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ROSEN–MORSE POTENTIALS FOR RELATIVISTIC SPINLESS PARTICLES; APPROXIMATE SOLUTIONS

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We deal with the D-dimensional radial Klein–Gordon equation for Rosen–Morse type potential with unequal scalar and vector potentials. We apply a proper approximation to the centrifugal term and, by proposing an ansatz solution to the resulting equation, we obtain the bound-state solution. To check the accuracy of our results, we compare our obtained quasi-analytical energies with the exact numerical ones.

Рассматривается D-мерное радиальное уравнение Клейна–Гордона с потенциалом типа Розена–Морзе с различными скалярными и векторными потенциалами. Уравнение дает решение в виде связанного состояния, если использовать подходящее приближение для центробежного члена и анзац — для получающегося при этом уравнения. Для проверки точности результатов полученные квазианалитические энергии сравниваются с точным численным решением уравнения.

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INTRODUCTION

The Rosen–Morse potential (RMP for short) [1] was traditionally used for interatomic interactions and can be considered as a special case of the five-parameter exponential-type potential model [2]. Unlike the one-dimensional nonrelativistic problem [3], it cannot be exactly solved in the higher dimensions due to the inverse square centrifugal term. Within the present research, we consider the radial Klein–Gordon equation for RMP with unequal scalar and vector potentials

$$V(r) = V_1 \sec h^2(\alpha r) - V_2 \tanh(\alpha r), \tag{1a}$$

$$S(r) = S_1 \sec h^2(\alpha r) - S_2 \tanh(\alpha r), \tag{1b}$$

where V_1, V_2, S_1, S_2 and α are constant coefficients. In the recent years, the Klein–Gordon equations under different potentials have been considered [4, 5]. There are very recent papers

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which discuss this interesting interaction in the relativistic regime [6, 7] via common analytical techniques of mathematical physics [8, 9].

The outline of our work is as follows. In Sec. 1, using an appropriate approximation to the centrifugal term and making a suitable ansatz, we calculate the ground-state solutions. We compare the approximate quasi-analytical results with the exact numerical ones as well. In Sec. 2 we give our concluding remarks.

1. THE QUASI-EXACT SOLUTION

The spherically symmetric stationary Klein–Gordon equation in the D-dimensional space has the form

$$-\Delta_D \psi_{n,l,m}(r,\Omega_N) = \left\{[E_{n,l} - V(r)]^2 - [m + S(r)]^2\right\} \psi_{n,l,m}(r,\Omega_N), \tag{2}$$

with

$$\Delta_D \equiv \nabla_D^2 = \frac{1}{r^{D-1}} \frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right) - \frac{\Lambda_D^2(\Omega_D)}{r^2} \tag{3}$$

and $\dfrac{\Lambda_D^2(\Omega_D)}{r^2}$ is a generalization of the centrifugal barrier for the D-dimensional space. One can write $\psi_{n,l,m}(r,\Omega_N)$ as

$$\psi_{n,l,m}(r,\Omega_N) = R_{n,l}(r) Y_l^m(\Omega_D), \tag{4}$$

where $Y_l^m(\Omega_D)$, $R_{n,l}(r)$, $E_{n,l}$ and l represent the hyperspherical harmonics, the hyperradial part, the energy eigenvalues and orbital angular momentum, respectively.

By choosing $R_{n,l}(r) = r^{-\frac{(D-1)}{2}} u_{n,l}(r)$, we find

$$\left[\frac{d^2}{dr^2} + E_{n,l}^2 + V^2(r) - 2E_{n,l}V(r) - m_0^2 - S^2(r) - \right.$$
$$\left. -2m_0\, S(r) - \frac{(D+2l-1)(D+2l-3)}{4r^2}\right] u_{n,l}(r) = 0. \tag{5}$$

To deal with the centrifugal term, we make use of the appropriate approximation [10–12]

$$f(r) = \frac{1}{r^2} \approx \frac{\alpha^2}{\sinh^2(\alpha r)}, \tag{6}$$

which is valid for sufficiently small values of α (see Fig. 1). When $\alpha \to 0$ the approximation approaches $1/r^2$. Substitution of Eqs. (1) and (6) into Eq. (5) leads to the following equation:

$$\left[\frac{d^2}{dr^2} + B\sec h^4(\alpha r) + C\tanh^2(\alpha r) + F\sec h^2(\alpha r)\tanh(\alpha r) + \frac{P}{\sinh^2(\alpha r)} + \right.$$
$$\left. + g\sec h^2(\alpha r) + h\tanh(\alpha r) + A\right] u_{n,l}(r) = 0, \tag{7}$$

where

$$
\begin{aligned}
A &= E_{n,l}^2 - m_0^2, \quad B = V_1^2 - S_1^2, \quad C = V_2^2 - S_2^2, \\
F &= -2V_1V_2^{+}2S_1S_2, \quad g = -(2E_{n,l}V_1 + 2m_0S_1), \\
P &= -\frac{\alpha^2}{4}(D+2l-1)(D+2l-3), \quad h = 2E_{n,l}V_2 + 2m_0S_2.
\end{aligned} \tag{8}
$$

