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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

MAGNETIC MONOPOLES

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These are notes of the first part of the lectures given at the JINR–ISU Baikal Summer School on Physics of Elementary Particles and Astrophysics (July 2010). I review classical monopole solutions of the $SU(2)$ Yang–Mills–Higgs theory, providing a pedagogical introduction into to the theory of non-Abelian monopoles both in the BPS limit and beyond of it. I briefly discuss monopole dynamics, the idea of the moduli space and some of the basic properties which are connected with the field theoretical aspects of these classical solutions.

Это первая часть лекций, прочитанных на Байкальской школе по физике элементарных частиц и астрофизике, организованной ОИЯИ и ИГУ (июль 2010 г.). Представлен обзор решений с монополями $SU(2)$ теории Янга–Миллса. Дается педагогическое введение в неабелеву теорию монополей в пределе BPS и за его рамками. Кратко обсуждаются динамика монополей, идея пространства модулей и некоторые из основных идей, связанных с теоретическими аспектами этих классических решений.

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’t HOOFT–POLYAKOV MONOPOLE

We consider non-Abelian classical Lagrangian of a Yang–Mills–Higgs theory with the gauge group $SU(N)$, which describes coupled gauge and Higgs fields:

$$L = -\frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \mathrm{Tr}\, D_\mu\phi D^\mu\phi - V(\phi) = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}(D^\mu\phi^a)(D_\mu\phi^a) - V(\phi). \quad (1)$$

Here, $A_\mu = A^a_\mu T^a$ is an $SU(N)$ connection with field strength $F_{\mu\nu} = F^a_{\mu\nu}T^a$, $\phi = \phi^a T^a$ and we use standard normalization of the Hermitian generators of the gauge group: $\mathrm{Tr}\,(T^aT^b) = 1/2\delta_{ab}$, $a, b = 1, 2, 3$, which satisfy the Lie algebra $[T^a, T^b] = i\varepsilon_{abc}T^c$. The scalar field $\phi = \phi^a T^a$ transforms in the adjoint representation of $SU(N)$ with the covariant derivative defined by $D_\mu\phi = \partial_\mu\phi + ie[A_\mu, \phi]$. The nonzero vacuum expectation value of the scalar field corresponds to the symmetry-breaking Higgs potential $V(\phi)$:

$$V(\phi) = \lambda(|\phi|^2 - v^2)^2, \quad (2)$$

where the group norm of the scalar field is defined as $|\phi|^2 = 2\ \mathrm{Tr}\,\phi^2 = \phi^a\phi^a$.

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In the simplest nontrivial case of $SU(2)$ group we have $T^a = \sigma^a/2$ and $V(\phi) = \lambda(\phi^a\phi^a - v^2)^2$. The energy of the configuration is minimal if the following conditions are satisfied:

$$\phi^a\phi^a = v^2, \quad F^a_{mn} = 0, \quad D_n\phi^a = 0. \tag{3}$$

These conditions define the vacuum. Note that the very definition (3) forces the classical vacuum of the $SU(2)$ Yang–Mills–Higgs theory to be degenerated. Indeed, the condition $V(\phi) = 0$ means that $|\phi| = v$, i.e., the set of vacuum values of the Higgs field forms a sphere $S^2_{\rm vac}$. All the points on this sphere are equivalent because there is a well-defined $SU(2)$ gauge transformation which connects them. If $v^2 \neq 0$ the $SU(2)$ symmetry is spontaneously broken to $U(1)$.

Thus, the solutions of the classical field equations map the vacuum manifold $\mathcal{M} = S^2_{\rm vac}$ onto the boundary of 3-dimensional space, which is also a sphere S^2. These maps are characterized by a *winding number* $n = 0, \pm1, \pm2\ldots$ which is the number of times $S^2_{\rm vac}$ is covered by a single turn around the spatial boundary S^2. The celebrated 't Hooft–Polyakov solution [3, 4] corresponds to the «hedgehog» asymptotic of the scalar field:

$$\phi^a \underset{r\to\infty}{\longrightarrow} \frac{vr^a}{r}. \tag{4}$$

Such a behavior obviously mixes the spatial and isotopic indices and defines a single mapping of the vacuum $\mathcal{M}$ onto the spatial asymptotic.

