ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

«MASSIVE» PERTURBATIVE QCD,
REGULAR IN THE IR LIMIT

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*Dedicated to the memory
of Petr Stepanovich Isaev.*

The goal of the research is to devise a modification of the perturbative QCD (pQCD) that should be regular in the low-energy region and could serve as a practical means for the analysis of data below 1 GeV down to the IR limit. Recent observation of the four-loop pQCD series «blow-up» in the region below 1 GeV for the Bjorken Sum Rule gave an impetus to this attempt.

The proposed «massive» analytic pQCD has two sources. It can be treated as the common logarithmic pQCD with only one parameter added, the effective «glueball mass» $m_{\mathrm{gl}} \lesssim 1$ GeV, serving as an IR regulator. At the same time, it looks like a modification of Analytic Perturbation Theory (APT) comprising nonpower perturbative expansion that makes it compatible with linear integral transformations.

Figuratively (with minor reservations), the proposed MAPT differs from the minimal APT by simple ansatz $Q^2 \to Q^2 + m_{\mathrm{gl}}^2$.

Цель исследования состоит в нахождении такой модификации пертурбативной КХД (pQCD), которая, будучи регулярной при низких энергиях, могла бы служить практическим средством анализа данных в области ниже 1 ГэВ вплоть до инфракрасного предела. Недавнее обнаружение «взрывного» характера поведения 4-петлевого ряда теории возмущений для формфактора правила сумм Бьёркена в области ниже 1 ГэВ дает стимул для этой попытки.

Предлагаемая «массивная» теория возмущений (МТВ) имеет два корня. Она может трактоваться как обычная безмассовая pQCD с одним добавленным параметром — «массой глюбола» $m_{\mathrm{gl}} \lesssim 1$ ГэВ, служащей инфракрасным регуляризатором. В то же время ее можно рассматривать как модификацию аналитической теории возмущений (АТВ), с сохранением ее нестепенной формы разложения, которая обеспечивает совместимость с линейными интегральными преобразованиями.

Выражаясь образно (с небольшими оговорками), предлагаемая МТВ отличается от минимальной АТВ заменой $Q^2 \to Q^2 + m_{\mathrm{gl}}^2$.

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1. MOTIVATION AND OUTLINE

As is well known, the so-called perturbative QCD (pQCD) or the renormalization group (RG)–improved QCD perturbation expansion *taken in the UV limit* is a firmly established part of particle interaction theory. This piece is not only respectable but worthy of admiration as, starting with gauge-noninvariant quantization, it correlates several dozens of experiments at quite different scales from a few up to hundreds of GeV.

At the same time, the pQCD meets serious troubles in the low-energy (large-distance) domain below a few GeV at the scales marked by the QCD parameter $\Lambda \sim 400$ MeV. This Achilles' heel is related to its UV origin.

To avoid the unwanted singularity in the LE region, several modifications [1] of the pQCD have been proposed. Recently, one of them, the Analytic Perturbation Theory [2] (APT), was good enough [4] in describing the polarized $\Gamma_1^{p-n}(Q^2) = \Gamma_1(Q^2)$ form factor of the Bjorken Sum Rule (BjSR) amplitude *down to a few hundred MeV*.

The Bjorken moment was presented there as a sum of PT and higher-twist (HT) non-perturbative contributions

$$\Gamma_1(Q^2) = \frac{g_A}{6}\left[1 - \Delta_{\mathrm{Bj}}^{\mathrm{PT}}(Q^2)\right] + \Gamma_{\mathrm{HT}}, \qquad \Gamma_{\mathrm{HT}} = \sum_{i=2}^{\infty} \frac{\mu_{2i}}{Q^{2i-2}}, \tag{1}$$

with $\Delta_{\mathrm{Bj}}^{\mathrm{PT}}$ including the forth-order term $\sim (\alpha_s(Q^2))^4$. However, an attempt to fit JLab data by expression (1) with appropriate HT coefficients failed as the perturbative part exploded in the region 0.5–1 GeV and the extracted (via comparison with fitted JLab data) μ_{2i} values turned out to be unstable with respect to higher-loop terms in first PT sum. This prevents the description of data below 1 GeV.