After the transformation $z = \tanh(\alpha r)$ $(0 < z < 1)$, Eq. (7) takes the form

$$
\begin{aligned}
\Bigg[\frac{d^2}{dz^2} - \frac{2z}{1-z^2}\frac{d}{dz} + C'\frac{z^2}{(1-z^2)^2} + F'\frac{z}{1-z^2} + g'\frac{1}{1-z^2} + \\
+ h'\frac{z}{(1-z^2)^2} + A'\frac{1}{(1-z^2)^2} + P'\frac{1}{z^2(1-z^2)} + B'\Bigg]u_{n,l}(z) = 0,
\end{aligned} \tag{9}
$$

where

$$
A' = \frac{A}{\alpha^2}, \quad B' = \frac{B}{\alpha^2}, \quad C' = \frac{C}{\alpha^2}, \quad F' = \frac{F}{\alpha^2}, \quad P' = \frac{P}{\alpha^2}, \quad h' = \frac{h}{\alpha^2}, \quad g' = \frac{g}{\alpha^2}. \tag{10}
$$

Further, we also use the transformation

$$
u_{n,l}(z) = \frac{\phi_{n,l}(z)}{\sqrt{1-z^2}}, \tag{11}
$$

which brings Eq. (9) into the form

$$
\begin{aligned}
\Bigg[\frac{d^2}{dz^2} + (1+C')\frac{z^2}{(1-z^2)^2} + F'\frac{z}{1-z^2} + (g'+1)\frac{1}{1-z^2} + h'\frac{z}{(1-z^2)^2} + \\
+ \frac{A'}{(1-z^2)^2} + \frac{P'}{z^2(1-z^2)} + B'\Bigg]\phi_{n,l}(z) = 0.
\end{aligned} \tag{12}
$$

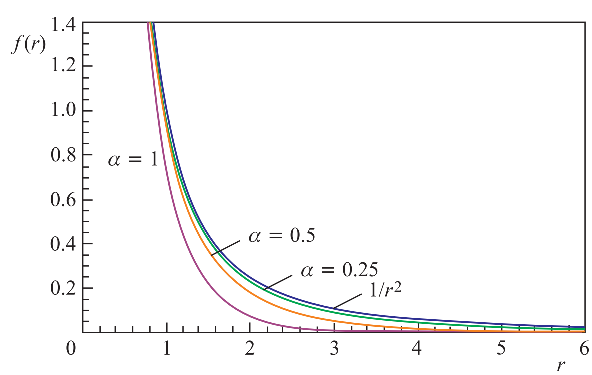

Fig. 1. The centrifugal term $1/r^2$ and its approximation (6) for $\alpha = 1$, $\alpha = 0.5$ and $\alpha = 0.25$

After decomposition of fractions, we have

$$\left[\frac{d^2}{dz^2}+\left(-\frac{C'}{4}+\frac{F'}{2}+\frac{g'}{2}+\frac{P'}{2}+\frac{A'}{4}+\frac{1}{4}\right)\frac{1}{1-z}+\right.$$
$$+\left(\frac{C'}{4}+\frac{A'}{4}+\frac{h'}{4}+\frac{1}{4}\right)\frac{1}{(1-z)^2}+\left(-\frac{C'}{4}-\frac{F'}{2}+\frac{g'}{2}+\frac{P'}{2}+\frac{A'}{4}+\frac{1}{4}\right)\frac{1}{1+z}+$$
$$\left.+\left(\frac{C'}{4}-\frac{h'}{4}+\frac{A'}{4}+\frac{1}{4}\right)\frac{1}{(1+z)^2}+\frac{P'}{z^2}+B'\right]\phi_{n,l}(z)=0. \quad (13)$$

