Письма в ЭЧАЯ. 2011. Т. 8, № 3(166). С. 535–544

FRACTAL CALCULUS (H) AND SOME APPLICATIONS

N. Makhaldiani[1]

Joint Institute for Nuclear Research, Dubna

A short history, background, and some applications of the fractal calculus are presented.

PACS: 01.65.+g; 05.45.Df

If you are only a poet,
You are not even that.
Piet Hein

INTRODUCTION

In the Universe, matter has mainly two geometric structures, homogeneous [11] and hierarchical [9]. The homogeneous structures are naturally described by real numbers with an infinite number of digits in the fractional part and usual Archimedean metrics. The hierarchical structures are described with p-adic numbers with an infinite number of digits in the integer part and non-Archimedean metrics [3].

A discrete, finite, regularized version of the homogeneous structures are homogeneous lattices with constant steps, and distance rising as arithmetic progression. The discrete version of the hierarchical structures is hierarchical lattice-tree with scale rising in geometric progression.

There is an opinion that present-day theoretical physics needs (almost) all mathematics, and the progress of modern mathematics is stimulated by fundamental problems of theoretical physics.

1. REAL, p-ADIC AND q-UANTUM FRACTAL CALCULUS

Every (good) school boy/girl knows what is

$$\frac{d^n}{dx^n} = \partial^n = (\partial)^n, \tag{1}$$

but what is its following extension:

$$\frac{d^\alpha}{dx^\alpha} = \partial^\alpha, \quad \alpha \in \Re \ ? \tag{2}$$

[1]E-mail: mnv@jinr.ru

1.1. Euler, . . . Liouville, . . . Holmgren, . . . Let us consider the integer derivatives of the monomials

$$\frac{d^n}{dx^n}x^m = m(m-1)\dots(m-(n-1))x^{m-n}, \quad n \leqslant m, \\ = \frac{\Gamma(m+1)}{\Gamma(m+1-n)}x^{m-n}. \tag{3}$$

L. Euler (1707–1783) invented the following definition of the fractal derivatives:

$$\frac{d^\alpha}{dx^\alpha}x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}. \tag{4}$$

J. Liouville (1809–1882) takes exponents as a base functions,

$$\frac{d^\alpha}{dx^\alpha}\,\mathrm{e}^{ax} = a^\alpha\,\mathrm{e}^{ax}. \tag{5}$$

J. H. Holmgren invented (in 1863) the following integral transformation:

$$D_{c,x}^{-\alpha}f = \frac{1}{\Gamma(\alpha)}\int\limits_c^x |x-t|^{\alpha-1}f(t)dt. \tag{6}$$

It is easy to show that

$$D_{c,x}^{-\alpha}x^m = \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)}(x^{m+\alpha}-c^{m+\alpha}), \\ D_{c,x}^{-\alpha}\mathrm{e}^{ax} = a^{-\alpha}(\mathrm{e}^{ax}-\mathrm{e}^{ac}), \tag{7}$$

so, $c = 0$, when $m+\alpha \geqslant 0$, in Holmgren's definition of the fractal calculus, corresponds to Euler's definition, and $c = -\infty$, when $a > 0$, corresponds to Liouville's definition. Holmgren's definition of the fractal calculus reduces to Euler's definition for finite c, and to Liouville's definition for $c = \infty$,

$$D_{c,x}^{-\alpha}f = D_{0,x}^{-\alpha}f - D_{0,c}^{-\alpha}f, \quad D_{\infty,x}^{-\alpha}f = D_{-\infty,x}^{-\alpha}f - D_{-\infty,\infty}^{-\alpha}f. \tag{8}$$

We considered the following modification of the $c = 0$ case [7]:

$$D_{0,x}^{-\alpha}f = \frac{|x|^\alpha}{\Gamma(\alpha)}\int\limits_0^1 |1-t|^{\alpha-1}f(xt)dt, \quad = \frac{|x|^\alpha}{\Gamma(\alpha)}B(\alpha,\ \partial x)f(x), \\ = |x|^\alpha\frac{\Gamma(\partial x)}{\Gamma(\alpha+\partial x)}f(x), \quad f(xt) = t^{x\frac{d}{dx}}f(x). \tag{9}$$