Along with Eq. (1), in [4] the PT sum was changed to the APT one:

$$\Delta_{\mathrm{Bj}}^{\mathrm{PT}} = \sum_{k \leqslant 4} c_k \left(\alpha_s(Q^2)\right)^k \Rightarrow \Delta_{\mathrm{Bj}}^{\mathrm{APT}}(Q^2) = \sum_{k \leqslant 4} c_k \, \mathcal{A}_k(Q^2), \tag{2}$$

with $\mathcal{A}_k(Q^2)$, the APT ghost-free expansion functions. The positive result consists in good fitting of the precise JLab data down to a few hundred MeV with stable HT parameters.

This achievement rises hope for the possibility of a global fitting down to the IR limit. Unhappily, none of the above-mentioned ghost-free modifications is suitable for this goal. The common drawback is the use of UV logs in the IR region.

To approach the global fitting of data (like ones for the BjSR form factor), one needs to have a theoretical framework with two essential features:

• Correspondence with common pQCD in the UV (that is above a few GeV);

• Correlation with lattice simulation results for the effective coupling $\alpha_s(Q)$ smooth behavior in the low-energy domain.

As a primary launch pad for this construction, the above-mentioned APT seems good. It satisfies the first of the conditions and, qualitatively, the second one. To exempt the APT-like scheme from its last drawback — the singularity (with an infinite derivative) in the IR limit, one has to disentangle it from the UV logs. To this end, a mass-dependent RG-invariant modification inspired by our paper [5] will be used.

In Sec. 3, on the basis of the massive renormalization group (see Sec. 2), the nonsingular version of pQCD with one additional (besides Λ) parameter, an effective «gluonic» mass, m_{gl}, a *massive pQCD*, MPT for short, is formulated.

2. MASSIVE PERTURBATION THEORY

In [5], a particular way of constructing the QCD invariant coupling $\alpha_s(Q^2)$ free of unphysical singularities was proposed. In contrast to the APT, it does not involve explicit nonperturbative contributions. Instead, the Q^2 algebraic (non-log) dependence appeared there due to threshold effects, and an essential technical ingredient was the assumption of the *finite gluon mass* formal presence.

The model expression for $\alpha_s(Q^2)$ was obtained there by the RG summation of the *mass-dependent* diagram contribution — see below Eqs. (4) and (6). It depends upon the gluon m_{gl} and light-quark m_{lq} (lq $= u, d, s$) masses; in the IR region $Q^2 > 0$ has no singularities with a finite limiting value $\alpha_s(0)$ and, as $Q^2/m^2 \to \infty$, smoothly transits into the usual asymptotic freedom formula.

2.1. Mass-Dependent 1-Loop Diagram. At the one-loop level, the starting element is the massive (mass-dependent) 1-loop contribution. For example, to the virtual dissociation of a vector particle (photon, gluon) into a massive fermion–antifermion pair ($e^+ + e^-$, $q + \bar{q}$) in the s-wave state there corresponds a function $I_s(Q^2/m^2)$ representable via spectral integral[1]

$$I_s(z) = z \int\limits_1^\infty \frac{k_s(\sigma)\, d\sigma}{\sigma(\sigma + z)}, \quad k_s(\sigma) = \sqrt{\frac{\sigma - 1}{\sigma}}, \quad I_s(0) = 0.492, \quad I_s'(0) = 2/3, \qquad (3)$$

which in the space-like region $z > 0$ is a *positive, monotonically growing function* with the log asymptotic behavior $I_s^{\mathrm{UV}}(z = Q^2/m^2) \simeq \ln z - C_s + O(1/z)$; $C_s = 2(1 - \ln 2)$ and the regular IR limit with a finite derivative.

