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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

TOPOLOGICAL SOLITONS. KINKS IN MODELS ϕ^4 AND ϕ^6

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These are notes of the first part of the lectures on topological solitons, presented at Baikal Summer School on Physics of Elementary Particles and Astrophysics (July 2011). Some of the basic properties of topological solitons are reviewed on a simple model example of simple one-dimensional kink solution of the nonintegrable scalar ϕ^4 model. Both perturbative and nonperturbative sectors of the model, oscillon solution and resonance structures in the soliton collision are discussed.

Данный обзор основан на первой части лекций о топологических солитонах, прочитанной на Байкальской летней школе по физике элементарных частиц и астрофизике в 2011 г. Основные динамические свойства топологических солитонов описаны на примере простейшего кинкового решения неинтегрируемой модели ϕ^4. Рассмотрены переходы между пертурбативным и непертурбативным секторами, осциллонное решение и резонансная структура в столкновениях солитонов.

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INTRODUCTION

Soliton solutions in the nonlinear field theories have been studied along several decades, it is evident that these spatially localized nonperturbative configurations, such as kink [1], vortex [2] and monopole [3,4] play an important role in a wide variety of physical systems. The study of the interaction between the solitons and their dynamical properties has attracted a lot of attention in many different contexts.

Simplest example of the topological solitons in one dimension is the class of the kink (K) solutions [1] which appear in the model with a potential with two or more degenerated minima. Double-well potential corresponds to the nonintegrable ϕ^4 model. This model has a number of applications in condensed matter physics [5], field theory [6,7] and cosmology [8].

Dynamical properties of kinks, the processes of their scattering, radiation and annihilation have already been discussed in a number of papers, see, e.g., [10–17]. In integrable theories, like the sine-Gordon model, there is no energy loss to radiation and kinks do not annihilate antikinks. However, in the nonintegrable ϕ^4 model, the radiation effects in the process of kink–antikink ($K\tilde{K}$) collision become very important and depending on the impact velocity, the collision may produce various results, e.g., an oscillating bound state can be formed, also the soliton and antisoliton may bounce and reflect from each other.

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In the first part of the present review we discuss the model and describe the spectrum of perturbative fluctuation on the background of the kink. Then, we discuss the process of production of kink–antikink pairs in the collision of particle-like states related to resonance excitation of the oscillon configuration. Finally, in the third part we revisit the mechanism of the $K\bar{K}$ resonant bouncing scattering related to the energy exchange between the kink internal mode and its translational mode.

1. KINKS AND PERTURBATIVE EXCITATIONS

We start from the Lagrangian of the well-known $(1+1)$-dimensional ϕ^4 model

$$L = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 - \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2 - \varepsilon\frac{m^3}{\sqrt{\lambda}}\phi, \tag{1}$$

where the dimensionless parameter $\varepsilon \ll 1$. The potential of the model

$$V[\phi] = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2 + \varepsilon\frac{m^3}{\sqrt{\lambda}}\phi \tag{2}$$

has two nongenerated minima. Evidently, the Lagrangian (1) corresponds to the so-called thin-wall approximation of the well-known problem of the spontaneous vacuum decay [18, 19]. Let us recall only that at $\varepsilon = 0$ the two vacua are degenerate and there is a topological nontrivial kink solution ϕ_0 that interpolates between these vacua (see, e.g., [20]).

Note, that in the classical case this system also has a very simple interpretation in the solid-state physics: it is the continuum representation of the model of a structurally unstable ion lattice, having a double-well local potential and nearest-neighbor coupling. The kink configuration in this picture corresponds to the domain wall and the continuum modes are just phonons.

Let us consider the evolution of the kink after the metastable vacuum decay. In order to solve the field equation corresponding to the Lagrangian (1)

$$\ddot{\phi} - \phi'' - m^2\phi + \lambda\phi^3 + \varepsilon\frac{m^3}{\sqrt{\lambda}} = 0, \tag{3}$$

we can use an expansion in powers of ε: $\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \ldots$

In general, for calculation of the corrections to the kink field in the nth order of ε we have the equation

$$\left(\frac{d^2}{dt^2} + D^2\right)\phi_n + F(\phi_{n-1}, \ldots, \phi_0) = 0, \tag{4}$$

where the operator D^2 is

$$D^2 = -\frac{d^2}{dx^2} - m^2 + 3m^2\tanh^2\,\frac{mx}{\sqrt{2}}, \tag{5}$$

and $F(\phi_{n-1}, \ldots, \phi_0)$ is a function of all low-order corrections ϕ_k, $k < n$. Note, that parity of $F(\phi_{n-1}, \ldots, \phi_0)$, as well as ϕ_n, are interchanging from one order of ε to another. Thus, one can pick out the asymptotic of the nth order correction by the definition

$$\phi_{2n-1}(x) = B_{2n-1} + \chi_{2n-1}(x), \quad \phi_{2n}(x) = B_{2n} \tanh \frac{mx}{\sqrt{2}} + \chi_{2n}(x),$$

where $B_k = \text{const}$. The boundary condition is that the functions $\chi_k(x)$ tend to zero at $x \to \infty$.

Thus, the zero-order approximation gives the classical equation

$$\ddot{\phi}_0 - \phi_0'' - m^2\phi_0 + \lambda\phi_0^3 = 0, \tag{6}$$

with the above-mentioned kink solution [1,23]

$$\phi_0 = \frac{m}{\sqrt{\lambda}} \tanh \frac{mx}{\sqrt{2}}. \tag{7}$$

The first-order corrections to the solution (7) can be obtained from the next equation

$$\frac{d^2}{dt^2}\phi_1 + D^2\phi_1 + \frac{m^3}{\sqrt{\lambda}} = 0, \tag{8}$$

where D^2 is the operator (5).