The above equation cannot be solved by the exact analytical tools of quantum mechanics. Therefore, we follow the quasi-exact ansatz technique. Here, for the sake of simplicity, we try to find only the 0th node solution. We assume an ansatz of the form

$$\phi_{0,l}(z)=\exp\left[\delta z+\beta\ln(1-z)+\gamma\ln(1+z)+\xi\ln(z)\right]=\mathrm{e}^{\delta z}(1-z)^{\beta}z^{\xi}(1+z)^{\gamma}, \quad (14)$$

with $\beta,\xi>0$ so that $\phi(z)\to 0$ as $z\to 0$ and $z\to 1$, which ensures that the radial functions $\phi(\tanh(\alpha r))$ fulfill the boundary conditions consistent with the requirements of quantum mechanics, i.e., $\phi(\tanh(\alpha r))\to 0$ as $r\to 0$ and $r\to\infty$. Calculating the second-order derivative of (14), we easily get

$$\left[\frac{d^2}{dz^2}-(-2\beta\delta-\beta\gamma-2\beta\xi)\frac{1}{1-z}-(\beta^2-\beta)\frac{1}{(1-z)^2}-(2\delta\gamma-\beta\gamma-2\gamma\xi)\frac{1}{1+z}-\right.$$
$$\left.-(\gamma^2-\gamma)\frac{1}{(1+z)^2}-(2\delta\xi+2\gamma\xi-2\beta\xi)\frac{1}{z}-(\xi^2-\xi)\frac{1}{z^2}-\delta^2\right]\phi_{0,l}(z)=0. \quad (15)$$

Hence, by comparing Eqs. (13) and (15), we find the correspondence

$$-\frac{C'}{4}+\frac{F'}{2}+\frac{g'}{2}+\frac{P'}{2}+\frac{A'}{4}+\frac{1}{4}=-(-2\beta\delta-\beta\gamma-2\beta\xi), \quad (16a)$$
$$\frac{C'}{4}+\frac{A'}{4}+\frac{h'}{4}+\frac{1}{4}=-(\beta^2-\beta), \quad (16b)$$
$$-\frac{C'}{4}-\frac{F'}{2}+\frac{g'}{2}+\frac{P'}{2}+\frac{A'}{4}+\frac{1}{4}=-(2\delta\gamma-\beta\gamma-2\gamma\xi), \quad (16c)$$
$$\frac{C'}{4}-\frac{h'}{4}+\frac{A'}{4}+\frac{1}{4}=-(\gamma^2-\gamma), \quad (16d)$$
$$\delta+\gamma-\beta=0, \quad (16e)$$
$$P'=-(\xi^2-\xi), \quad (16f)$$
$$B'=-\delta^2. \quad (16g)$$

Having in mind Eq. (1), for fixed values of V_1,S_1,l,D,α,m_0, in particular, the system of seven equations (16) determines the sets of variables $E_{0,l},S_2,V_2,\delta,\gamma,\xi,\beta$. In this sense, Eq. (16) enables one to obtain exact analytical solutions to Eq. (7) at particular values of the control parameters of the system. In other words, Eq. (7) appears quasi-solvable.

1.1. The Solution for $V_1 \neq S_1$, $V_2 \neq S_2$. From Eqs. (16b), (16d), (16f) and (16g), we can find δ, β, ξ and γ as below

$$\beta = \frac{1}{2}(1 \pm \sqrt{-(A' + C' + h')}), \tag{17a}$$

$$\gamma = \frac{1}{2}(1 \pm \sqrt{-(A' + C' - h')}), \tag{17b}$$

$$\xi = \frac{1}{2}(1 \pm \sqrt{1 - 4P'}), \tag{17c}$$

$$\delta = \pm\sqrt{-B'}. \tag{17d}$$

Equation (16e) gives

$$\delta + \gamma = \beta \quad \text{or} \quad \gamma = \beta - \delta. \tag{18}$$

By using Eqs. (16a) and (16c), we have

$$-\frac{C'}{4} + \frac{g'}{2} + \frac{P'}{2} + \frac{A'}{4} + \frac{1}{4} - \beta\gamma = -(-2\beta\delta - 2\beta\xi) - \frac{F'}{2}, \tag{19a}$$

$$-\frac{C'}{4} + \frac{g'}{2} + \frac{P'}{2} + \frac{A'}{4} + \frac{1}{4} - \beta\gamma = -(2\delta\gamma - 2\gamma\xi) + \frac{F'}{2}. \tag{19b}$$