2.2. Massive Renormalization Group Summation. For the QCD coupling modification at small space-like $Q^2 \lesssim \Lambda^2$, we involve the mass-dependent Bogoliubov renormalization group (mRG) formulated in the pioneer RG papers [8] in the mid-fifties. As has been known since that time [9], the mRG, like the common massless RG, sums iterations of a one-loop contribution[2] $\alpha_s(z)_{\mathrm{pt}}^{[1]} = \alpha_s - \alpha_s^2 A_1(z, y) + \ldots$, into the geometric progression

$$\alpha_s(Q^2)_{\mathrm{rg}}^{[1]} = \frac{\alpha_s}{1 + \alpha_s A_1(z, y)}, \quad z = \frac{Q^2}{m^2}, \quad y = \frac{\mu^2}{m^2}. \qquad (4)$$

For the 2-loop case, with A_2, the genuine second-loop contribution

$$\alpha_s(z, y)_{\mathrm{pt}}^{[2]} = \alpha_s - \alpha_s^2 A_1(z, y) + \alpha_s^3(A_1^2 - A_2(z, y)) + \ldots, \qquad (5)$$

an analogous (approximate) RG-invariant «massive» sum

$$\alpha_s(z, y)_{\mathrm{rg}}^{[2]} = \frac{\alpha_s}{1 + \alpha_s A_1(z, y) + \alpha_s \dfrac{A_2(\ldots)}{A_1(\ldots)}(1 + \alpha_s A_1(\ldots))} \qquad (6)$$

was also devised later (Eq. (8) of paper [10]). There,

$$A^{[\ell=1,2]}(z, y) = I^\ell\left(\frac{Q^2}{m^2}\right) - I^\ell\left(\frac{\mu^2}{m^2}\right)\Bigg|_{\mathrm{UV}} \to \beta_{\ell-1} \ln\left(\frac{Q^2}{\mu^2}\right), \qquad (7)$$

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[1]For an explicit expression, see Sec. 24.1 in the textbook [6] and Sec. 35.1 in the monograph [7].

[2]Here and below, the superscript in square brackets $\mathcal{A}^{[\ell]}$ denotes the order of loop approximation.

with transition to the QCD scale

$$\alpha_s(Q^2)^{[1]}_{\text{rg}} = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)} \tag{8}$$

performed via the relation

$$\frac{1}{\beta_0\alpha_s} + \ln\left(\frac{Q^2}{\mu^2}\right) = \ln\left(\frac{Q^2}{\Lambda^2}\right). \tag{9}$$

3. THE MPT CONSTRUCTION

3.1. One-Parameter Massive Model. A simple idea is also to change the usual UV logarithm $\ln x$ ($x = Q^2/\Lambda^2$) that is also singular in the IR for the «long logarithm» $L_\xi(x) = \ln(\xi + x)$ which reproduces qualitatively the smooth LE behavior of function (3), being regular at $Q^2 = 0$. It is worthy of note that the new parameter is expressed via the coupling constant (at $Q^2 = 0$) by the relation $\xi = e^{1/\beta_0\alpha_s}$ nonanalytic at $\alpha_s = 0$. It corresponds to the «effective gluonic mass» $m_{\text{gl}} = \sqrt{\xi}\Lambda$, an old notion (see a recent survey by Simonov [11] and references therein) used as an IR regulator. In short, our ansatz is

$$\boxed{Q^2 \to Q^2 + m_{\text{gl}}^2.} \tag{A}$$

One-loop case. The «1-loop structure» in the denominator of Eq. (8) is changed now to the «long logarithm» with ξ, an adjustable parameter:

$$\ln x \to L_\xi(x) = \ln(\xi + x), \quad x = \frac{Q^2}{\Lambda^2}. \tag{10}$$

At moderate LE scales the form

$$L_\xi(x) = \ln\xi + \ln(1 + \phi x) = \frac{1}{\beta_0\alpha_s} + \ln(1 + \phi x), \quad \phi = 1/\xi \tag{11}$$

is more adequate. In terms of this LE form, one has

$$\mathcal{A}^{[1]}_{1\text{MPT}}(x;\xi) = \frac{1}{\beta_0 \ln(\xi + x)} = \frac{\alpha_s}{1 + \alpha_s\beta_0 \ln(1 + x\phi)}, \quad \phi = e^{-1/\beta_0\alpha_s}, \tag{12}$$

with $\alpha_s = \alpha_s(x = 0)$; $\alpha_s|_{\xi=10\pm2} = 0.61 \mp 0.05$. The finite derivative at $Q = 0$ is also of interest, $\mathcal{A}^{[1]'}_{1,\text{MPT}}(0,\xi) = -\beta_0\alpha_s^2$.