As an example, consider Euler B-function,

$$B(\alpha,\beta) = \int\limits_0^1 dx|1-x|^{\alpha-1}|x|^{\beta-1} = \Gamma(\alpha)\Gamma(\beta)D_{01}^{-\alpha}D_{0x}^{-\beta}1 = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{10}$$

We can define also FC as

$$D^{\alpha} f = (D^{-\alpha})^{-1} f = \frac{\Gamma(\partial x + \alpha)}{\Gamma(\partial x)}(|x|^{-\alpha} f), \quad \partial x = \delta + 1, \quad \delta = x\partial. \tag{11}$$

For Liouville's case,

$$D^{\alpha}_{-\infty,x} f = (D_{-\infty,x})^{\alpha} f = (\partial_x)^{\alpha} f, \tag{12}$$

$$\partial_x^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int\limits_0^{\infty} dt\, t^{\alpha-1} \mathrm{e}^{-t\partial_x} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_0^{\infty} dt\, t^{\alpha-1} f(x-t) =$$

$$= \frac{1}{\Gamma(\alpha)} \int\limits_{-\infty}^{x} dt (x-t)^{\alpha-1} f(t) = D^{-\alpha}_{-\infty,x} f. \tag{13}$$

The integrals can be calculated as

$$D^{-n} f = (D^{-1})^n f, \tag{14}$$

where

$$D^{-1} f = x \frac{\Gamma(\partial x)}{\Gamma(1+\partial x)} f = x \frac{1}{\partial x} f = x(\partial x)^{-1} f = (\partial)^{-1} f = \int\limits_0^x dt f(t). \tag{15}$$

Let us consider Weierstrass' C. T. W. (1815–1897) fractal function

$$f(t) = \sum_{n \geqslant 0} a^n\, \mathrm{e}^{i(b^n t + \varphi_n)}, \quad a < 1, \quad ab > 1. \tag{16}$$

For fractals we have no integer derivatives,

$$f^{(1)}(t) = i \sum (ab)^n\, \mathrm{e}^{i(b^n t + \varphi_n)} = \infty, \tag{17}$$

but the fractal derivative,

$$f^{(\alpha)}(t) = \sum (ab^{\alpha})^n\, \mathrm{e}^{i(b^n t + \pi\alpha/2 + \varphi_n)}, \tag{18}$$

when $ab^{\alpha} = a' < 1$, is another fractal (16).

1.2. p-Adic Fractal Calculus. p-adic analog of the fractal calculus (6),

$$D_x^{-\alpha} f = \frac{1}{\Gamma_p(\alpha)} \int\limits_{Q_p} |x-t|_p^{\alpha-1} f(t)\, dt, \tag{19}$$

where $f(x)$ is a complex function of the p-adic variable x, with p-adic Γ-function

$$\Gamma_p(\alpha) = \int\limits_{Q_p} dt\, |t|_p^{\alpha-1} \chi(t) = \frac{1-p^{\alpha-1}}{1-p^{-\alpha}}, \tag{20}$$

was considered by V. S. Vladimirov [10].

The following modification of p-adic FC is given in [7]:

$$D_x^{-\alpha} f = \frac{|x|_p^\alpha}{\Gamma_p(\alpha)} \int\limits_{Q_p} |1-t|_p^{\alpha-1} f(xt)\, dt = |x|_p^\alpha \frac{\Gamma_p(\partial|x|)}{\Gamma_p(\alpha+\partial|x|)} f(x). \tag{21}$$

1.3. Fractal q-Calculus. The basic object of q-calculus [1] is q-derivative

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x} = \frac{1-q^{x\partial}}{(1-q)x} f(x), \tag{22}$$

where either $0<q<1$ or $1<q<\infty$. In the limit $q \to 1, D_q \to \partial_x$.