As was mentioned by 't Hooft [3], the configurations which are characterized by different winding numbers cannot be continuously deformed into each other. Indeed, the gauge transformation of the form $U = {\rm e}^{i(\sigma_k\hat{\varphi}_k)\theta/2}$ rotates the isovector to the third axis. However, if we try to «comb the hedgehog», that is, to rotate the scalar field everywhere in space to a given direction (so-called *unitary* or *Abelian gauge*), the singularity of the gauge transformation on the south pole does not allow us to do it globally. Thus, there is no well-defined global gauge transformation which connects the trivial and the «hedgehog» configurations and this singularity results in the infinite barrier separating them.

Note that the electric charge of the massive vector bosons $A^\pm$ is given by the unbroken $U(1)$ subgroup. In general, this is a subgroup H of the gauge group G, the action of which leaves the Higgs vacuum invariant. Obviously, that is a little group of the rotation in isospace about the direction given by the vector ϕ^a. The generator of it, $(\phi^a T^a)/a$, must be identified with the operator of electric charge Q. Thus, the expression for the covariant derivative can be written in the form

$$D_\mu = \partial_\mu + ieA^a_\mu T^a = \partial_\mu + iQA^{\rm em}_\mu \tag{5}$$

that allows us to define the «electromagnetic projection» of the gauge potential

$$A^{\rm em}_\mu = \frac{1}{a}A^a_\mu\phi^a, \qquad Q = e\frac{1}{a}\phi^a T^a. \tag{6}$$

Taking into account the definition of the generators T^a of the gauge group, we can easily see that the minimal allowed eigenvalues of the electric charge operator now are $q = \pm e/2$.

Thus, it would be rather natural to introduce the electromagnetic potential as a projection of the $SU(2)$ gauge potential A^a_μ onto that direction, see Eq. (6). Furthermore [5], a general solution of the equation $D_\mu\phi^a = 0$, for $\phi^a\phi^a = a^2$, can be written as

$$A^a_\mu = \frac{1}{a^2e}\varepsilon_{abc}\phi^b\partial_\mu\phi^c + \frac{1}{a}\phi^a\Lambda_\mu, \tag{7}$$

where Λ_μ is an arbitrary 4-vector. It can be identified with the electromagnetic potential because Eq. (7) yields for $\phi^a\phi^a = a^2$

$$\frac{\phi^a}{a}A^a_\mu = \Lambda_\mu \equiv A^{\text{em}}_\mu.$$

Inserting Eq. (7) into the definition of the field strength tensor yields

$$F^a_{\mu\nu} = F_{\mu\nu}\frac{\phi^a}{a}, \quad \text{where} \;\; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{a^3 e}\varepsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c. \tag{8}$$

This gauge-invariant definition of the electromagnetic field strength tensor $F_{\mu\nu}$ suggested by 't Hooft [3] has a very deep meaning. It is rather obvious that in the topologically trivial sector the last term in Eq. (8) vanishes and then we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

This is precisely the case of standard Maxwell electrodynamics. Of course, in this sector there is no place for a monopole because the Bianchi identities are satisfied: $\partial^\mu \widetilde{F}_{\mu\nu} \equiv 0$.

