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NILPOTENT QUANTUM MECHANICS AND SUSY

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Formalism where fundamental variables are nilpotent, but in contrast to the supermathematics, not anticommutative but commutative, gives another version of realization of the Pauli exclusion principle. We discuss some aspects of nilpotent quantum mechanics realized in the generalized Hilbert space of functions of nilpotent commuting variables. The qubits are natural objects described by such a formalism. Supersymmetric system of qubit and fermion is presented.

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INTRODUCTION

In the previous edition of the «Supersymmetry and Quantum Symmetry», SQS'07, we have presented elements of the classical or prequantum theory of nilpotent systems. Despite apparently very simple properties of the basic variables in terms of which the theory is described, resulting formalism turns out to be interesting and nontrivial [1, 2]. Presently, we show the extension of the above formalism to the quantum case. Quantum systems described by nilpotent commuting variables are not related to fundamental object but rather composed ones, in description of which we do not enter their intrinsic structure, but treat them as nondecomposable. In a natural way, the formalism we present, suits the analysis of entanglement in multiqubit systems. The applicability of nilpotent commuting variables is not restricted only to the quantum mechanics, but they are also of use in the quantum field theory and statistical physics [3,4]. In the context of the description of qubit systems, the formalism of functions of η variables — functions of the first order nilpotent commuting variables, $\eta^2 = 0$, gives natural language to address the entanglement questions for multiqubit systems. It allows one to single out the set of appropriate invariants, which are coming from the η-functional determinants. Many interesting states known from quantum optics, represented in the η-function language, turn out to be just elementary functions of several η variables [5]. In the present contribution, we show how supersymmetry can be implemented into the systems of fermions and qubits, where we stick to the choice that qubit, when in ensamble, is not a fermionic object [6].

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1. NILPOTENT QUANTUM MECHANICS AND GENERALIZED HILBERT SPACE

To study the first quantized theory of η-nilpotent systems, it is natural to consider an analog of the Schrödinger quantization and the generalization of the Hilbert space formalism. Let us introduce an η-Hilbert space of the L^2 functions. In the set of functions $F(\eta_1, \eta_2, \ldots, \eta_n)$, we define generalized scalar product given by the integral

$$\langle F,\ G\rangle_{\mathcal{N}} = \int F^*(\boldsymbol{\eta})\, G(\boldsymbol{\eta})\, \mathrm{e}^{\langle \boldsymbol{\eta}^*, \boldsymbol{\eta}\rangle}\, d\boldsymbol{\eta}^*\, d\boldsymbol{\eta}. \tag{1}$$

It has desirable properties

$$\langle \nu F, G\rangle = \langle F, \nu^* G\rangle, \quad \nu \in \mathcal{N}, \tag{2}$$

$$\langle F, G\rangle = 0 \quad \forall G \in \mathcal{H} \Rightarrow F = 0, \tag{3}$$

$$b(\langle F, G\rangle)^* = b(\langle G, F\rangle), \tag{4}$$

$$b(\langle F, F\rangle) \geqslant 0, \quad \forall F \in \mathcal{H}, \tag{5}$$

where the $b(\langle F, G\rangle)$ denotes the body of $\mathcal{N}$ number, i.e., a numerical (real or complex) part of an $\mathcal{N}$ number. The conjugation is defined in such a way

$$F^*(\boldsymbol{\eta}) = \sum_{k=0}^{n} \sum_{I_k} F^*_{I_k} \eta^{I_k\,*}. \tag{6}$$

The one-qubit states are realized in this formalism by the η functions of one variable. In particular, η-scalar product of $F(\eta)$ and $G(\eta)$ functions takes the simple form

$$\langle F,\ G\rangle_{\mathcal{N}} = F_0^* G_0 + F_1^* G_1. \tag{7}$$

For example, two-qubit trigonometric states are given by the following normalized functions:

$$\psi_{GHZ-} = \frac{1}{\sqrt{2}} \cos(\eta_1 + \eta_2) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \tag{8}$$

$$\psi_W = \frac{1}{\sqrt{2}} \sin(\eta_1 + \eta_2) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle). \tag{9}$$

The last expressions in the above formulas show, how the trigonometric η functions for two qubits read in the so-called binary basis.

