeated. Following the standard formalism for free classical fields [9], let us work in the momentum representation

$$\mathbf{A}(x) = \int \frac{d^4k}{(2\pi)^{3/2}} \, \exp{(ikx)}\, \mathbf{A}(k), \qquad \mathbf{B}(x) = \int \frac{d^4k}{(2\pi)^{3/2}} \, \exp{(ikx)}\, \mathbf{B}(k). \tag{3}$$

Choose a special reference frame $\mathbf{e}_i$: $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, $[\mathbf{e}_i \times \mathbf{e}_j] = \epsilon_{ijk}\mathbf{e}_k$, $\mathbf{e}_3 = \mathbf{k}/|\mathbf{k}|$. Let the decomposition of the fields over the basis be $\mathbf{A}(k) = a_i(k)\mathbf{e}_i$, $\mathbf{B}(k) = b_i(k)\mathbf{e}_i$. Then the action, Eq. (2), can be expressed as

$$\begin{aligned}\mathcal{S}_T = \pi \int d^4k \, \Big\{ \sum_{i=1}^{2} \big[a_i^*(k)\big(k_0^2+\mathbf{k}^2\big)a_i(k) + b_i^*(k)\big(k_0^2+\mathbf{k}^2\big)b_i(k)\big] + \\ +\; 2k_0\; |\mathbf{k}|\; \big[a_1^*(k)b_2(k) + b_2^*(k)a_1(k) - a_2^*(k)b_1(k) - b_1^*(k)a_2(k)\big] + \\ +\; a_3^*(k)\big(k_0^2-\mathbf{k}^2\big)a_3(k) + b_3^*(k)\big(k_0^2-\mathbf{k}^2\big)b_3(k)\Big\}.\end{aligned} \tag{4}$$

An additional rotation

$$\begin{aligned} a_1(k) &= \frac{1}{\sqrt{2}}\big[c_1(k)+d_2(k)\big], \qquad a_2(k) = \frac{1}{\sqrt{2}}\big[c_2(k)+d_1(k)\big], \qquad a_3(k) = c_3(k); \\ b_1(k) &= \frac{1}{\sqrt{2}}\big[d_1(k)-c_2(k)\big], \qquad b_2(k) = \frac{1}{\sqrt{2}}\big[d_2(k)-c_1(k)\big], \qquad b_3(k) = d_3(k) \end{aligned}$$

diagonalizes action (4)

$$\begin{aligned}\mathcal{S}_T = \pi \int d^4k \; \Big[c_1^*(k)\big(k_0-|\mathbf{k}|\big)^2 c_1(k) + c_2^*(k)\big(k_0+|\mathbf{k}|\big)^2 c_2(k) + \\ + c_3^*(k)\big(k_0-|\mathbf{k}|\big)\big(k_0+|\mathbf{k}|\big)c_3(k) + (c \to d)\Big].\end{aligned} \tag{5}$$

The principle of extreme action leads to the following field equations:

$$\begin{gathered}\big(k_0-|\mathbf{k}|\big)^2 c_1(k_0,\mathbf{k}) = 0, \qquad \big(k_0+|\mathbf{k}|\big)^2 c_2(k_0,\mathbf{k}) = 0, \\ \big(k_0-|\mathbf{k}|\big)\big(k_0+|\mathbf{k}|\big)\; c_3(k_0,\mathbf{k}) = 0.\end{gathered} \tag{6}$$

The same equations hold for $d_i(k)$. The general solutions to Eqs. (6) are of the form

$$\begin{aligned} c_1(k_0,\mathbf{k}) &= \delta\big(k_0-|\mathbf{k}|\big)c_1(\mathbf{k}) + \delta'\big(k_0-|\mathbf{k}|\big)\widetilde{c}_1(\mathbf{k}), \\ c_2(k_0,\mathbf{k}) &= \delta\big(k_0+|\mathbf{k}|\big)c_2(\mathbf{k}) + \delta'\big(k_0+|\mathbf{k}|\big)\widetilde{c}_2(\mathbf{k}), \\ c_3(k_0,\mathbf{k}) &= \delta\big(k_0-|\mathbf{k}|\big)\overline{c}_3(\mathbf{k}) + \delta\big(k_0+|\mathbf{k}|\big)c_3(\mathbf{k}). \end{aligned} \tag{7}$$

We want to note here that the transverse components $c_1(k_0,\mathbf{k})$ and $d_1(k_0,\mathbf{k})$ involve only positive frequencies $k_0=|\mathbf{k}|$. Other transverse components $c_2(k_0,\mathbf{k})$ and $d_2(k_0,\mathbf{k})$ involve only negative frequencies $k_0=-|\mathbf{k}|$.