However, for the configuration with nontrivial boundary conditions (4), (14), also the Higgs field gives a nonvanishing contribution to the electromagnetic field strength tensor (8). Then, the second pair of Maxwell equations becomes modified:

$$\partial^\mu \widetilde{F}_{\mu\nu} = k_\nu. \tag{9}$$

Note that, if the electromagnetic potential A^{em}_μ is regular, the magnetic, or topological current k_μ is expressed via the scalar field alone:

$$k_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = \frac{1}{2a^3 e}\varepsilon_{\mu\nu\rho\sigma}\varepsilon_{abc}\partial^\nu\phi^a\partial^\rho\phi^b\partial^\sigma\phi^c. \tag{10}$$

At first glance this current is independent of any property of the gauge field. It is conserved by its very definition:

$$\partial_\mu k_\mu \equiv 0, \tag{11}$$

unlike a Noether current which is conserved because of some symmetry.

The static regular solution of the corresponding field equations was constructed numerically by employing the spherically symmetric ansatz [3, 4] for the gauge and the Higgs fields, respectively:

$$\phi^a = \frac{r^a}{er^2}H(\xi), \quad A^a_n = \varepsilon_{amn}\frac{r^m}{er^2}[1 - K(\xi)], \quad A^a_0 = 0, \tag{12}$$

where $H(\xi)$ and $K(\xi)$ are functions of the dimensionless variable $\xi = ver$.

The condition of vanishing covariant derivative of the scalar field on the spatial asymptotic (3), together with the choice of the nontrivial hedgehog configuration, implies that at $r \to \infty$

$$\partial_n\left(\frac{r^a}{r}\right) - e\varepsilon_{abc}A^b_n\frac{r^c}{r} = 0. \tag{13}$$

The simple transformation

$$\partial_n\left(\frac{r^a}{r}\right) = \frac{r^2\delta_{an} - r_a r_n}{r^3} = \frac{1}{r}\left(\delta_{an}\delta_{ck} - \delta_{ak}\delta_{nc}\right)\frac{r_c r_k}{r^2} = -\varepsilon_{abc}\varepsilon_{bnk}\frac{r_c r_k}{r^3}$$

then provides an asymptotic form of the gauge potential

$$A_k^a(r) \underset{r\to\infty}{\longrightarrow} \frac{1}{e}\varepsilon_{ank}\frac{r_n}{r^2}. \tag{14}$$

This corresponds to the non-Abelian magnetic field

$$B_n^a \underset{r\to\infty}{\longrightarrow} \frac{r_a r_n}{er^4}. \tag{15}$$

Therefore, the boundary conditions (4), (14) are compatible with the existence of a long-range gauge field associated with an Abelian subgroup which is unbroken in the vacuum. Since this field decays like $1/r^2$, which is typical behavior of the Coulomb-like field of a point-like charge, and since the electric components of the field strength tensor vanish, we can recognize a monopole in such a «hedgehog» configuration with a finite energy.

The explicit forms of the shape functions of the scalar and gauge field can be found numerically. It turns out that the functions $H(\xi)$ and $K(\xi)$ approach rather fast the asymptotic values. Thus, there is almost vacuum outside of some region of the order of the characteristic scale R_c, which is called the *core* of the monopole. One could estimate this size by simple arguments [7]. The total energy of the monopole configuration consists of two components: the energy of the Abelian magnetic field outside the core and the energy of the scalar field inside the core:

$$E = E_{\text{mag}} + E_s \sim 4\pi g^2 R_c^{-1} + 4\pi v^2 R_c \sim \frac{4\pi}{e^2}\left(R_c^{-1} + m_v^2 R_c\right).$$

This sum is minimal if $R_c \sim m_v^{-1}$. In other words, inside the core at distances shorter than the wavelength of the vector boson $m_v^{-1} \sim (ve)^{-1}$, the original $SU(2)$ symmetry is restored. However, outside the core this symmetry is spontaneously broken down to the Abelian electromagnetic subgroup.