The 2-loop 1-parameter model. Starting with the 2-loop massive RG-summed result, Eq. (6) corresponding to the mass-dependent PT expansion, Eq. (5), we generalize Eq. (12) by using *the same «long logarithm» model for the second-loop contribution*

$$A_2(x\phi) = \beta_1 \ln(1 + \phi x), \tag{13}$$

which results in

$$\mathcal{A}^{[2]}_{1,\text{MPT}}(x,\xi) = \frac{\alpha_s}{1 + \alpha_s\beta_0 \ln(1 + \phi x) + \alpha_s\dfrac{\beta_1}{\beta_0} \ln[1 + \alpha_s\beta_0 \ln(1 + \phi x)]}, \tag{14}$$

written down in the denominator representation [12].

Now, the condition $\alpha_s(M_\tau^2) = 0.34$ can be used for a rough evaluation of an «effective MPT-QCD scale» value. In the NLO case, the $\Lambda^{[2]}(\xi)$ dependence on ξ turns out to be rather weak[1]:

$$\Lambda^{[2]}(10 \pm 2) \sim (315 \mp 10) \text{ MeV}. \tag{15}$$

The related $m_{\text{gl}} = \sqrt{\xi}\Lambda^{[2]}$ value could be close to the nucleon mass at $\xi \sim 10$.

3.2. Higher MPT Expansion Functions. In the construction under devising, we intend to preserve an essential APT feature, namely, the nonpolynomiality of the modified «perturbative» MPT expansion, expansion over a set[2] of functions $\{\mathcal{A}_k(Q^2, \xi)\}$ connected by the same differential relations as in the APT (with the dotted notation for logarithmic derivative $\dot{F}(x) = xF'(x)$)

$$-\frac{x}{k}\frac{\partial}{\partial x}\mathcal{A}_k(x, \xi) := -\frac{1}{k}\dot{\mathcal{A}}_k(x, \xi) = \beta_0 \mathcal{A}_{k+1}(x, \xi) + \beta_1 \mathcal{A}_{k+2}(x, \xi) + \dots \tag{16}$$

To the arguments ascending to the 80s [13] and related to the π^2-terms summation procedure in the s-channel (see also [2]), one can add a fresher reasoning [14].

The second MPT function. This recurrence property ensures compatibility [15] with linear transformations involved in transition to the distance picture (Fourier-conjugated with the momentum-transfer one) and to the annihilation s-channel.

In a particular case $k = 1$, with (12), and neglecting the second r.h.s. term, i.e., using the one-loop relation

$$\mathcal{A}_{k+1}(x, \xi) = -\frac{1}{k\beta_0}\dot{\mathcal{A}}_k(x, \xi) \tag{17}$$

as a definition for higher functions, one gets the second expansion function[3]

$$\mathcal{A}_2(x, \xi) = (\mathcal{A}_1(x, \xi))^2 R(x, \xi), \quad R(x) = \dot{L}_\xi(x) = \frac{\phi x}{1 + \phi x}, \quad \phi = 1/\xi, \tag{18}$$

that turns to zero in the IR limit. Besides, $\mathcal{A}'_2(0, \xi) = \alpha_s^2 \phi$.

The third MPT expansion function obtained by Eqs. (17) and (18) can be represented in the form

$$-2\beta_0 \mathcal{A}_{3,\text{MPT}}^{[1]}(x, \xi) = \dot{\mathcal{A}}_{2,\text{MPT}}(x, \xi) = 2\mathcal{A}_{1,\text{MPT}}\dot{\mathcal{A}}_{1,\text{MPT}} R(x) + (\mathcal{A}_{1,\text{MPT}})^2 \dot{R}(x), \tag{19}$$

which is sufficient for perceiving IR properties $\mathcal{A}_3^{[1]}(0, \xi) = 0$; $\mathcal{A}_3^{[1]'}(0, \xi) = -\alpha_s^2 \phi$.