In order to find the corrections to the kink solution, we can use the expansion of ϕ_1 on the normalizable eigenfunctions $\eta_n(x)$ of the operator D^2 which describe the scalar field fluctuations on the kink background, i.e., one can write

$$\phi_1 = \sum_{n=0}^{\infty} C_n(t)\, \eta_n(x), \tag{9}$$

where the solutions of the eigenvalue problem $D^2\eta_n(x) = \omega_n^2\eta_n(x)$ are (see, e.g., [20])

$$\begin{aligned} \eta_0(z) &= \frac{1}{\cosh^2 z}, \quad \eta_1(z) = \frac{\sinh z}{\cosh^2 z}, \\ \eta_k(z) &= \mathrm{e}^{ikz}\,(3\tanh^2 z - 3ik\tanh z - 1 - k^2) \end{aligned} \tag{10}$$

and $z = mx/\sqrt{2}$. The corresponding eigenvalues are

$$\omega_0^2 = 0, \quad \omega_1^2 = \frac{3}{2}m^2, \quad \omega_k^2 = m^2\left(2 + \frac{k^2}{2}\right). \tag{11}$$

So, there are a zero mode (η_0), which corresponds to kink translation, a vibrational mode (η_1), connected with the time-dependent deformation of the kink profile, and continuum modes (η_k), which in quantum theory correspond to scalar particle excitations on the kink background. These functions form a complete set which spans the space of any function of x. The corresponding orthogonality relations are

$$\begin{aligned} &\int_{-\infty}^{\infty} \eta_0^2\, dx = \frac{4\sqrt{2}}{3m}, \quad \int_{-\infty}^{\infty} \eta_1^2\, dx = \frac{2\sqrt{2}}{3m}, \\ &\int_{-\infty}^{\infty} \eta_k^*\eta_{k'}\, dx = \frac{2\sqrt{2}\pi}{m}(1+k^2)(4+k^2)\delta(k-k'). \end{aligned} \tag{12}$$

If we substitute the expansion (9) into Eq. (8), we obtain

$$\sum_{n=0}^{\infty}\Big(\ddot{C}_n(t)+\omega_n^2\,C_n(t)\Big)\eta_n(x)+\frac{m^3}{\sqrt{\lambda}}=0. \tag{13}$$

Using the orthogonality relations (12) one can make a projection of Eq. (13) onto the modes $\eta_n(z)$. The projection onto the zero modes gives the equation (here we take into account that $\int\limits_{-\infty}^{\infty}\eta_0\,dx=2\sqrt{2}/m$)

$$\frac{4\sqrt{2}}{3m}\ddot{C}_0+\frac{2\sqrt{2}m^2}{\sqrt{\lambda}}=0, \tag{14}$$

with the solution

$$C_0=-\frac{3m^3}{4\sqrt{\lambda}}t^2+V_0t+x_0.$$

It means that the correction to the kink solution due to zero-mode excitation is (here we suppose that $V_0=x_0=0$)

$$\phi=\phi_0(x)+\varepsilon C_0(t)\eta_0(x)=\frac{m}{\sqrt{\lambda}}\tanh\,\frac{mx}{\sqrt{2}}-\varepsilon\frac{3m^3}{4\sqrt{\lambda}}\frac{t^2}{\cosh^2\ z}=\phi_0(x+\delta x^{(1)}), \tag{15}$$

where the shift of the kink to the first order is given by

$$\delta x^{(1)}=-\varepsilon\frac{3m}{2\sqrt{2}}t^2.$$

The meaning of this correction is quite obvious: because the external force F we introduced in (1) in the first order is (here E is the energy density)

$$F=-\int\limits_{-\infty}^{\infty}dx\frac{dE}{dx}=-\int\limits_{-\infty}^{\infty}dx\ \frac{d}{dx}\left[\frac{1}{2}\dot{\phi}^2+\frac{1}{2}\phi'^2+\frac{\lambda}{4}\left(\phi^2-\frac{m^2}{\lambda}\right)^2-\varepsilon\frac{m^3}{\sqrt{\lambda}}\phi_0\right]=$$

$$=-\frac{dM}{dx}-\varepsilon\int\limits_{-\infty}^{\infty}dx\ \frac{d}{dx}\frac{m^3}{\sqrt{\lambda}}\phi_0=-2\varepsilon\frac{m^4}{\lambda},$$

where the kink energy $\mathcal{E}$ or its classical mass M is

$$\mathcal{E}\equiv M=\int\limits_{-\infty}^{\infty}dx\left[\frac{1}{2}\dot{\phi}^2+\frac{1}{2}\phi'^2+\frac{\lambda}{4}\left(\phi^2-\frac{m^2}{\lambda}\right)^2+\varepsilon\frac{m^3}{\sqrt{\lambda}}\phi\right]=\frac{2\sqrt{2}m^3}{3\lambda}, \tag{16}$$

we see that really the acceleration of the kink is given by the relation

$$a=-\varepsilon\frac{3m}{\sqrt{2}}=\frac{F}{M},$$

that is exactly the Newton formula.

The meaning of other corrections can be found in the same way. After the projection of Eq. (13) onto the vibrational mode $\eta_1(z)$ we have

$$\ddot{C}_1 + \omega_1^2 C_1 = 0, \quad \text{i.e.,} \quad C_1 = \text{ const } \mathrm{e}^{i\omega_1 t}, \tag{17}$$

i.e., there is no interaction between this mode and the external force.