Note that the solution given by the 't Hooft–Polyakov ansatz (12) corresponds to the condition $A_0^a = 0$. One could consider a more general case, where this time component of the vector potential is not equal to zero, but is also a function of the spatial coordinates [8]:

$$A_0^a = \frac{r^a}{er^2}J(r). \tag{16}$$

This field configuration corresponds to the non-Abelian *dyon*, which has both magnetic and electric charges. The electric charge of the system of the fields can be defined as

$$q = \frac{1}{v}\int dS_n\, E_n = \frac{1}{v}\int dS_n\, E_n^a\phi^a = \frac{1}{v}\int d^3x\, E_n^a D_n\phi^a. \tag{17}$$

Here, we invoked the field equations, according to which $D_n E_n^a = 0$, and made use of the relation $\varepsilon_{abc}\phi^b D_0\phi^c = 0$, which is valid for the ansatz under consideration. The asymptotic behavior of the profile function $J(r)$ is very similar to that of the scalar field:

$$J(r) \to 0 \quad \text{as} \quad r \to 0, \qquad J(r) \to Cr \quad \text{as} \quad r \to \infty. \tag{18}$$

The arbitrary constant C is connected with the electric charge of the dyon (17) [8]. The charge vanishes if $C = 0$.

Indeed, substituting the ansatz (12) into the integral (17), after some algebra we obtain

$$q = \frac{4\pi C}{e} = Cg. \tag{19}$$

However, on the classical level there is no reason for the electric charge (17), unlike the magnetic charge, to be quantized and the constant C in (19) remains an arbitrary parameter.

Finally, we note that the time component of the vector-potential (16) is in isospace parallel to the direction of the Higgs field. Moreover, one can consider it as an additional triplet of the scalar fields. This is the so-called *Julia–Zee correspondence* $\phi^a \rightleftharpoons A_0^a$.

THE BOGOMOL'NYI LIMIT

Unfortunately, the system of nonlinear coupled differential equations on the functions $H(\xi)$ and $K(\xi)$ in general has no analytical solution. The only known exception is the very special case of vanishing scalar potential $V = 0$ [9–11]. This is the so-called *Bogomol'nyi–Prasad–Sommerfield (BPS) limit*.

In the BPS limit of vanishing Higgs potential the scalar field also becomes massless and the energy of the static field configuration is taking the form

$$E = \int \left\{ \frac{1}{4}\mathrm{Tr}\left((\varepsilon_{ijk}F_{ij} \pm D_k\phi)^2 \right) \mp \frac{1}{2}\varepsilon_{ijk}\mathrm{Tr}\left(F_{ij}D_k\phi\right) \right\} d^3r. \tag{20}$$

Thus, the absolute minimum of the energy corresponds to the static configurations which satisfy the first-order Bogomol'nyi equations:

$$\varepsilon_{ijk}F_{ij} = \pm D_k\phi, \tag{21}$$

which are solved by

$$K = \frac{\xi}{\sinh \xi}, \quad H = \xi \coth \xi - 1. \tag{22}$$

Definitely, the solution to the first-order BPS equation (21) automatically satisfies the system of field equations of the second order.

BPS monopoles are very remarkable objects. Let us briefly recapitulate the properties of these solutions:

• The BPS equation, together with the Bianchi identity, means that $D_nD_n\phi^a = 0$. Therefore, the condition $D_n\phi^a D_n\phi^a = (1/2)\partial_n\partial_n(\phi^a\phi^a)$ holds and the energy of the monopole configuration in the BPS limit is independent of the properties of the gauge field:

$$E = \frac{1}{2}\int d^3x\partial_n\,\partial_n(\phi^a\phi^a) = \frac{4\pi v}{e} = gv. \tag{23}$$

If the configuration has both electric and magnetic charges, the monopole mass becomes

$$M = v\sqrt{g^2 + q^2}. \tag{24}$$

This yields the so-called *Bogomol'nyi bound* on the monopole mass.

• In comparison with the 't Hooft–Polyakov solution, the behavior of the Higgs field of the monopole in the BPS limit has changed drastically: as we can see from (22), alongside with the exponentially decaying component it also obtains a long-distance Coulomb tail

$$\phi^a \to v\hat{r}^a - \frac{r^a}{er^2} \quad \text{as} \quad r \to \infty. \tag{25}$$

The reason for this is that in the limit $V(\phi) = 0$, the scalar field becomes massless.