The fact that $\mathbf{A}(x)$ and $\mathbf{B}(x)$ are real leads to the relations

$$\begin{aligned}
&c_2(\mathbf{k}) = sc_1^*(-\mathbf{k}) - cd_1^*(-\mathbf{k}),\\
&\widetilde{c}_2(\mathbf{k}) = -s\widetilde{c}_1{}^*(-\mathbf{k}) + c\widetilde{d}_1{}^*(-\mathbf{k}),\quad \overline{c}_3(\mathbf{k}) = -c_3^*(-\mathbf{k}),\\
&d_2(\mathbf{k}) = cc_1^*(-\mathbf{k}) + sd_1^*(-\mathbf{k}),\\
&\widetilde{d}_2(\mathbf{k}) = -c\widetilde{c}_1{}^*(-\mathbf{k}) - s\widetilde{d}_1{}^*(-\mathbf{k}),\ \overline{d}_3(\mathbf{k}) = -d_3^*(-\mathbf{k}),
\end{aligned} \tag{8}$$

where $c=\mathbf{e}_1(\mathbf{k})\cdot\mathbf{e}_1(-\mathbf{k}) = -\mathbf{e}_2(\mathbf{k})\cdot\mathbf{e}_2(-\mathbf{k}),\ s=\mathbf{e}_1(\mathbf{k})\cdot\mathbf{e}_2(-\mathbf{k}) = \mathbf{e}_2(\mathbf{k})\cdot\mathbf{e}_1(-\mathbf{k})$, $c^2+s^2=1$. Thus, $\overline{c}_3$, $\overline{d}_3$, and all the amplitudes with index 2 are not independent and can be eliminated.

The energy-momentum 4-vector for the tensor matter field is defined as

$$\mathcal{P}_\mu = \int d^3\mathbf{x}\,\Big[(\partial_\mu T_{\alpha\beta})\frac{\partial\mathcal{L}_T}{\partial(\partial_0 T_{\alpha\beta})} - g_{\mu 0}\mathcal{L}_T\Big].$$

By a direct calculation we find

$$\begin{aligned}
\mathcal{P}_0 = \int d^3\mathbf{x}\,\Big[\frac{1}{2}(\partial_0\mathbf{A})^2 - \frac{1}{2}(\partial_i\mathbf{A})^2 + (\partial_i A_i)^2 + (\mathbf{A}\to\mathbf{B})\Big] &=\\
= \int d^3\mathbf{k}\,\Big\{\widetilde{c}_1{}^*(\mathbf{k})\widetilde{c}_1(\mathbf{k}) - |\mathbf{k}|\,\big[\widetilde{c}_1{}^*(\mathbf{k})c_1(\mathbf{k}) + c_1^*(\mathbf{k})\widetilde{c}_1(\mathbf{k})\big] &+\\
+ 2\mathbf{k}^2 c_3^*(\mathbf{k})c_3(\mathbf{k}) + (c\to d)\Big\}&,
\end{aligned} \tag{9}$$

$$\begin{aligned}
\mathcal{P}_i = \int d^3\mathbf{x}\,\Big\{(\partial_i\mathbf{A})\cdot(\partial_0\mathbf{A}) + (\partial_i\mathbf{B})\cdot(\partial_0\mathbf{B}) + 2(\partial_i\mathbf{A})\cdot[\boldsymbol{\partial}\times\mathbf{B}]\Big\} &=\\
= \int d^3\mathbf{k}k_i\Big\{\widetilde{c}_1{}^*(\mathbf{k})c_1(\mathbf{k}) + c_1^*(\mathbf{k})\widetilde{c}_1(\mathbf{k}) + 2|\mathbf{k}|c_3^*(\mathbf{k})c_3(\mathbf{k}) + (c\to d)\Big\}&.
\end{aligned} \tag{10}$$

The obtained energy and momentum cannot be made simultaneously diagonal. The straightforward diagonalization of Eq. (9) reveals two positive- and two negative-energy transverse modes which cannot be interpreted as relativistic particles. These facts are due to the presence of δ'-type solutions in Eqs. (7). In the co-ordinate representation (3) such solutions do not correspond to ordinary plane waves, but rather grow linearly with the time and are unbounded at the infinity. If we set their amplitudes to zero, that is, choose to restrict ourselves to the plane-wave sector, then the transverse plane-wave amplitudes drop out of the energy-momentum. Everything is reduced to the longitudinal waves. Both energy (9) and momentum (10) can be written uniformly:

$$\mathcal{P}_\mu = \int d^3\mathbf{k}k_\mu\big[c_3^+(\mathbf{k})c_3^-(\mathbf{k}) + d_3^+(\mathbf{k})d_3^-(\mathbf{k})\big], \tag{11}$$

where, as usual [9], $c_3^+(\mathbf{k}) = c_3^*(\mathbf{k})/\sqrt{2k_0}$, $c_3^-(\mathbf{k}) = c_3(\mathbf{k})/\sqrt{2k_0}$, $k_0=|\mathbf{k}|$. The Hamiltonian $\mathcal{P}_0$ becomes positive-definite on the plane-wave solutions. This important property would not take place, if one introduced a mass term like $T^{\mu\nu}T_{\mu\nu}$ in Eq. (2). Thus, the free tensor field describes massless relativistic particles.