As for the kth mode belonging to the continuum, one obtains

$$\ddot{C}_k + \omega_k^2 C_k + \frac{m^3 \int \eta_k(z)\, dz}{2\sqrt{\lambda}\pi(1+k^2)(4+k^2)} = 0. \tag{18}$$

Calculation of the integral here gives

$$\int_{-\infty}^{\infty} dz\ \eta_k(z) = 2\pi(2-k^2)\,\delta(k), \tag{19}$$

and we have

$$\ddot{C}_k + \omega_k^2 C_k + \frac{m^3(2-k^2)\delta(k)}{\sqrt{\lambda}(1+k^2)(4+k^2)} = 0. \tag{20}$$

In case of the lowest mode of the continuum ($k_0 = 0$) it is just the equation for an oscillator in external field with the solution

$$C_{k_0} = \mathrm{e}^{i\omega_{k_0} t} - \frac{m}{4\sqrt{\lambda}} \equiv \widetilde{C}_0 - \frac{m}{4\sqrt{\lambda}}. \tag{21}$$

For all other continuum modes with $k \neq 0$ we have the trivial oscillator equation

$$\ddot{C}_k + \omega_k^2\, C_k = 0, \quad \text{i.e.,} \quad C_k = \text{const}\, \mathrm{e}^{i\omega_k t}. \tag{22}$$

Using the above-mentioned arguments one can write the arbitrary constants as $(m/\sqrt{\lambda})a_k$, where the parameters a_k are fixed by the initial conditions.

Thus, collecting the contributions from all modes of excitation (14), (17), (21), and (22), we find the first-order correction to the kink configuration

$$\phi_1 = \frac{m}{\sqrt{\lambda}} \left\{ -\frac{3}{4} m^2 t^2 \eta_0 - \frac{1}{4}\eta_{k_0} + a_1\, \mathrm{e}^{i\omega_1 t}\, \frac{\sinh z}{\cosh^2 z} + \sum_{k=0}^{\infty} a_k \widetilde{C}_k(t)\, \eta_k(x) \right\}, \tag{23}$$

where $\widetilde{C}_k(t) = \mathrm{e}^{i\omega_k t}$ and $\eta_{k_0} = 3\tanh^2 z - 1$. The last two terms in this expression correspond to the fluctuation corrections to the kink solution and can be excluded if we take the initial condition at $t = 0$ as $a_1 = 0, a_k = 0$ for all k.

The first term, connected with the zero-mode contribution, describes the motion of the kink with a constant acceleration, as mentioned above. The meaning of the second term can be clarified if one considers the corresponding correction in the asymptotic region ($x \to \pm\infty$), where we have (up to fluctuation corrections)

$$\phi(\pm\infty) = \frac{m}{\sqrt{\lambda}} \left(\pm 1 - \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2) \right). \tag{24}$$

Indeed, the potential (2) has the minima at $\phi = \phi(\pm\infty)$ given by Eq. (24). Thus, this term corresponds to a shift of the vacuum value of the scalar field.

Expression (23) allows one to calculate the first-order corrections to the kink energy $\mathcal{E}$. Substituting $\phi = \phi_0 + \varepsilon\phi_1$ into Eq. (16) we have, as one could expect,

$$\mathcal{E} = M + \varepsilon^2 \int\limits_{-\infty}^{\infty} dx \frac{1}{2}\dot{\phi}_1^{\,2} + \mathcal{O}(\lambda) = M + \varepsilon^2 \frac{3m^5}{\lambda\sqrt{2}} t^2 + \mathcal{O}(\lambda) = M + \frac{MV^2}{2} + \mathcal{O}(\lambda), \quad (25)$$

where $V = \varepsilon 3mt/\sqrt{2} = at$ is the kink velocity.

Note, that the changing of the kink kinetic energy is equal to the changing of the potential energy of the field due to linear perturbation, because

$$\Delta\mathcal{V} = \varepsilon \frac{m^3}{\sqrt{\lambda}} \int dx \left(\phi_0 + \varepsilon\phi_1 + \ldots\right), \quad (26)$$

and, in the same second order, for the large $mt \gg 1$ we have

$$\Delta^{(2)}\mathcal{V} = -\varepsilon^2 \frac{3m^4}{4\lambda} m^2 t^2 \int \frac{dx}{\cosh^2 z} = -\varepsilon^2 \frac{3m^5}{\lambda\sqrt{2}} t^2 \equiv \frac{MV^2}{2}. \quad (27)$$

In the same way, one can evaluate the second-order corrections to the kink solution [15].

2. PRODUCTION OF KINKS IN THE COLLISION OF PARTICLE-LIKE STATES

The most interesting aspects of the topological solitons are related to their dynamical properties, the processes of their scattering, radiation and annihilation [11–14, 16, 17, 21, 22]. In integrable theories, like the sine-Gordon model, there is no energy loss to radiation and kinks do not annihilate antikinks. However, in the nonintegrable ϕ^4 model, the radiation effects in the process of kink–antikink ($K\tilde{K}$) collision become very important and depending on the impact velocity, the collision may produce various results, e.g., an oscillating bound state can be formed, also the soliton and antisoliton may bounce and reflect from each other.

It is known that the collision of a kink and an antikink is chaotic, i.e., for some values of the impact velocity the solitons bounce back, while for some different impact velocity, smaller or larger, they annihilate [12, 17]. This behavior is related to a resonance effect between the oscillations of the $K\tilde{K}$ pair and excitation of the discrete vibrational mode of the kink.