• The long-range monopole–monopole interaction is composed of two parts originating from the long-range scalar force and the standard electromagnetic interaction, which could be either attractive or repulsive [12]. Mutual compensation of both contributions leaves the pair of BPS monopoles static but the monopole and antimonopole would interact with double strength.

• The Bogomol'nyi equation may be treated as a three-dimensional reduction of the integrable self-duality equations. Indeed, the Julia–Zee correspondence means that

$$\begin{aligned} D_n\phi^a &\rightleftharpoons D_n A_0^a \equiv F_{0n}^a, \\ B_n^a = D_n\phi^a &\rightleftharpoons \widetilde{F}_{0n} = F_{0n}^a. \end{aligned} \tag{26}$$

Therefore, if we suppose that all the fields are static, the Euclidean equations of self-duality $F_{\mu\nu}^a = \widetilde{F}_{\mu\nu}^a$ reduce to Eqs. (21) and the monopole solutions in the Bogomol'nyi limit could be considered as a special class of self-dual fields.

• The analogy between the Euclidean Yang–Mills theory and the BPS equations can be traced up to the solutions. It was shown [13, 14] that the solutions of these equations are exactly equal to an infinite chain of instantons directed along the Euclidean time axis t in $d = 4$. More precisely, the BPS monopole is equivalent to an infinite chain of instantons having identical orientation in isospace and separated by an interval $\tau_0 = 2\pi$. An alternative configuration is a chain of correlated instanton–antiinstanton pairs, which corresponds to an infinite monopole loop.

GAUGE ZERO MODE AND THE ELECTRIC DYON CHARGE

In the BPS limit the Julia–Zee dyonic solutions have a very interesting interpretation [15, 16]. First, we note that for the *static* ansatz (12), (16) and the choice $A_0 = 0$, the kinetic energy of the configuration

$$T = \int d^3x \,\mathrm{Tr}\,(E_n E_n + D_0\phi D_0\phi) \tag{27}$$

is equal to zero. Moreover, in this case the Gauss law

$$D_n E_n - ie\,[\phi, D_0\phi] = 0 \tag{28}$$

can be satisfied trivially, with $E_n = D_0\phi = 0$.

Let us now consider time-dependent fields $A_n(\mathbf{r}, t)$, $\phi^a(\mathbf{r}, t)$, but suppose that their time dependence arises as a result of a gauge transformation of the original static configuration:

$$A_n(\mathbf{r}, t) = U(\mathbf{r}, t)A_n(\mathbf{r}, 0)U^{-1}(\mathbf{r}, t) - \frac{i}{e}\,U(\mathbf{r}, t)\partial_n U^{-1}(\mathbf{r}, t)\,. \tag{29}$$

Here, $U(\mathbf{r}, t) = \mathrm{e}^{ie\omega t}$ with $\omega(\mathbf{r})$ a parameter of the transformation. If the time interval δt is very small, we can expand

$$U(\mathbf{r}, \delta t) \approx 1 + ie\omega\delta t + \dots \tag{30}$$

Now it follows from (29) that

$$A_n(\mathbf{r}, \delta t) \approx A_n(\mathbf{r}) + (ie[\omega, A_n(\mathbf{r})] - \partial_n\omega)\,\delta t, \tag{31}$$

and we have

$$\partial_0 A_n = ie[\omega, A_n(\mathbf{r})] - \partial_n\omega = -D_n\omega. \tag{32}$$

In a similar way we obtain for the time dependence of the scalar field:

$$\partial_0\phi = ie\,[\omega, \phi]\,. \tag{33}$$

These gauge transformations simultaneously affect the time component of the gauge potential, which for the monopole configurations (12), (16) is a pure gauge:

$$A_0(\mathbf{r}, t) = -\frac{i}{e}\,U(\mathbf{r}, t)\,\partial_0 U^{-1}(\mathbf{r}, t) = -\omega. \tag{34}$$