Description of an η-nilpotent system in the above space we call the η-Schrödinger representation. The classical theory based on nilpotent commuting variables, the so-called nilpotent classical mechanics, provides configuration and phase space description of nilpotent systems [1]. It is worth noting that there exists path integral formalism, discussed by Palumbo et al. [8–10]. The nilpotent quantum mechanical systems can be obtained by a kind of an «η-canonical quantization» of the classical nilpotent systems. We shall use here the restricted η-Schrödinger quantization in the following sense. A classical observable of the nilpotent system is taken in the «normal ordered» form, i.e., momentum variables are to the right of the coordinate variables. The position and momentum operators are realized as follows:

$$\eta_k \longrightarrow \hat{\eta} = \eta_k \cdot, \quad p \longrightarrow \hat{p}_k = \frac{\partial}{\partial \eta_k} \tag{10}$$

in the $\mathcal{N}$-Hilbert space of η functions depending on η_k, $k = 1, 2, \ldots, n$. Taking $\psi \in \mathcal{F}[x, \boldsymbol{\eta}]$, we have

$$i\hbar \frac{d}{dt}\psi(x, \boldsymbol{\eta}, t) = \hat{H}\psi(x, \boldsymbol{\eta}, t), \tag{11}$$

where $\hat{H}$ is the quantized Hamiltonian $H(x, p_x, \eta, p_\eta, t)$ of the system. The two-level systems considered in literature typically are taken with explicit time dependence of the Hamiltonian. For example, in the $n = 1$ case the Hamilton function is singular in its nilpotent part in the sense that it contains terms linear in p_η, i.e., $H = \frac{1}{2m}p_x^2 + b(t)p_\eta + c(t)\eta p_\eta + V(x, \eta, t)$. After quantization, this Hamiltonian can be written in the following form $\hat{H} = \frac{1}{2m}\hat{p}_x^2 + V(x) + \mathbf{B}(t) \cdot \boldsymbol{\sigma}$, where nilpotent part is

$$\hat{H}_{\rm nilp} = (B_x(t) + iB_y(t))\eta + (B_x(t) - iB_y(t))\frac{\partial}{\partial\eta} - 2B_z(t)\eta\frac{\partial}{\partial\eta} + B_z. \tag{12}$$

One can describe the properties of the nilpotent part of the system alone, neglecting the question of the simultaneous x-coordinate dependence [5]. The stationary η-Schrödinger equation for nilpotent quantum system can be written as follows:

$$\hat{H}\psi(\boldsymbol{\eta}) = \lambda\psi(\boldsymbol{\eta}). \tag{13}$$

The set of eigenstates for multiqubit systems turns out to be nontrivial — they are entangled.

The above-mentioned η-Schrödinger equation can be related to the one studied in [7]. In the latter one, the authors restrict themselves to the case when η-wave functions have invertible values in the algebra $\mathcal{N}$, i.e., $\psi(\boldsymbol{\eta}) = \psi_0 + \psi_i\eta_i + \ldots$ and $\psi_0 \neq 0$. One can take function $\tilde{\psi}(\boldsymbol{\eta}) = \frac{1}{\psi_0}\psi(\boldsymbol{\eta})$, and there exists its logarithm $f(\boldsymbol{\eta}) = ln\,\tilde{\psi}$. Using relations

$$i\frac{d}{dt}f(\boldsymbol{\eta}) = i\frac{d}{dt}\,\ln\,\tilde{\psi}(\boldsymbol{\eta}) = i\tilde{\psi}^{-1}(\boldsymbol{\eta})H\tilde{\psi}(\boldsymbol{\eta}) \tag{14}$$

and $\tilde{\psi}(\boldsymbol{\eta}) = {\rm e}^{f(\boldsymbol{\eta})}$, one can write

$$i\frac{d}{dt}f(\boldsymbol{\eta}) = {\rm e}^{-f(\boldsymbol{\eta})}H\,{\rm e}^{f(\boldsymbol{\eta})}, \tag{15}$$

which is the form of the equation employed in [7]. But let us note that there are many states that have η-wave functions with noninvertible values (like Werner-like states) and for them such equation is not valid, but η-Schrödinger equation (11) can be used.