The opposite process of the production of $K\tilde{K}$ pairs in the collision of particles also will have similar fractal character due to resonance effect between the oscillon created in the particle collision and the oscillation of the correlated $K\tilde{K}$ pair [21].

Thereafter we consider the rescaled model (1) without perturbation, i.e., we set $\varepsilon = 0$ and make use of the classical scale-invariance of the model to absorb the values of the vacuum expectation value and the scalar coupling into rescaled scalar field and spacial coordinate. Hence, we consider the Lagrangian

$$L = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\left(\phi^2 - 1\right)^2. \quad (28)$$

The perturbative sector of the model consists of small linear perturbations around one of the vacua with the mass $m = 2$. The static kink solution for this model interpolates between the vacua $\phi_0 = -1$ and $\phi_0 = 1$ as x increases from $-\infty$ to ∞: $\phi_K(x,t) = \tanh x$.

Note, that there is a lot of similarity between the nonintegrable model (34) and its integrable sine-Gordon counterpart. However, the states of the perturbative sector are different in these theories. Evidently, in both models there are zero translational modes in the spectrum of the linear perturbation about the kinks, but a single ϕ^4 kink has in addition a normalizable discrete vibrational mode which oscillates harmonically with frequency $\omega_1 = \sqrt{3}$. The continuum modes on the kink background have higher frequencies $\omega > 2$. This is the feature which makes the ϕ^4 model be nonintegrable.

Evidently, if the amplitude of the oscillation is large enough, such a periodically expanding and contracting kink can be treated as kink–antikink–kink bound state and this excitation can be considered as an intermediate step in the process of creation of the $K\tilde{K}$ pair on the kink background [13, 16].

Another situation is related to the possibility of production of $K\tilde{K}$ pairs on the trivial background. Indeed, the linear excitation spectrum around the trivial vacuum contains the radiation modes and within the ϕ^4 model the collision of these particle-like states may produce $K\tilde{K}$ pairs. Evidently, the $K\tilde{K}$ production may proceed even in the case when there are no kink-like states in the initial state at all [21, 31].

Note, that nonlinear field theories usually contain several types of topological and non-topological excitations. Indeed, besides the solitonic configurations there is another spatially localized nonperturbative oscillon solution which, although unstable, is extremely long-lived [24–26]. The oscillon states naturally appear in various models [27–29].

In the ϕ^4 model the oscillon solutions are almost periodic. One can find the oscillon numerically by solving the field equation in the Fourier series in time:

$$\phi = 1 + \eta_0(x) + \eta_1(x)\cos(\Omega t) + \eta_2(x)\cos(2\Omega t) + \ldots \tag{29}$$

If $\Omega < m = 2$, the oscillations are below the threshold and cannot propagate as modes of the continuum, so the oscillon remains relatively stable and the η_1 term dominates.

It was pointed out recently that an oscillation mode of the ϕ^4 model may decay into a $K\tilde{K}$ pair [21, 30].

To investigate the process of production of the kinks, we consider two widely separated identical wave trains propagating from both sides on the trivial background towards a collision point:

$$\phi(x,t) = 1 + C[F(x+vt)\sin(\omega t + kx) + F(x-vt)\sin(\omega t - kx)], \tag{30}$$

where k is the wavenumber of the incoming wave; $\omega = \sqrt{k^2+4}$ is the frequency and $v = k/\omega$ is the velocity of propagation of the wave train. We consider the envelope of the train $F(x) = [\tanh(x-a_1) - \tanh(x-a_2)]$, also the Gaussian envelope $F(x) = \mathrm{e}^{-(x-a_3)^2}$ was used to prove that our results are independent of the particular choice of the initial state. The parameters a_1, a_2, and a_3 define the length of the train and the initial separation between the trains. Typically, in our numerical simulations we used the values $a_1 = 10$, $a_2 = 30$, $a_3 = 20$. The amplitude C and the wavenumber k are the impact parameters, which can be changed freely. To find a numerical solution of the PDE describing the evolution of the system, we used the pseudospectral method. For the time-stepping function we used symplectic (or geometric) integrator of the 4th order to ensure that the energy is conserved.

In our numerical analysis we found that after small-amplitude collisions, the two wave trains separate and move in opposite directions and the radiation is created due to the interaction between these trains. In the center of collision an oscillating lump remains. For small amplitudes, the frequency of the oscillation is just a bit above the mass threshold. This indicates that the lump could be identified with low wavenumber linear excitation of the trivial vacuum. For large-amplitude collisions, the remaining lump oscillates with frequency within the mass gap, so such a state can be identified as an oscillon.

Furthermore, for a certain range of values of the impact parameters, C and k, we observed the creation of $K\tilde{K}$ pairs. During this process also an oscillon is created in the collision center (Fig. 1).

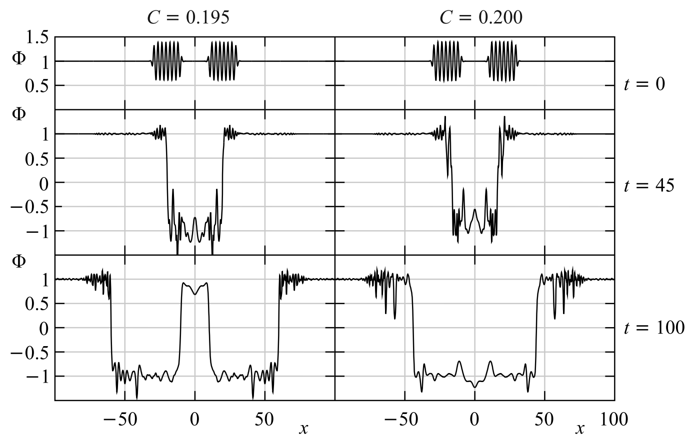

Fig. 1. Production of the kinks in the collision of two identical wave trains. The initial and final field configurations are plotted at $t = 0$, $t = 45$, and $t = 100$, respectively

The most important feature of this process is that in the space of parameters, the regions of creation of the solitons and the regions where this process is not taking place, are separated by a fractal-like boundary (Fig. 2).