Now we define the fractal q-calculus,

$$D_q^\alpha f(x) = (D_q)^\alpha f(x) = \\ = ((1-q)x)^{-\alpha} (f(x) + \sum_{n\geqslant 1} (-1)^n \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} f(q^n x)). \tag{23}$$

For the case $\alpha = -1$, we obtain the integral

$$D_q^{-1} f(x) = (1-q)x(1-q^{x\partial})^{-1} f(x) = (1-q)x \sum_{n\geqslant 0} f(q^n x). \tag{24}$$

In the case of $1<q<\infty$, we can give a good analytic sense to these expressions for prime numbers $q = p = 2, 3, 5, \ldots, 29, \ldots, 137, \ldots$ This is an *algebra-analytic quantization* of the q-calculus and corresponding physical models. Note also, that p-adic calculus is the natural tool for the physical models defined on the fractal spaces like Bete lattice (or Brua–Tits trees, in mathematical literature).

1.4. Fractal Finite-Difference Calculus. Usual finite-difference calculus is based on the following (left) derivative operator:

$$D_- f(x) = \frac{f(x)-f(x-h)}{h} = \left(\frac{1-\mathrm{e}^{-h\partial}}{h}\right) f(x). \tag{25}$$

We define corresponding fractal calculus as

$$D_-^\alpha f(x) = (D_-)^\alpha f(x). \tag{26}$$

In the case of $\alpha = -1$, we have usual finite difference sum as regularization of the Riemann integral

$$D_-^{-1} f(x) = h(f(x) + f(x-h) + f(x-2h) + \ldots). \tag{27}$$

I believe that the fractal calculus (and geometry) are the proper language for the quantum (field) theories, and discrete versions of the fractal calculus are proper regularizations of the fractal calculus and field theories.

2. HYPERGEOMETRIC FUNCTIONS

A hypergeometric series, in the most general sense, is a power series in which the ratio of successive coefficients indexed by n is a rational function of n,

$$f(x) = \sum_{n \geqslant 0} a_n x^n, \quad a_{n+1} = R(n) a_n, \quad R(n) = \frac{P(\alpha, n)}{Q(\beta, n)}, \tag{28}$$

so

$$\begin{gathered} P(\alpha, \delta) f(x) = Q(\beta, \delta) \frac{f(x) - f(0)}{x}, \\ f(x) - f(0) = x R(\delta) f(x), \quad f(x) = (1 - x R(\delta))^{-1} f(0), \quad \delta = x \partial_x. \end{gathered} \tag{29}$$

Hypergeometric functions have many particular special functions as special cases, including many elementary functions, the Bessel functions, the incomplete gamma function, the error function, the elliptic integrals and the classical orthogonal polynomials, because the hypergeometric functions are solutions to the hypergeometric differential equation, which is a fairly general second-order ordinary differential equation.

In the generalization given by Eduard Heine in the late nineteenth century, the ratio of successive terms, instead of being a rational function of n, is considered to be a rational function of q^n

$$\begin{gathered} f(x) = \sum_{n \geqslant 0} a_n x^n, \ a_{n+1} = R(q^n) a_n, \ R(n) = \frac{P(\alpha, q^n)}{Q(\beta, q^n)}, \\ P(\alpha, q^\delta) f(x) = Q(\beta, q^\delta) \frac{f(x) - f(0)}{x}, \\ f(x) - f(0) = x R(q^\delta) f(x), \quad f(x) = (1 - x R(q^\delta))^{-1} f(0), \quad \delta = x \partial_x. \end{gathered} \tag{30}$$

Another generalization, the elliptic hypergeometric series, are those series where the ratio of terms is an elliptic function (a doubly periodic meromorphic function) of n.

There are a number of new definitions of hypergeometric series, by Aomoto, Gelfand and others, and applications, for example, to the combinatorics of arranging a number of hyperplanes in complex N-space.

2.1. Lauricella Hypergeometric Functions (LFs). For LFs (see, e.g., [8]), we find the following formulas:

$$\begin{gathered} F_A(a; b_1, \ldots, b_n; c_1, \ldots, c_n; z_1, \ldots, z_n) = \frac{(a)_{\delta_1 + \ldots + \delta_n} (b_1)_{\delta_1} \cdots (b_n)_{\delta_n}}{(c_1)_{\delta_1} \cdots (c_n)_{\delta_n}} \mathrm{e}^{z_1 + \ldots + z_n} = \\ = \frac{(a)_{\delta_1 + \ldots + \delta_n}}{(a_1)_{\delta_1} \cdots (a_n)_{\delta_n}} F(a_1, b_1; c_1; z_1) \cdots F(a_n, b_n; c_n; z_n) = \\ = T^{-1}(a) F^n = \sum_{m \geqslant 0} \frac{(a)_{m_1 + \ldots + m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_n^{m_n}}{m_n!}, \quad |z_1| + \ldots + |z_n| < 1; \end{gathered}$$