The helicity — the projection of the spin onto the direction of motion — proves to be equal to zero. In fact, even without the restriction to plane waves, the 3-vector of spin

$$\mathbf{S} = \int d^3\mathbf{x}\Big([\mathbf{A}\times\partial_0\mathbf{A}] + [\mathbf{B}\times\partial_0\mathbf{B}] + \big[\mathbf{A}\times[\boldsymbol{\partial}\times\mathbf{B}]\big] - \big[\mathbf{B}\times[\boldsymbol{\partial}\times\mathbf{A}]\big]\Big) \tag{12}$$

vanishes on solutions (7).

There are global transformations of the vector and pseudovector fields

$$\begin{pmatrix}\mathbf{A}'\\ \mathbf{B}'\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}, \qquad \delta T_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\nu\kappa\lambda}\,T^{\kappa\lambda}\,\delta\alpha \tag{13}$$

that leave action (2) invariant. They induce a conserved axial charge

$$\begin{aligned}\mathcal{Q} &= \int d^3\mathbf{x}\,\Big(\mathbf{A}\cdot\partial_0\mathbf{B} - \mathbf{B}\cdot\partial_0\mathbf{A} - \mathbf{A}\cdot[\boldsymbol{\partial}\times\mathbf{A}] - \mathbf{B}\cdot[\boldsymbol{\partial}\times\mathbf{B}]\Big) = \\ &= -i\int d^3\mathbf{k}\big[c_1^*\widetilde{d}_1 - \widetilde{d}_1{}^*c_1 + d_1^*\widetilde{c}_1 - \widetilde{c}_1{}^*d_1 + 2|\mathbf{k}|(c_3^*d_3 - d_3^*c_3)\big],\end{aligned} \tag{14}$$

which in the plane-wave sector is reduced to

$$\mathcal{Q} = -i\int d^3\mathbf{k}\big[c_3^+(\mathbf{k})d_3^-(\mathbf{k}) - d_3^+(\mathbf{k})c_3^-(\mathbf{k})\big]. \tag{15}$$

Again, the transverse components fall out. The axial (chiral) symmetry (13) — as well as the positive definiteness of the energy — forbids the mass term for $T_{\mu\nu}$.

As the dynamical invariants for bounded solutions do not depend on the transverse components of the tensor field, it is natural to suppose that only the longitudinal excitations are physical. In contrast to the tensor gauge field (which has only one degree of freedom on shell [2]), two physical states (the longitudinal components of the vector $\mathbf{A}$ and pseudovector $\mathbf{B}$) are left for the tensor matter field. On the mass shell the transverse components play no role. They become important as interactions with other fields and a self-interaction are added [10].

It is also worth comparing the tensor matter field with the vector gauge field. The transverse plane-wave components of the tensor field are similar to the time and longitudinal polarizations of the photon. However, in the latter case there is a gauge symmetry which is responsible for the cancellation of unphysical degrees of freedom. Introducing the scalar (anti)ghost fields with unusual commutation rules, one can write the more general BRST transformations [11] which leave the action invariant even after gauge fixing. The ghost degrees of freedom are subtracted from the total number, thus leaving only physical degrees of freedom.

In the case of the tensor matter field there is no gauge invariance. However, like for the tensor gauge fields, we can add a pyramid of ghost fields (ghosts for ghosts): anticommuting C_μ, $\overline{C}_\mu$ with the ghost numbers ±1, commuting D, $\overline{D}$ with the zero ghost number, and E, $\overline{E}$ with the ghost numbers ±2. Then, naively counting the degrees of freedom, $6-2\times4+2\times2$, we get just two longitudinal physical degrees of freedom for the tensor matter field. The action for the ghosts is of the form

$$\mathcal{S}_{\rm ghost} = \int d^4x \Big[(\partial^\mu \overline{C}^\nu)\partial_\mu C_\nu + (\partial^\mu \overline{D})\partial_\mu D + (\partial^\mu \overline{E})\partial_\mu E\Big]. \tag{16}$$

The total action $\mathcal{S}_T + \mathcal{S}_{\rm ghost}$ is left invariant under the following nilpotent transformations with the anticommuting constant parameter Λ whose ghost number equals 1:

$$\begin{gathered} \delta T_{\mu\nu} = -(\partial_\mu \overline{C}_\nu - \partial_\nu \overline{C}_\mu)\Lambda, \qquad \delta C_\nu = (\partial^\mu T_{\mu\nu} + \partial_\nu D)\Lambda, \\ \delta \overline{D} = \partial^\nu \overline{C}_\nu \Lambda, \qquad \delta \overline{C}_\nu = \partial_\nu \overline{E}\Lambda, \qquad \delta E = -\partial^\nu C_\nu \Lambda, \qquad \delta D = \delta \overline{E} = 0. \end{gathered} \tag{17}$$

These transformations are valid for the theory of the free fields. In case of an interaction, Eqs. (17) should be generalized to involve the coupling constants.

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