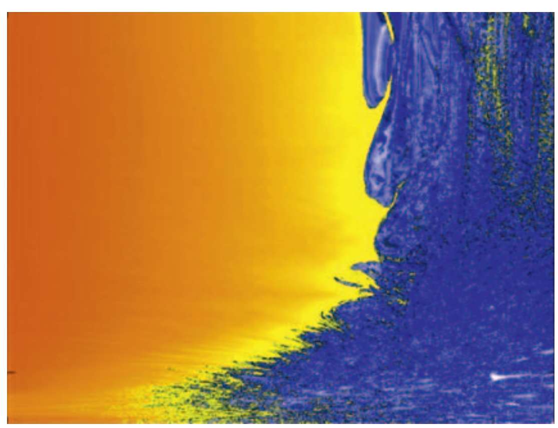

Fig. 2 (color online). Fractal structure in the C, k plane. Shading (or color) represents the measured minimum of average of the field $\langle A\rangle = 1/20 \int_{-10}^{10} dx\phi(x,t)$. The dark regions (blue in color), where $\langle A\rangle < -1$, indicate the creation of $K\tilde{K}$ pairs

Indeed, this diagram is made of elementary plaquettes (boxes). The total number of these boxes corresponds to spatial resolution of the diagram, in our calculations we typically used the pixel resolution 640×480. If N is the number of boxes covering the boundary between the regions of creation and the trivial sector and l is the side-length of the boxes, the box-counting (fractal) dimension is defined as ratio $d = \lim_{l \to 0} \log N / \log l$. For finite wave trains we do not expect that this boundary would be a real fractal but some properties of scaling are observed. We have measured the fractal dimension to be $d = 1.770 \pm 0.011$, which is much more than 1. In the case of the Gaussian envelope we found $d = 1.865 \pm 0.007$. The interesting peculiarity of the latter case is that the $K\tilde{K}$ pairs can be created even if $k = 0$ (standing wave perturbation).

For certain values of impact parameters, an oscillon remaining in the collision center decays into the second $K\tilde{K}$ pair. Sometimes the second pair moves even faster than the first pair, and it may annihilate with the first one creating two moving oscillons. We know that the process of collision $K\tilde{K}$ pair also leads to fractal structure in the velocity space [12, 17]. In our process, instead of creation on $K\tilde{K}$ pair, two oscillons could also be ejected from the collision center and after a while they could decay into two pairs of $K\tilde{K}$. We observed some evidences that these processes also yield the fractal dependency of impact parameters.

In order to capture the most important steps in the process of the creation of $K\tilde{K}$ pair, in the collision of two identical bunches of particles, we use the collective coordinate method which allows us to identify the physical degrees of freedom of the system under consideration. This approach has been applied to describe the dynamics of the kink–antikink system [32].

First, we describe the process of creation of the oscillon in the collision of the incoming wave trains. We assume an initial field configuration on the trivial background

$$\phi(x,t) = 1 + \frac{A(t)}{\cosh(x/x_0)} + \xi(x,t), \tag{31}$$

which corresponds to the profile of the oscillon solution [25] with some additional perturbation $\xi(x,t)$. The Gaussian approximation to the oscillon configuration [26] was also used to check the results. Here, the variable $A(t)$ is introduced as the collective coordinate of the oscillon and the parameter x_0 represents the oscillon width. From the expansion (29) we know that when $\xi = 0$ the oscillon should, in the first approximation, oscillate as $A(t) = A_0 \cos(\Omega t)$, where A_0 is the amplitude of the oscillations, $\Omega < 2$ and the value of the parameter x_0 depends on the amplitude A_0. In the presence of the external field ξ, the amplitude of the oscillon changes. However, for the sake of simplicity, we set $x_0 = 1.5$ as it is the width of the oscillon oscillating with amplitude $A_0 = 0.4$. Substituting (31) into (34) and after integration over all space x gives the effective Lagrangian which can be split into three parts: Lagrangian of the free oscillon, Lagrangian of the perturbation ξ, and Lagrangian of interaction between the oscillon and the perturbation,

$$L(A, \dot{A}) = L_A + L_\xi + L_{\text{int}}. \tag{32}$$

The Lagrangian of the free oscillon has the form

$$\frac{L_A}{x_0} = (\dot{A})^2 - \frac{2}{3}A^4 - \pi A^3 - \left(4 + \frac{1}{3x_0^2}\right)A^2. \tag{33}$$

This is the Lagrangian of an anharmonic oscillator with frequency $\Omega_0 = \sqrt{4 + \dfrac{1}{3x_0^2}} > 2$. Since the frequency of the oscillon must be smaller than $m = 2$, the amplitude of the oscillations must be large enough to decrease the oscillation frequency below the mass threshold [21, 26], so the nonlinearities are crucial for the existence of the oscillon. We assume that the field ξ is a solution to the equation of motion of the Lagrangian L_ξ. The perturbation ξ should represent two wave trains coming from $\pm\infty$. For the sake of simplicity, we take the perturbation of the form (30).