$$F_B(a_1,\ldots,a_n;b_1,\ldots,b_n;c;z_1,\ldots,z_n)=\frac{(a_1)_{\delta_1}\cdots(a_n)_{\delta_n}(b_1)_{\delta_1}\cdots(b_n)_{\delta_n}}{(c)_{\delta_1+\ldots+\delta_n}}\,\mathrm{e}^{z_1+\ldots+z_n}=$$
$$=\frac{(c_1)_{\delta_1}\cdots(c_n)_{\delta_n}}{(c)_{\delta_1+\cdots+\delta_n}}F(a_1,b_1;c_1;z_1)\cdots F(a_n,b_n;c_n;z_n)=T(c)F^n=$$
$$=\sum_{m\geqslant 0}\frac{(a_1)_{m_1}\cdots(a_n)_{m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}}\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!},\quad |z_1|<1,\ldots,|z_n|<1; \tag{31}$$
$$F_C(a;b;c_1,\ldots,c_n;z_1,\ldots,z_n)=\frac{(a)_{\delta_1+\ldots+\delta_n}(b)_{\delta_1+\ldots+\delta_n}}{(c_1)_{\delta_1}\cdots(c_n)_{\delta_n}}\,\mathrm{e}^{z_1+\ldots+z_n}=$$
$$=\frac{(a)_{\delta_1+\ldots+\delta_n}(b)_{\delta_1+\ldots+\delta_n}}{(a_1)_{\delta_1}\cdots(a_n)_{\delta_n}(b_1)_{\delta_1}\cdots(b_n)_{\delta_n}}F(a_1,b_1;c_1;z_1)\cdots F(a_n,b_n;c_n;z_n)=$$
$$=T^{-1}(a)T^{-1}(b)F^n=T^{-1}(b)F_A=$$
$$=\sum_{m\geqslant 0}\frac{(a)_{m_1+\ldots+m_n}(b)_{m_1+\ldots+m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}}\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!},\quad |z_1|^{1/2}+\ldots+|z_n|^{1/2}<1;$$
$$F_D(a;b_1,\ldots,b_n;c;z_1,\ldots,z_n)=\frac{(a)_{\delta_1+\ldots+\delta_n}(b_1)_{\delta_1}\cdots(b_n)_{\delta_n}}{(c)_{\delta_1+\ldots+\delta_n}}\,\mathrm{e}^{z_1+\ldots+z_n}=$$
$$=\frac{(a)_{\delta_1+\ldots+\delta_n}(c_1)_{\delta_1}\cdots(c_n)_{\delta_n}}{(a_1)_{\delta_1}\cdots(a_n)_{\delta_n}(c)_{\delta_1+\ldots+\delta_n}}\;F(a_1,b_1;c_1;z_1)\cdots F(a_n,b_n;c_n;z_n)=$$
$$=T^{-1}(a)T(c)F^n=T(c)F_A=T^{-1}(a)F_B=$$
$$=\sum_{m\geqslant 0}\frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}}\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!},\quad |z_1|<1,\ldots,|z_n|<1.$$

2.2. Lomidze B_n Function (LBn). In paper [4], the following formula was proposed:

$$B_n(r_0,r_1,\ldots,r_n)=$$
$$=\det\left[x_j^{i-1}\int\limits_{x_{j-1}/x_j}^{1}u^{i-1}(1-u)^{r_j-1}\prod_{k=0,k\neq j}^{n}\left(\frac{x_ju-x_k}{x_j-x_k}\right)^{r_k-1}du\right]/\det\,[x_j^{i-1}]=$$
$$=\frac{\Gamma(r_0)\Gamma(r_1)\cdots\Gamma(r_n)}{\Gamma(r_o+r_1+\ldots+r_n)},\quad 0=x_0<x_1<x_2<\ldots<x_n,\quad n\geqslant 1. \tag{32}$$