By comparing the above equations, we have

$$-(-2\beta\delta - 2\beta\xi) - \frac{F'}{2} = -(2\delta\gamma - 2\gamma\xi) + \frac{F'}{2}, \tag{20}$$

or

$$(\delta + \xi)\beta + (\delta - \xi)\gamma = \frac{F'}{2}. \tag{21}$$

By considering Eq. (18), we can find the energy of the system from Eq. (21) as below

$$E_{0,l} = V_2 \pm \sqrt{V_2^2 - \upsilon}, \tag{22}$$

where

$$\upsilon = C + \alpha^2\left(1 + \delta + \xi - \frac{F'}{2\delta}\right)^2 - m^2 - 2mS_2, \tag{23}$$

or equivalently

$$E_{0,l} = V_2 \pm \left\{(m + S_2)^2 - \alpha^2\left[\frac{3}{2} + \frac{1}{2}\sqrt{1 + (D + 2l - 1)(D + 2l - 3)} - \right.\right.$$
$$\left.\left. - \frac{1}{\alpha}\sqrt{S_1^2 - V_1^2} + \frac{1}{\alpha\sqrt{S_1^2 - V_1^2}}(-V_1V_2 + S_1S_2)\right]^2\right\}^{1/2}. \tag{24}$$

1.2. The Solution for $V_1 = S_1$, $V_2 = S_2$. In the case that $V_2 = S_2$ and $V_1 = S_1$, we obtain

$$\begin{aligned} A =& E_{n,l}^2 - m_0^2, \quad B = 0, \quad C = 0, \quad F = 0, \quad P = 0, \\ g =& -2(E_{n,l} + m_0)V_1, \quad h = 2(E_{n,l} + m_0)V_2. \end{aligned} \tag{25}$$

Therefore, Eq. (13) changes into

$$\left[\frac{d^2}{dz^2}+\left(\frac{g'}{2}+\frac{A'}{4}+\frac{1}{4}\right)\frac{1}{1-z}+\left(\frac{A'}{4}+\frac{h'}{4}+\frac{1}{4}\right)\frac{1}{(1-z)^2}+\right.$$
$$\left.+\left(\frac{g'}{2}+\frac{A'}{4}+\frac{1}{4}\right)\frac{1}{1+z}+\left(-\frac{h'}{4}+\frac{A'}{4}+\frac{1}{4}\right)\frac{1}{(1+z)^2}\right]\phi_{n,l}(z)=0. \quad (26)$$

To solve Eq. (26), we assume an ansatz of the form

$$\phi_{0,l}(z)=\mathrm{e}^{\beta\ln(1-z)+\gamma\ln(1+z)}=(1-z)^{\beta}(1+z)^{\gamma}. \quad (27)$$

By doing the same procedure as in the previous section, we find the equations

$$\beta\gamma=\frac{g'}{2}+\frac{A'}{4}+\frac{1}{4}, \quad (28a)$$

$$\beta^2-\beta=-\left(\frac{A'}{4}+\frac{h'}{4}+\frac{1}{4}\right), \quad (28b)$$

$$\gamma^2-\gamma=-\left(-\frac{h'}{4}+\frac{A'}{4}+\frac{1}{4}\right). \quad (28c)$$

From Eqs. (28b) and (28c), one can find

$$\beta=\frac{1}{2}(1\pm\sqrt{-(A'+h')}), \quad (29a)$$

$$\gamma=\frac{1}{2}(1\pm\sqrt{-(A'-h')}). \quad (29b)$$

By using Eq. (28a), the energy of the system for $|0,0\rangle$ state obtained is

$$\frac{1}{4}\left(1\pm\frac{1}{\alpha}\sqrt{-(E_{0,0}^2-m_0^2+2(E_{0,0}+m_0)V_2)}\right)\times$$
$$\times\left(1\pm\frac{1}{\alpha}\sqrt{-(E_{0,0}^2-m_0^2-2(E_{0,0}+m_0)V_2)}\right)=$$
$$=\frac{1}{\alpha^2}\left(-\frac{2(E_{0,0}+m_0)V_1}{2}+\frac{E_{0,0}^2-m_0^2}{4}\right)+\frac{1}{4}. \quad (30)$$

For further comparison, we have reported some numerical results for different values of V_2 and V_1 in Table 4 and compared with [6].

2. DISCUSSIONS

As an example, in Table 1, we present for $V_1=0.2$, $S_1=0.3$, $l=4$, $D=3$, $\alpha=0.01$, $m_0=1$ the resulting solutions of (16) corresponding to the radial functions $\phi(\tanh(\alpha r))$ fulfilling the boundary conditions $\phi(\tanh(\alpha r))\to 0$ as $r\to 0$ and $r\to\infty$. For the sake of illustration, we also show in Fig. 2 the behavior of $\phi(\tanh(\alpha r))$ for two different sets of the parameters taken from Table 1.