The initial condition is that $A(0) = 0$. As the wave train approaches the point of the collision, the oscillon mode is excited. If the amplitude of the perturbation is relatively small, then the oscillon, created in the collision, oscillates with a constant amplitude around $A = 0$. However, if the amplitude is large enough, or the incoming perturbations are close to one of the (Mathieu) resonances, the amplitude of the oscillon rapidly increases and it starts to oscillate around $A = -1$ (or, in other words, around $\phi = 0$) with amplitude of order 1. This clearly breaks our effective approach, but it also means that the system has changed the ground state. Such a resonant oscillation with a large amplitude, on the other hand, shifts the center of the oscillation. This transition can be related to the creation of $K\tilde{K}$ pairs although the corresponding collective coordinates are not presented in our simplified model.

Again, when we examined this effective model, we found a fractal structure in the plane A, k. This fractal structure was less complicated and more localized than in the case of full PDE. That means that although our effective model works and captures qualitatively the most important features of the full system it also fails to reproduce some of the details, which is not a surprise for such a complicated dynamical process. We have also introduced an approximation for $\alpha(t)$, $\beta(t)$, and $\gamma(t)$, and again, we could reproduce both the resonance excitation of the oscillon and the generation of fractal structure.

This result shows, that even after performing so many simplifications we could reproduce (at least qualitatively) the most important features of the evolution of the system. This result confirms our conjecture about the mechanism of the creation of the $K\tilde{K}$ pair in a three-stage process (i.e., excitation of the oscillon, resonance and oscillon decay into the $K\tilde{K}$ pair). Secondly, we conclude that the interaction between the incoming wave trains and the oscillon is the underlying reason for the generation of the fractal structure. Thirdly, given the generality of our approach, we expect that the effective nonlinear interactions of the same type can be found in many different models, so the fractal structure should not be limited only to the case of the ϕ^4 model.

3. RESONANCE STRUCTURE IN THE KINK–ANTIKINK COLLISIONS IN THE ϕ^6 MODEL

Let us consider now a bit different model, the $(1+1)$-dimensional ϕ^6 theory, defined by the rescaled Lagrangian density [33]

$$L = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\phi^2\left(\phi^2 - 1\right)^2. \tag{34}$$

The model has three vacua $\phi_v \in -1, 0, 1$. The static kink solution ϕ_K interpolates between the vacua, e.g., $\phi_v = 0$ and $\phi_v = 1$ as x increases from $-\infty$ to ∞: $\phi_K(x) = \phi_{(0,1)}(x) = \sqrt{\dfrac{1+\tanh x}{2}}$. Other static (anti)kinks can be obtained from this solution by using the discrete symmetries of the model $\phi \to -\phi$, $x \to -x$, so $\phi_{\bar{K}}(x) = \phi_{(1,0)}(x) = \phi_{(0,1)}(-x)$, $\phi_{(0,-1)}(x) = -\phi_{(0,1)}(x)$ and $\phi_{(-1,0)}(x) = -\phi_{(0,1)}(-x)$. The mass of the kinks is $M = 1/4$.

The perturbative sector of the model consists of small linear perturbations («mesons») around one of the kink solutions $\phi(x,t) = \phi_K(x) + \eta(x)\,\mathrm{e}^{i\omega t}$. Linearized field equations are $-\eta_{xx} + U(x)\eta = \omega^2\eta$, where [33]

$$U(x) = 15\phi^4 - 12\phi^2 + 1. \tag{35}$$

Considering an isolated kink $\phi(x) = \phi_K$ one can see that there are no bound states but the usual translational zero mode. The states of the continuum spectrum can be written in terms of the hypergeometric functions [33].

Since the model (34) contains two different classes of the kink solutions, we have to analyze $K\bar{K}$ collisions in the sector with vacuum state $\phi_v = 0$ and in the sector with vacuum state $\phi_v = 1$ separately (note mirror symmetry of the sectors $\phi_v = \pm 1 f$). In the former case the initial configuration of the colliding kinks, which we denote as $(0,1) + (1,0)$, can be taken as a superposition $\phi(x) = \phi_K(x+a) + \phi_{\bar{K}}(x-a) - 1$, where a is the separation parameter; in the latter case the initial $K\bar{K}$ configuration $(1,0) + (0,1)$ is $\phi(x) = \phi_K(x-a) + \phi_{\bar{K}}(x+a)$. Note, that in both cases there is no internal vibrational mode bounded to the kinks, so we could expect the $K\bar{K}$ collision will be elastic and no resonant structure will be observed.

However, our numerical results reveal completely different picture. Indeed, in the case of the $(0,1) + (1,0)$ $K\bar{K}$ collisions we observe no resonance windows and the process is regular as expected (Fig. 3, *b*). For $v < v_{\mathrm{cr}} \approx 0.289$ the pair annihilates into the vacuum $\phi_v = 0$ with small amount of radiation emitted, while for $v > v_{\mathrm{cr}}$ the collision yields a mirror pair of solitons escaping to infinity with no bouncing: $(0,1) + (1,0) \to (0,-1) + (-1,0)$. By contrast, the collision of the $(1,0) + (0,1)$ configuration reveals the fractal structure with a sequence of bouncing windows presented in Fig. 3, *a*.

For velocities $v > v_{\mathrm{cr}} = 0.0457$ the kinks would always have enough energy to separate, however, for smaller impact velocities we observe regular n-bounce windows. Evidently, this pattern is very much similar to the well-known observations in the ϕ^4 model although, as in the case of the $(0,1) + (1,0)$ $K\bar{K}$ collisions, there is no internal mode into which the kinetic energy can be transferred. Thus, we have to look for another mechanism which would explain such a behavior.