We find a simple proof of this formula. Let us put the formula in the following factorized form:

$$LB_n(x,r)\equiv\det\left[x_j^{i-1}\int\limits_{x_{j-1}/x_j}^{1}du\,u^{i+r_0-2}(1-u)^{r_j-1}\prod_{k=1,k\neq j}^{n}\left(\frac{x_ju-x_k}{x_j-x_k}\right)^{r_k-1}\right]=$$
$$=\det V_n(x)B_n(r),\quad V_n(x)=[x_j^{i-1}],\quad B_n(r)=\frac{\Gamma(r_0)\Gamma(r_1)\cdots\Gamma(r_n)}{\Gamma(r_o+r_1+\ldots+r_n)}. \tag{33}$$

Now, it is enough to proof this formula for general values of x_i and particular values of r_i, e.g., $r_i=1$, and for general values of r_i and particular values of x_i, e.g., $x_i=p^i$, $1\leqslant i\leqslant n$.

In the case of $r_i = 1$, the right-hand side of the formula is equal to the Vandermonde determinant divided by $n!$ The left-hand side is the determinant of the matrix with elements $A_{ij} = x_j^{i-1}(1-(x_{j-1}/x_j)^i)/i$. When we calculate determinant of this matrix, from the row i, we factorize $1/i, 2 \leqslant i \leqslant n$ which gives the $1/n!$ — the rest matrix we calculate transforming the matrix to the form of the Vandermonde matrix.

This is the half way of the proof. Let us take the concrete values of $x_i = p^i,\ 1 \leqslant i \leqslant n$, where p is positive integer and general complex values for $r_i, 0 \leqslant i \leqslant n$, and calculate both sides of the equality. For Vandermonde determinant we find for high values of p the following asymptotics:

$$\det V = p^N, \quad N = \sum_{k=2}^{n} k(k-1) = \frac{n(n^2-1)}{3}. \tag{34}$$

The matrix elements are

$$\begin{aligned} B_{ij} &= x_j^{i-1} \int\limits_{x_{j-1/x_j}}^{1} u^{i+r_0-2}(1-u)^{r_j-1} \prod_{k=1,k\neq j}^{n} \left(\frac{x_j u - x_k}{x_j - x_k}\right)^{r_k-1} du = \\ &= x_j^{i-1} \prod_{1\leqslant k<j} \left(\frac{x_j}{x_j - x_k}\right)^{r_k-1} \prod_{j<k\leqslant n} \left(\frac{x_k}{x_k - x_j}\right)^{r_k-1} \int\limits_{x_{j-1/x_j}}^{1} u^{i+r_0-2}(1-u)^{r_j-1} \times \\ &\quad \times \prod_{1\leqslant k<j} \left(\frac{u - x_k}{x_j}\right)^{r_k-1} \prod_{j<k\leqslant n} \left(\frac{1 - x_j}{x_k u}\right)^{r_k-1} du = \\ &= p^{(i-1)j} \int\limits_{0}^{1} u^{i+r_0-2+\sum\limits_{k=1}^{j-1}(r_k-1)} (1-u)^{r_j-1} du = \\ &= p^{(i-1)j} B\left(i + \sum_{k=0}^{j-1}(r_k - 1), r_j\right). \end{aligned} \tag{35}$$

For $n = 2$ we have

$$\begin{aligned} B_{11} &= \int\limits_{0}^{1} u^{r_0-1}(1-u)^{r_1-1}\, du = \frac{\Gamma(r_0)\Gamma(r_1)}{\Gamma(r_0+r_1)}, \\ B_{22} &= p^2 \int\limits_{0}^{1} u^{r_0+r_1-1}(1-u)^{r_2-1} du = \frac{\Gamma(r_0+r_1)\Gamma(r_2)}{\Gamma(r_0+r_1+r_2)}, \\ \frac{LB_2}{V_2} &= \frac{B_{11}B_{22}}{p^2} = \frac{\Gamma(r_0)\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_0+r_1+r_2)}. \end{aligned} \tag{36}$$