2. TWO-LEVEL SUPERSYMMETRIC SYSTEMS

Let us describe a system composed of qubit and fermion. Both components forming this system are two-level. The algebra of sypersymmetry consists of odd supercharges and end-even Hamiltonian, as in the conventional boson–fermion case

$$Q_{\rm QF} = i\sqrt{\omega}\,d \otimes f^+, \tag{16}$$

$$Q_{\rm QF}^+ = -i\sqrt{\omega}\,d^+ \otimes f. \tag{17}$$

Graded commutation relations for this generators are

$$[Q_{\rm QF}^{+},\ Q_{\rm QF}]_{+} = \omega\left(d^{+}d\otimes 1 + (1-2N_d)\otimes f^{+}f\right) = H_{\rm QF}^{(0)}, \tag{18}$$

$$H_{\rm QF}^{(F)} = Q_{\rm QF}^{+} + Q_{\rm QF}, \tag{19}$$

$$H_{\rm QF} = H_{\rm QF}^{(0)} + \kappa H_{\rm QF}^{(F)}, \tag{20}$$

$$[H_{\rm QF},\ Q_{\rm QF}]_{-} = \kappa H_{\rm QF}^{(0)}, \tag{21}$$

and

$$[H_{\rm QF},\ Q_{\rm QF}^{+}]_{-} = \kappa H_{\rm QF}^{(0)}. \tag{22}$$

Both parts of this system are two-level systems and they are related to nontrivial supersymmetry transformations.

The Hamiltonian $H_{\rm QF}^{(0)}$ in the η-Schrödinger representation can be written in the following form:

$$\hat{H}_{\rm QF}^{(0)} = \omega(\eta\partial_\eta + \theta\partial_\theta - 2\eta\theta\partial_\eta\partial_\theta). \tag{23}$$

Generalized stationary η-Schrödinger equation $\hat{H}_{\rm QF}^{(0)}F(\eta,\theta) = \lambda F(\eta,\theta)$ yields the eigenspace related to the zero energy, and another one related to $\lambda = \omega$. So the vacuum is degenerated and invariant under supersymmetry transformations. The subspace with nonzero energy is degenerated, as it should be in supersymmetric system. The nonunique ground state is peculiar. So, we have $\phi_0 = 1$ and $\psi_0 = \eta\theta$ even and odd, respectively, ground states (in the sense of the Grassmannian parity), and $\phi_\omega = \eta$, $\psi_\omega = \theta$ even and odd excited states.

$$\hat{H}_{\rm QF}^{(0)}\phi_\omega = \omega\phi_\omega, \tag{24}$$

$$\hat{H}_{\rm QF}^{(0)}\psi_\omega = \omega\psi_\omega, \tag{25}$$

$$\hat{H}_{\rm QF}^{(0)}\phi_0 = 0, \tag{26}$$

$$\hat{H}_{\rm QF}^{(0)}\psi_0 = 0; \tag{27}$$

$$\hat{Q}\phi_\omega = \psi_\omega, \tag{28}$$

$$\hat{Q}\psi_\omega = 0, \tag{29}$$

$$\hat{Q}\phi_0 = 0, \tag{30}$$

$$\hat{Q}\psi_0 = 0, \tag{31}$$

$$\hat{Q}^{+}\phi_\omega = 0, \tag{32}$$

$$\hat{Q}^{+}\psi_\omega = \phi_\omega, \tag{33}$$

$$\hat{Q}^{+}\phi_0 = 0, \tag{34}$$

$$\hat{Q}^{+}\psi_0 = 0. \tag{35}$$

In the above example we use notation that ϕ_i are even and ψ_i are odd, $i = 0,\, \omega$.

From the point of view of the supersymmetric quantum mechanics, we have here pair of nontrivial zero modes, and therefore the full even and odd spectra are identical, which means that the analog of Witten index vanishes

$$\Delta = n_Q^{(E=0)} - n_F^{(E=0)} = 0.$$

Such effect is not new and is known from SUSY quantum mechanical models with a periodic potential [11, 12] or with local and nonlocal potentials [13] as well as in the model of spin $1/2$ particle in a rotating magnetic field and constant scalar potential [14]. Here in the qubit–fermion system, the effect has algebraical origin and comes from the structure of the model, not from the particular properties of the potential.

3. FINAL COMMENTS

We have sketched the formalism allowing quantum description of qubits with the use of nilpotent commuting variables. For functions of such variables, we can conveniently formulate criteria for factorization, which in the same time answer the questions of entanglement of multiqubit pure states. An extremely interesting is simple supersymmetric model of qubit and fermion. We have given basic properties of such a system. Finally, let us enumerate main points of the formalism: Pauli principle is incorporated, but described objects otherwise show bosonic behavior; the formalism is analogous to supermathematical and is of use in quantum mechanics and field theory, there exists «classical» limit (like for fermions) for the qubit systems. Quantum η formalism is a development of the classical η formalism providing a way of description of the nilpotent systems.

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