A special feature of the spectrum of linear perturbation around the ϕ^6 kink is that, unlike the ϕ^4 model, the potential $U(x)$ is not symmetric with respect to reflections $x \to -x$. Therefore, the mass of the meson states is different on the opposite sides of the kink. This peculiarity means that there is a wide potential well in the system of well-separated $(1,0) + (0,1)$ $K\bar{K}$ pair (Fig. 4) with two local minima associated with positions of the solitons.

By contrast, in the case of $(0,1) + (1,0)$ configuration these two minima are separated by a barrier. If the velocity of the kinks is relatively small, the adiabatic approximation can

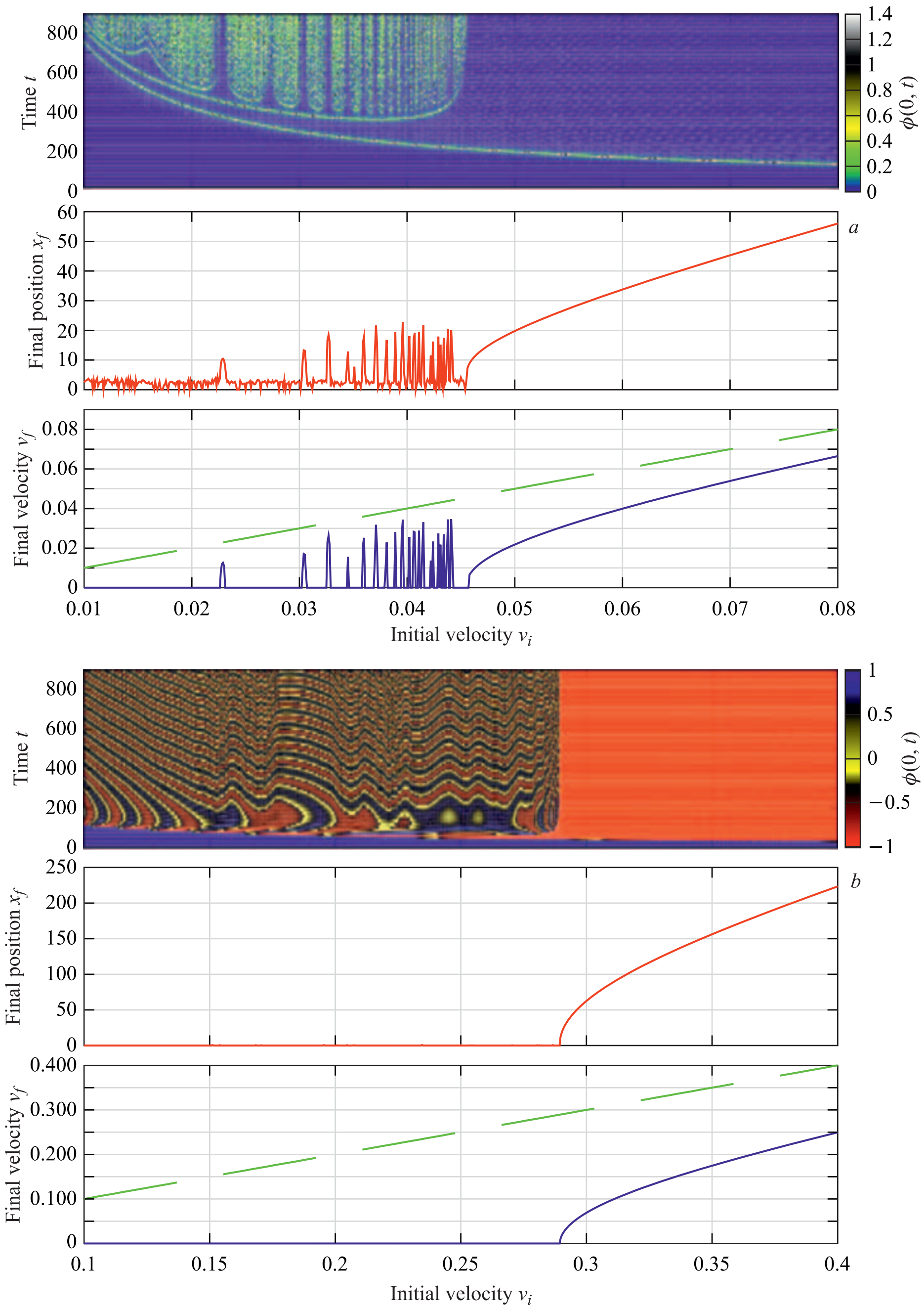

Fig. 3. The $K\bar{K}$ collision in the sectors $(1,0)+(0,1)$ (*a*) and $(0,1)+(1,0)$ (*b*), respectively. The plots represent the field values measured at the collision center (up), position of the kink (middle) and fitted velocity of the kink (bottom), respectively $(1,0)+(0,1)$