Since the gauge transformations (32) and (33) do not change the potential energy of the configuration, the parameter ω can be identified with the *gauge zero mode*. This is one of four collective coordinates (they are also called *moduli*) of the one-monopole configuration [16]. The other three specify the position of the monopole in space. Their appearance reflects an obvious breaking of translational invariance of the original Lagrangian (1) by the monopole configuration: the position of the monopole in $\mathbb{R}^3$ can be chosen arbitrarily.

However, defined in this way, the gauge zero mode is not physical, since the gauge transformations (32) and (33) do not affect the non-Abelian electric field:

$$\begin{aligned} E_n^a &= \partial_0 A_n - D_n A_0 = -D_n\omega + D_n\omega \equiv 0, \\ D_0\phi &= \partial_0\phi + ie[A_0, \phi] = ie\,[\omega, \phi] - ie\,[\omega, \phi] \equiv 0. \end{aligned} \tag{35}$$

Thus, as before, the Gauss law is satisfied trivially and the kinetic energy of the monopole (27) is still equal to zero.

Now let us suppose that the time dependence of the fields again appears as a result of the gauge transformations (32) and (33), but that the corresponding gauge zero mode $(\partial_0 A_n, \partial_0\phi)$ satisfies the *background gauge condition*:

$$D_n(\partial_0 A_n) - ie\,[\phi, (\partial_0\phi)] = 0. \tag{36}$$

Then the Gauss law (28) is satisfied, if $A_0 = 0$ and there is a nontrivial solution of Eqs. (32), (33) and (36) [15], where ω is proportional to ϕ and an additional time dependence is allowed:

$$\omega = \dot{\Upsilon}(t)\phi,$$

which corresponds to the gauge transformation

$$U(\mathbf{r}, t) = \exp\{ie\Upsilon(t)\phi(\mathbf{r})\} \approx 1 + ie\dot{\Upsilon}\phi\delta t. \tag{37}$$

Here $\Upsilon(t)$ is an arbitrary function of time. Indeed, in this case we have $\partial_0 A_n = \dot{\Upsilon} D_n \phi$ and $\partial_0 \phi = 0$, and, since in the Bogomol'nyi limit $D_n D_n \phi = 0$, the background gauge condition (36) is satisfied by the ansatz (37). However, this solution corresponds to the generation of a non-Abelian electric field

$$E_n = \partial_0 A_n = \dot{\Upsilon}(t) D_n \phi = \dot{\Upsilon}(t) B_n, \quad D_0 \phi = 0, \tag{38}$$

so the kinetic energy of the monopole (27) is no longer zero:

$$T = \frac{1}{2}\dot{\Upsilon}^2 \int d^3x D_n \phi^a D_n \phi^a = \frac{1}{2}\dot{\Upsilon}^2 \int d^3x B_n^a B_n^a = 2\pi v g \dot{\Upsilon}^2 = \frac{1}{2} M \dot{\Upsilon}^2, \tag{39}$$

where we make use of the definition of the magnetic charge and take into account that the mass of the BPS monopole is

$$M = \frac{4\pi v}{e}.$$

Since the potential energy of the configuration is time-independent, the gauge transformations (32) and (33), supplemented with the condition $A_0 = 0$, define a physical collective coordinate $\Upsilon(t)$, that is a gauge zero mode. Its excitation corresponds to the generation of an electric charge $Q = \dot{\Upsilon} g$. Thus, such a gauge-induced time dependence of the fields transforms the monopole into a dyon.