For $n = 3$,

$$
\begin{aligned}
B_{11} &= \int_0^1 u^{r_0-1}(1-u)^{r_1-1} = \frac{\Gamma(r_0)\Gamma(r_1)}{\Gamma(r_0+r_1)} = B(r_0, r_1),\\
B_{22} &= p^2 \int_0^1 u^{r_0+r_1-1}(1-u)^{r_2-1} = p^2 \frac{\Gamma(r_0+r_1)\Gamma(r_2)}{\Gamma(r_0+r_1+r_2)},\\
B_{33} &= p^6 \int_0^1 u^{r_0+r_1+r_2-1}(1-u)^{r_3-1} = p^6 \frac{\Gamma(r_0+r_1+r_2)\Gamma(r_3)}{\Gamma(r_0+r_1+r_2+r_3)},\\
\frac{LB_3}{V_3} &= \frac{B_{11}B_{22}B_{33}}{p^8} = \frac{\Gamma(r_0)\Gamma(r_1)\Gamma(r_2)\Gamma(r_3)}{\Gamma(r_0+r_1+r_2+r_3)}.
\end{aligned} \tag{37}
$$

Now, it is obvious the last step of the proof [5]

$$
\begin{aligned}
LB_n(x,r) &= \det V_n(x) B(r_0, r_1) \cdots B(r_0 + r_1 + \ldots + r_{n-1}, r_n) = \det V_n(x) B_n(r),\\
V_n(x) &= [x_j^{i-1}],\ B_n(r) = \frac{\Gamma(r_0)\Gamma(r_1)\cdots\Gamma(r_n)}{\Gamma(r_0+r_1+\ldots+r_n)}.
\end{aligned} \tag{38}
$$

3. FIELD THEORY APPLICATIONS OF FC

Let us consider the following action:

$$
S = \frac{1}{2}\int_{Q_v} dx\, \Phi(x) D_x^\alpha \Phi, \quad v = 1, 2, 3, 5, \ldots, 29, \ldots, 137, \ldots \tag{39}
$$

Q_1 is a real number field, Q_p, p-prime, are p-adic number fields. In the momentum representation

$$
\begin{aligned}
S &= \frac{1}{2}\int_{Q_v} du\, \tilde{\Phi}(-u)|u|_v^\alpha \tilde{\Phi}(u), \quad \Phi(x) = \int_{Q_v} du\, \chi_v(ux)\, \tilde{\Phi}(u),\\
&D^{-\alpha}\chi_v(ux) = |u|_v^{-\alpha}\chi_v(ux).
\end{aligned} \tag{40}
$$

The statistical sum of the corresponding quantum theory is

$$
Z_v = \int d\Phi \exp^{\left(-\frac{1}{2}\int \Phi D^\alpha \Phi\right)} = \det{}^{-1/2} D^\alpha = \left(\prod_u |u|_v\right)^{-\alpha/2}. \tag{41}
$$

3.1. String Theory Applications. For (symmetrized, 4-tachyon) Veneziano amplitude we have (see, e.g., [2])

$$
\begin{aligned}
B_s(\alpha,\beta) &= B(\alpha,\beta) + B(\beta,\gamma) + B(\gamma,\alpha) = \int_{-\infty}^{\infty} dx |1-x|^{\alpha-1}|x|^{\beta-1},\\
&\alpha + \beta + \gamma = 1.
\end{aligned} \tag{42}
$$

For the p-adic Veneziano amplitude we take

$$B_p(\alpha,\beta) = \int\limits_{Q_p} dx|1-x|_p^{\alpha-1}|x|_p^{\beta-1} = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)}. \tag{43}$$

Now we obtain the N-tachyon amplitude using fractal calculus. We consider the dynamics of particle given by multicomponent generalization of the action (39), $\Phi \to x^\mu$. For the closed trajectory of the particle passing through N points, we have

$$\begin{aligned} A(x_1,x_2,\ldots,x_N) = \int dt \int dt_1 \cdots \int dt_N \delta(t-\Sigma t_n) = \\ v(x_1,t_1;x_2,t_2)v(x_2,t_2;x_3,t_3)\cdots v(x_N,t_N;x_1,t_1) = \\ = \int dx(t)\Pi\left(\int dt_n \delta(x^\mu(t_n)-x_n^\mu)\right)\exp\left(-S[x(t)]\right) = \int \Pi(dk_n^\mu \chi(k_n x_n))\tilde{A}(k), \end{aligned} \tag{44}$$

where

$$\begin{aligned} \tilde{A}(k) = \int dx V(k_1)V(k_2)\cdots V(k_N)\exp\left(-S\right), \\ V(k_n) = \int dt \chi(-k_n x(t)) \end{aligned} \tag{45}$$

is vertex function.