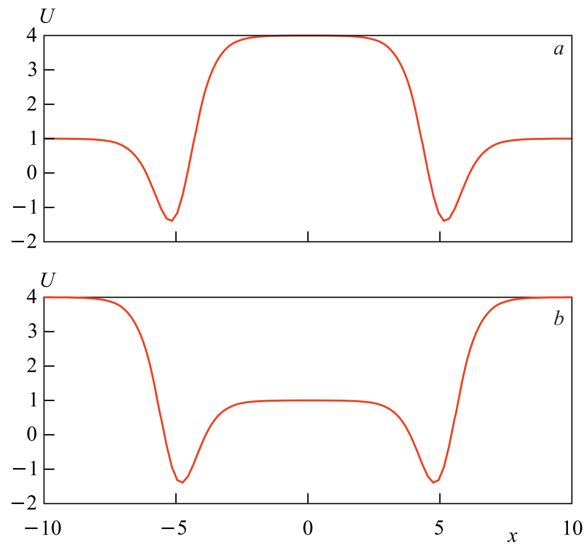

Fig. 4. The potential $U(x)$ of the linear perturbations on the background of the composed $(0,1)+(1,0)$ (*a*) and $(1,0)+(0,1)$ (*b*) $K\bar{K}$ configurations

be used and then we can find a numerical solution of Eq. (35) for a configuration $\phi(x) = \phi_K(x-a) + \phi_{\bar{K}}(x+a)$ as a function of separation parameter a. Evidently, two lowest states are the quasi-zero modes of the $K\bar{K}$ pair which rapidly approach zero eigenvalue from opposite sides as separation grows. The meson states, trapped by the potential well generated by the $K\bar{K}$ pair, then have separation-dependent energy.

So, the difference from the kink collision in the ϕ^4 model is that although there is no internal mode of the kinks, the collective meson states can be excited by slowly approaching solitons absorbing the kinetic energy of the $(0,1)+(1,0)$ $K\bar{K}$ configuration and giving it back in some sort of the resonance process. Note, that the critical velocity in this case is much smaller than in the collision of the $K\bar{K}$ pair in the ϕ^4 model [12, 17]. Since the mass of the excitations around the vacua $\phi_v = 0$ and $\phi_v = 1$ is $m_0 = 1$ and $m_1 = 2$, respectively, it is also much easier to excite the radiation modes in the collision of the $(1,0)+(0,1)$ pair. It implies that less energy will be lost to radiation than it happened in the case of the collision of the $(0,1)+(1,0)$ pair, so the critical velocity in the former case is lower.

Furthermore, a very precise scan of the narrow region of the velocities range just below the critical velocity $v_{\rm cr}$, reveals the fractal structure of this process similar to the intricate pattern of the $K\bar{K}$ interaction in the ϕ^4 model [17]. Thus, considering two-bounce windows, we can label each window by an integer n denoting the number of oscillations between two collisions. This number increases by one for the next consequent windows.

Note, that there is an important difference from the resonant structure observed in the ϕ^4 $K\bar{K}$ collision [12, 17], where the first two-bounce window is associated with a single oscillation of the internal mode. In our case, the separation between the kinks must be much larger to create a trapping potential, so the first two-bounce window already has a large number of oscillations.

Our numerical results clearly suggest that these oscillations are associated with the lowest collective mode. Indeed, for all two-bounce windows we observed, the kinks are separated by $2a \approx 12$. Considering the dependence of the time $T(v)$ between the collision vs. number of oscillations in the two-bounce window $n(v)$, we observe fitted linear function $T = 2\pi n\omega + \delta$, where $\delta = 11.7779$ and $\omega = 1.045$, which is the same frequency of the lowest collective mode as the value fitted from Fourier transform. Again, this closely resembles similar relations in the ϕ^4 model [12, 17] although the mechanism of the resonant reflection and the parameters ω, δ are different. Actually, the main difference is that in the case of the $(1,0)+(0,1)$ system the interaction between the kinks mediated by the collective meson state is long-ranged and relatively weak, whereas the internal meson state absorbing the energy of collision of the ϕ^4 $K\bar{K}$ collision, is exponentially localized and the kinks after the first collision can move as free particle-like configurations. In the ϕ^6 model the $K\bar{K}$ pair forms a potential well whose boundaries are interacting with the collective meson states exerting an extra pressure on the kinks. This could be observed as small acceleration of the bouncing kinks between the first and the second collisions. Also the energy of the collective meson states trapped by the adiabatically approaching ϕ^6 kinks is not constant.

Nevertheless, we can try to extend the similarity between these different models by consideration of the asymptotic attractive force, which can be approximated by corresponding Yukawa potentials as $F(a) \sim 2\,\mathrm{e}^{-a}$ for the $\bar{K}K$ pair, and as $F(a) \sim 2\,\mathrm{e}^{-2a}$ for the $K\bar{K}$ pair. This allows us to estimate the dependence $T(v)$ under assumption that the energy of the colliding kinks goes to the excitation of the collective meson state and then it returns back to the kinetic energy of the kink after the second bounce.

A novel feature of resonant ϕ^6 scattering is the «missing» window at $n = 13$. For resonant ϕ^4 scattering, two-bounce windows are also missing, for $n < 3$, but once they set in, they are found for all n, at least up to initial velocities very close to the critical escape velocity. By contrast, in ϕ^6 scattering we found the first two-bounce window at $n = 12$, then a gap at $n = 13$, and then windows for all higher values of n that we examined. A similar structure is reproduced when looking at the three-bounce windows next to a given two-bounce window, and we suspect that the pattern will continue at all higher levels. It is possible that an explanation for this behavior will be found in a careful treatment of the higher modes of the bound-state spectrum.

Our investigation of ϕ^6 kink collisions has shown that resonance phenomena have wider relevance to kink scattering than it had previously been thought. In particular, similar behavior should be seen in any model in which kinks interpolate between degenerate but nonequivalent vacua. The new mechanism enabling resonances to occur is the formation of meson bound states in the potential well created in the space between the constituents of a suitably ordered kink–antikink pair, and does not require the existence of an internal mode localized on a single kink. The resulting pattern of resonance windows is more complicated than that for ϕ^4 scattering, with gaps appearing at all levels. It remains a major challenge, deserving further study, to find a robust mechanism to explain these gaps theoretically.

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