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[1]With values less than the pQCD one $\Lambda^{n_f=3} = (420 \pm 10)$ MeV.

[2]The same symbol $\mathcal{A}$, as in the minimal APT with limiting relation $\mathcal{A}_k(x, \xi = 0) = \mathcal{A}_k(x)$ is used.

[3]For discussion of a more accurate definition of the two-loop second function, see Appendix.

4. GENERAL FEATURES OF THE MPT SCHEME

Here, we shortly discuss some important features of the proposed MPT construction.

4.1. Annihilation Channel. For the transition to the s-channel, $f(Q^2) \to F(s)$, one uses the spectral representation and the Adler-type relation

$$f(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{\rho(\sigma)\,d\sigma}{Q^2 + \sigma}, \quad f(Q^2) = Q^2 \int_0^\infty \frac{F(s)\,ds}{(s + Q^2)^2},$$

which, in turn, results in

$$F(s) = \frac{1}{\pi} \int_s^\infty \frac{d\sigma}{\sigma}\, \rho(\sigma), \quad \rho(\sigma) = \Im f(-\sigma - i\varepsilon). \tag{20}$$

In the simplest one-loop case (12),

$$\beta_0 \mathcal{A}_{1\mathrm{MPT}}^{[1]}(x;\xi) = \frac{1}{\ln(\xi + x)} = \frac{1}{\pi} \int_0^\infty \frac{\rho_1(\sigma)\,d\sigma}{x + \sigma}, \quad \rho_1(\sigma) = \frac{\pi\theta(\sigma - \xi)}{\ln^2(\sigma - \xi) + \pi^2}, \tag{21}$$

$$\beta_0 \mathfrak{A}_{1\mathrm{MPT}}^{[1]}(s;\xi) = \frac{1}{\pi} \int_s^\infty \frac{d\sigma}{\sigma}\, \rho_1(\sigma) = \frac{1}{\pi}\arctan\frac{1}{L_-(s)}, \quad L_-(s) = \ln\left(\frac{s - m_{\mathrm{gl}}^2}{\Lambda^2}\right). \tag{22}$$

This expression remains regular around $s \sim m_{\mathrm{gl}}^2 + \Lambda^2$ and constant below $s = m_{\mathrm{gl}}^2$.

To obtain the higher $\mathfrak{A}_{k\mathrm{MPT}}$ functions, one should use a differential recurrent relation, just as in APT. For the two-loop case, one can combine it with the *two-loop effective log L^**, a trick proposed in [3] and further developed in [16].

5. COMPARISON WITH APT AND DISCUSSION

To compare the new construction with the APT one, in Fig. 1 we give the curves at a few values of $\xi = 10 \pm 2$ in the region below 2 GeV for the first MPT function $\mathcal{A}_{1,\mathrm{MPT}}^{[2]}$ (14) together with the corresponding APT (dashed) curve.

It can be seen from the NLO curves that values $\xi = 8{-}10$ seem to be preferable. Indeed, for these values, the first MPT function, $\mathcal{A}_1^{[2]}(x,\xi)$ is reasonably close to the first APT one down to 1 GeV. At the same time, in the region around 500–700 MeV it deviates from APT but is more similar to the results of lattice simulations, especially to the Orsay group [17, 19] ones.