Note that this collective coordinate is an angular variable, which is defined on a circle S^1. Indeed, the points $\Upsilon = 2\pi n$, $n \in \mathbb{Z}$ correspond to the same gauge transformation $U(\mathbf{r}, t)$, which is unity on the spatial asymptotic [15]. However, the points $\Upsilon = 0$ and, for example, $\Upsilon = 2\pi$, correspond to different topological classes.

To sum up, the one-monopole configuration in the BPS limit could be characterized by four zero modes (moduli) that form the so-called *moduli space* $\mathcal{M}_1$. It is clear from the above discussion that $\mathcal{M}_1 = \mathbb{R}^3 \times S^1$.

Note that we can come back to the Julia–Zee description of a dyon configuration just by inverting the above discussion: we could start from a system of time-dependent fields and apply the gauge transformations (32) and (33) to compensate for that dependence. The price we would have to pay would be the appearance of a nonzero time component of the gauge potential A_0. This corresponds to the static ansatz (16).

CLASSICAL INTERACTION OF TWO WIDELY SEPARATED DYONS

Now we consider the mechanism of interaction between two widely separated monopoles. If they are close enough to each other, the cores overlap and we have quite a complicated picture of short-range interactions mediated by the gauge and scalar fields. However, if we consider well-separated monopoles, there is some simplification. We may suppose that the monopole core has a radius that is much smaller than the distance between the monopoles. Moreover, outside of this core the covariant derivatives of the scalar field vanish and thus the gauge fields obey the free Yang–Mills equations. This approximation is a standard assumption in the analysis of monopole interactions.

The result of both analytical and variational calculations confirm a rather surprising conclusion, first observed by Manton [12]: there is no interaction between two BPS monopoles at all, but the monopole–antimonopole pair attracts each other with double strength.

The reason for this unusual behavior is that the normal magnetostatic repulsion of the two monopoles is balanced by the long-range scalar interaction: in the BPS limit the quanta of the scalar field are also massless.

Indeed, we have already noted that there is a crucial difference between the asymptotic behavior of the Higgs field in the non-BPS and the BPS cases: there is a long-range tail of the BPS monopole $\phi^a \to v\hat{r}^a - r^a/er^2$ as $r \to \infty$. The result is that, in a system of two widely separated monopoles, the asymptotic value of the Higgs field in the region outside the core of the first monopole is distorted due to the long-range scalar field of the other monopole: the mass of the first monopole will decrease and the size of its core is increased. In other words, the additional long-range force appears as a result of violation of the original scale invariance of the model in the BPS limit $\lambda \to 0$. The scalar charge of a dyon is just $Q_D = \sqrt{g^2 + q^2}$ [17] and the corresponding Coulomb scalar potential of interaction is $\sim -\sqrt{g^2+q^2}/r$ (recall that the scalar interaction is always attractive). Then the relative motion of two well-separated BPS dyons is a geodesic motion in the four-dimensional space of collective coordinates $\mathcal{M}_0$ with one compact variable (moduli space). This dynamics is governed by the Taub–NUT (Newman–Unti–Tamburino) metric

$$ds^2 = \left(1 - \frac{2g^2}{Mr}\right) d\mathbf{r}^2 + \frac{\left(2g^2/M\right)^2}{1 - 2g^2/Mr}(d\Upsilon + \mathbf{a}\cdot d\mathbf{r})^2. \tag{40}$$

A general description of the low-energy dynamics of BPS monopoles on the moduli space is given by the Atiyah–Hitchin metric, whose asymptotic form is the Taub–NUT metric [18].

Let us stop our discussion at this point. Recent developments in the understanding of the low-energy dynamics of the supersymmetric monopoles, which basically used the same simple picture of geodesic motion on the underlying moduli space, have greatly improved our understanding of the structure of the vacuum of supersymmetric theories. The restricted volume of our review does not allow us to go into detail of many remarkable works. Because of lack of time we also do not discuss here the powerful Nahm formalism, which allows us to obtain many results in a very simple and elegant way. In this rapidly developing situation, we direct the reader to the original works and reviews.

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