Motion equation

$$D^\alpha x^\mu - i\Sigma k_n^\mu \delta(t-t_n) = 0 \tag{46}$$

in the momentum representation

$$|u|^\alpha \tilde{x}^\mu(u) - i\Sigma_n k_n^\mu \chi(-ut_n) = 0 \tag{47}$$

has the solution

$$\tilde{x}^\mu(u) = i\Sigma k_n^\mu \frac{\chi(-ut_n)}{|u|^\alpha}, \quad u \neq 0, \tag{48}$$

the constraint

$$\Sigma_n k_n = 0, \tag{49}$$

and the zero mod $\tilde{x}_n^\mu(0)$, which is arbitrary. Integration in (44) with respect to this zero mod gives the constraint (49). On the solution of equation (46)

$$x^\mu(t) = iD_t^{-\alpha}\sum_n k_n^\mu \delta(t-t_n) = \frac{i}{\Gamma(\alpha)}\sum_n k_n^\mu |t-t_n|^{\alpha-1}, \tag{50}$$

the action (39) takes value

$$S = -\frac{1}{\Gamma(\alpha)}\sum_{n<m} k_n k_m |t_n - t_m|^{\alpha-1}, \quad \tilde{A}(k) = \int \prod_{n=1}^N dt_n \exp\left(-S\right). \tag{51}$$

In the limit, $\alpha \to 1$, for p-adic case we obtain

$$x^\mu(t) = -i\frac{p-1}{p\ln p}\sum_n k_n^\mu \ln|t-t_n|,$$
$$S[x(t)] = \frac{p-1}{p\ln p}\sum_{n<m} k_n k_m \; ln|t_n - t_m|, \quad (52)$$
$$\tilde{A}(k) = \int \prod_{n=1}^{N} dt_n \prod_{n<m} |t_n - t_m|^{\frac{p-1}{p\ln p}k_n k_m}.$$

Now in the limit $p \to 1$ we obtain the proper expressions of the real case

$$x^\mu(t) = -i\sum_n k_n^\mu \ln|t-t_n|, \quad S[x(t)] = \sum_{n<m} k_n k_m \ln|t_n - t_m|,$$
$$\tilde{A}(k) = \int \prod_{n=1}^{N} dt_n \prod_{n<m} |t_n - t_m|^{k_n k_m}. \quad (53)$$

By fractal calculus and vector generalization of the model (39), fundamental string amplitudes were obtained in [6].

REFERENCES

1. *Gasper G., Rahman M.* Basic Hypergeometric Series. Cambridge: Cambridge Univ. Press, 1990.
2. *Kaku M.* Strings, Conformal Fields, and M-Theory. N. Y.: Springer, 2000.
3. *Koblitz N.* p-Adic Numbers, p-Adic Analysis, and Zeta-Functions. N. Y.; Heidelberg; Berlin: Springer-Verlag, 1977.
4. *Lomidze I.* On Some Generalizations of the Vandermonde Matrix and Their Relations with the Eiler Beta-Function // Georg. Math. J. 1994. V. 1. P. 405.
5. *Lomidze I., Makhaldiani N.* In progress.
6. *Makhaldiani N. V.* Number Fields Dynamics and the Compactification of the Space Problem in the Unified Theories of Fields and Strings. JINR Commun. P2-88-916. Dubna, 1988.
7. *Makhaldiani N.* Adelic Universe and Cosmological Constant. JINR Commun. E2-2003-215. Dubna, 2003; hep-th/0312291. 2003.
8. *Miller W., Jr.* Symmetry and Separation of Variables. London: Addison-Wesley PC, 1977.
9. *Okun L. B.* Leptons and Quarks. North Holland, 1982.
10. *Vladimirov V. S.* // Rus. Math. Surveys. 1988. V. 43. P. 19.
11. *Weinberg S.* Gravitation and Cosmology. N. Y., 1972.