Figure 2 exposes the second and third MPT functions as roughly estimated by the one-loop Eqs. (18) and (19). It is seen that instead of singular IR slopes of all APT thick dashed (red) curves, all the MPT functions are the IR regular ones. At the same time, just as in APT, the second and third MPT functions $\mathcal{A}_{2,3;\mathrm{MPT}}$ are noticeably smaller than square $(\mathcal{A}_{1,\mathrm{MPT}})^2$ and cube $(\mathcal{A}_{1,\mathrm{MPT}})^3$ of the first one. For example, at $Q = 0.5$ GeV and $Q = 1.0$ GeV the reduction factors for the $\mathcal{A}_{3;\mathrm{MPT}}$ are about 0.2 and between 0.1–0.2 correspondingly. Besides, the $\mathcal{A}_{3,\mathrm{MPT}}$ is negative below 1 GeV.
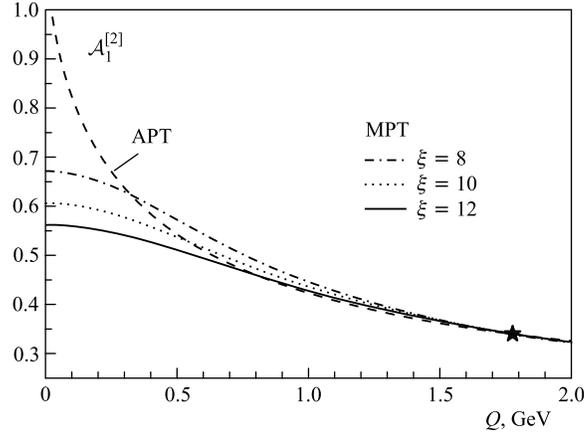
Fig. 1. The values of the first MPT function (for a few ξ values) in comparison with the APT one
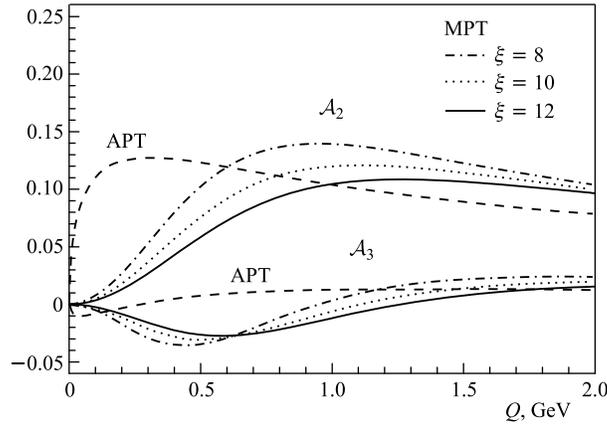

Fig. 2. (Color online) The values of the second and third MPT functions vs. the APT ones

Now, the «MPT-perturbative» expansion similar to Eq. (2), due to recurrent relation (17), can be represented in a form of Taylor series expanded over the parameter $\Delta \ln x = c_1 \pi / \beta_0 = 1.60$ with the final effect

$$\Delta^*_{\mathrm{MPT}} \simeq \frac{1}{\pi} \mathcal{A}_{1,\mathrm{MPT}}(x^*), \quad x^* = x\, e^{-\Delta \ln x} = \frac{Q^2}{(\Lambda^*)^2}, \quad \Lambda^* = 2.25\Lambda. \tag{23}$$

One should keep in mind that both the logarithm shift $\Delta \ln x$ and the «one-term approximation» (23) error $\delta\Delta_{\mathrm{MPT}} \sim \mathcal{A}_{3,\mathrm{MPT}}$ are scheme-dependent quantities. In the $\overline{\mathrm{MS}}$ scheme under consideration, $\delta\Delta_{\mathrm{MPT}}$ — due to the smallness of the reduction factor — is negligible.

However, one can get another angle on Eq. (23) and return to the old idea of the *effective coupling constant* [20], which is not so far from «RESIPE» [21] and from the «commensurate scale relations» [22] concepts. Then, the new scale can be treated as a specific one for the given process; $\Lambda^* = \Lambda_{\mathrm{Bj}}$.

The «glueball mass» values given in Table 1 for the LO and NLO cases also look attractive. They can be confronted with the glueball mass $M_{2g}^{Q\bar{Q}} \sim 1-2$ GeV of paper [11] and with gluon mass $M \sim 500$ MeV from the lattice estimate [17] as well as from solution of the Schwinger–Dyson equations (see [23] and references therein).

Table 1. **«Glueball mass» and** Λ_{Bj} **for a few values of** ξ

ξ	Λ_1	$m_{\mathrm{gl}}^{[1]}$	Λ_2	$m_{\mathrm{gl}}^{[2]}$	Λ_{Bj}
8	244	690	324	915	730
10	249	787	315	995	710
12	253	876	305	1160	686

Besides, as can be shown [24], the MPT perturbative sum Δ_{MPT}, together with a duly modified HT sum, allows one to fit the JLab data down to the very IR limit — see below Fig. 4 in Appendix B. There, the generic HT function was conjectured in the IR-regular form $\mu_4 \, (Q^2 + m_{\mathrm{ht}}^2)^{-1}$ with the only parameter. It is remarkable that its value $m_{\mathrm{ht}} \sim 0.7-1$ GeV is close to the m_{gl} one. This gives hope that ansatz (A) reflects some general physical essence.

Appendix A
2-LOOP MPT HIGHER FUNCTIONS

For a more accurate definition of the 2-loop higher expansion functions, one could use recurrent relation (16) at $k = 1$ and truncated Eq. (17) for the $k = 2$ case.

With the technical notation $\varphi(t = \ln x) = -(1/\beta_0) \, \dot{\mathcal{A}}_{1,\mathrm{MPT}}(x)$, $\mathcal{A}_{2,\mathrm{MPT}}^{[2]}(x) = y(t)$, one gets two relations of Eq. (16). Neglecting $\mathcal{A}_{4,\mathrm{MPT}}$, we come to a boundary value problem

$$y(t) - \theta \, \dot{y}(t) = \varphi(t), \quad y(\infty) = 0, \quad \theta = \frac{\beta_1}{2\beta_0^2} \tag{24}$$

and auxiliary relations

$$\mathcal{A}_{3,\mathrm{MPT}}^{[1]}(x, \xi) = -\frac{1}{2\,\beta_0} \, \dot{\varphi}(t), \quad \mathcal{A}_{4,\mathrm{MPT}}(x, \xi) = \frac{1}{6\beta_0^2} \, \ddot{\varphi}(t). \tag{25}$$

Solution of (24) $y(t) = \int\limits_0^\infty \mathrm{e}^{-s} \, \varphi(t + s\theta) \, ds$ being expanded in powers of θ yields the form

$$\mathcal{A}_{2,\mathrm{MPT}}^{[2]}(x, \xi) = -\frac{1}{\beta_0} \dot{\mathcal{A}}_{1,\mathrm{MPT}}^{[2]}(x, \xi) - \frac{\beta_1}{\beta_0} \mathcal{A}_{3,\mathrm{MPT}} + O(\mathcal{A}_{4,\mathrm{MPT}}), \tag{26}$$

completely correlating with Eq. (16).

On the other hand, one can use an approximate, «two-loop effective log trick» of papers [16]:

$$\ell = \ln x \to \mathcal{L}_2[\ell] := \ell + b \ln \sqrt{\ell^2 + 2\pi^2}, \quad b = \frac{\beta_1}{\beta_0}. \tag{27}$$

Combining this with Eqs. (8) and (10), one gets (with $L_\xi(x) = \ln(\xi + x)$)

$$\mathcal{A}_{1,\ell_2}^{[2]}(x,\xi) = \frac{1}{\beta_0 \mathcal{L}_2[L_\xi(x,\xi)]}, \quad \mathcal{A}_{2,\ell_2}^{[2]}(x,\xi) = \frac{\mathcal{R}(x)}{\beta_0^2 (\mathcal{L}_2[L_\xi(x)])^2}, \quad \dots$$

$$\mathcal{R}(x) = \mathcal{L}_2'[L_\xi(x)] R(x) = \left(1 + b\frac{L_\xi(x)}{L_\xi^2(x) + 2\pi^2}\right) R(x), \quad R(x) = \frac{x}{\xi + x}. \tag{28}$$

With due account for the numerical values $\beta_0(n_f = 3) = 0.716$, $b = 0.566$, one has

$$\beta_0 \, \alpha_s = \frac{1}{\ln \xi + b \ln \sqrt{(\ln \xi)^2 + 2\pi^2}}, \quad \Lambda_{2,\ell_2} = \frac{1.777}{\sqrt{23.33 - \xi}},$$

with[1] $\alpha_s|_{\xi=10\pm2} = 0.435 \mp 0.03$, $\Lambda_{2,\ell_2}(\xi = 10 \pm 2) = (490 \pm 35)$ MeV $\sim 1.95\Lambda_1$; for more detail, see Table 2.

Table 2

ξ	Λ_1	$m_{\mathrm{gl}}^{[1]}$	Λ_2	$m_{\mathrm{gl}}^{[2]}$	α_s	Λ_{2,ℓ_2}
8	244	690	324	915	0.438	455
10	249	787	315	995	0.435	490
12	253	876	305	1160	0.432	525

These expressions can be confronted with the previous ones (Eqs. (14) and (18)). For example, at $Q \sim 500$ MeV, $x \sim 1$ and $\mathcal{A}_{1,\ell_2}^{[2]}(1, \xi = 8) \simeq 0.45$, $\mathcal{A}_{1,\ell_2}^{[2]}(1, \xi = 12) \simeq 0.40$.

In the context of relation (23), the second term in the r.h.s. of the last expansion (26) reduces further the error $\delta\Delta^*$ of expression (23) for Δ_{MPT}^*.

Appendix B
THE ANSATZ (A) EFFECT ON THE BJORKEN SUM RULE ANALYSIS

The net effect of the Ansatz (A) used literally (but roughly) can be described as a transition to the new momentum-transfer scale in both perturbative (PT) and higher-twist (HT) items. Explicitly, in Eq. (1), this means

$$\Delta_{\mathrm{Bj}}^{\mathrm{PT}}(Q^2) \to \Delta^{\mathrm{MPT}} \sim \Delta_{\mathrm{Bj}}^{\mathrm{PT}}(Q^2 + m_{\mathrm{gl}}^2), \quad \Gamma_{\mathrm{HT}} = \frac{\mu_4}{Q^2} \to G_{\mathrm{HT}} = \frac{\mu_4}{Q^2 + m_{\mathrm{ht}}^2}. \tag{29}$$

Meanwhile, as was shown above, under a more detailed analysis (that includes differential recurrent relations) the correspondence is more intricate — see, e.g., Figs. 1 and 2.

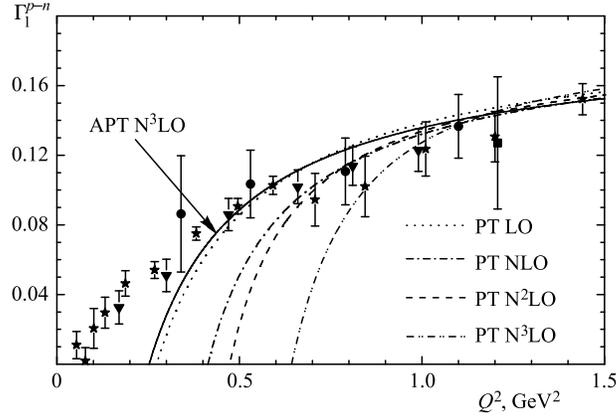[1]For the practical use of the last simple relation, see [25].
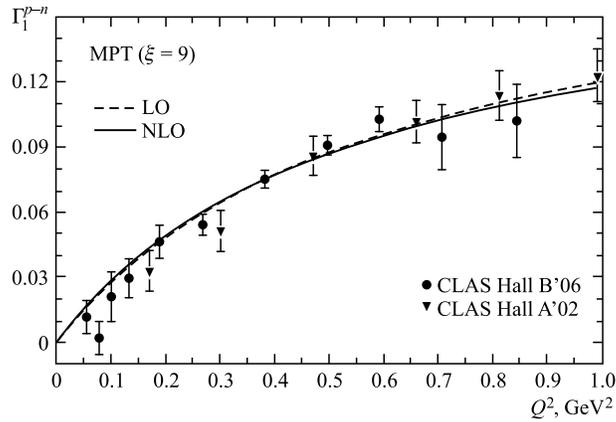
Fig. 3. Figure 5 from paper [4]

Fig. 4. The MPT fitting of the JLab data with change (29) used, according to [24]

Nevertheless, it is evident by observation that for $m_{\mathrm{gl}} \sim 500$ MeV the solid (green) curve from Fig. 3 (taken from paper [4]) visually corresponds to Fig. 4 curve (according to [24]) with «shifted» scale $Q_\xi^2 = Q^2 - m_{\mathrm{gl}}^2$.

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