Table 1. **The solutions to Eq. (16) for $V_1 = 0.2$, $S_1 = 0.3$, $m_0 = 1$ and $D = 3$, $\alpha = 0.01$, $l = 4$. In all the cases presented $\xi = 5$**

$E_{0,4}$	S_2	V_2	δ	γ	β
0.562449	0.391736	–0.377247	–22.3607	51.8298	29.4692
0.562449	–0.391736	0.377247	22.3607	29.4692	51.8298
–0.928072	–0.858734	–0.844052	22.3607	6.17816	28.5388
–0.928072	0.858734	0.844052	–22.3607	28.5388	6.17816
–1.2105	–6.7486	–4.68826	22.3607	229.365	251.726
–1.2105	6.7486	4.68823	–22.3607	251.726	229.365
–1.28501	–8.22855	–5.32105	22.3607	300.343	322.704
–1.28501	8.22855	5.32105	–22.3607	322.704	300.343

In order to reveal the effect of α on the accuracy of our results, we compare the energies by Eq. (7), calculated exactly with the use of Eq. (16), with those of the Klein–Gordon Eq. (5) with (1). Some of our results are presented in Tables 2 and 3 for $\alpha = 0.075$ and $\alpha = 0.01$, respectively. In our numerical calculation, we have used a «Shooting Method» based on a discrete variable representation (DVR) [13] of the Hamiltonian. As one could have expected, the energies associated with Eq. (7) tend to those of the Klein–Gordon equation as α decreases. In particular, we find from Table 3 that for $\alpha = 0.01$, $D = 3$ the energies obtained from Eq. (7) are in good agreement with those of the Klein–Gordon equation, at least up to $l = 4$.

Our results can be useful in places where relativistic study of spinless particles is of interest.

Table 2. **Eigenvalues of Eqs. (7) and (5) for $V_1 = 0.2$, $S_1 = 0.3$, $\alpha = 0.075$, $D = 3$, $m_0 = 1$ and some values of V_2, S_2**

l	V_2	S_2	$E_{0,l}$ (Eq. (7))	$E_{0,l}$ (Eq. (5))
1	2.00779	3.43296	–1.42193	–1.42445
2	1.55062	2.65464	–1.44165	–1.44910
3	1.30709	2.22126	–1.45372	–1.46847
4	1.16359	1.95178	–1.46064	–1.48504

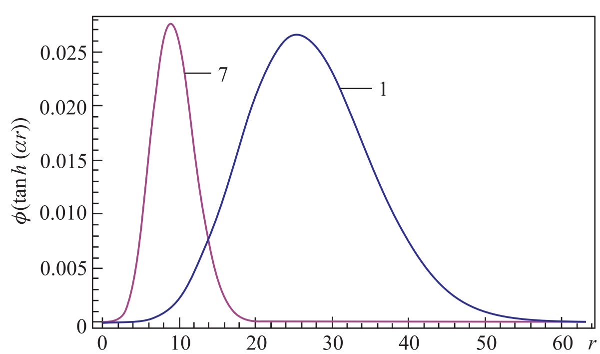

Fig. 2. Unnormalized radial wave functions corresponding to the control parameters of the first and seventh rows of Table 1

Table 3. **Eigenvalues of Eqs. (7) and (5) for $V_1 = 0.2$, $S_1 = 0.3$, $\alpha = 0.01$, $D = 3$, $m_0 = 1$ and some values of V_2, S_2**

l	V_2	S_2	$E_{0,l}$ (Eq. (7))	$E_{0,l}$ (Eq. (5))
1	11.4372	17.5484	–1.27461	–1.27467
2	8.23835	12.6834	–1.27878	–1.27894
3	6.45674	9.96592	–1.28218	–1.28218
4	5.32105	8.22855	–1.28501	–1.28557

Table 4. **The energy of the system for $\beta = \frac{1}{2}(1 - \sqrt{-(A' + h')})$, $\gamma = \frac{1}{2}(1 - \sqrt{-(A' - h')})$, $m = 1$, $\alpha = 0.01$**

Coefficients	$E_{0,0}$	$E_{0,0}$ [6]
$V_1 = -0.05$, $V_2 = 0.06$	0.8624804	0.8624804
$V_1 = -0.5$, $V_2 = -0.8$	–0.6462851	–0.6462851
$V_1 = -0.09$, $V_2 = -0.3$	–0.9999379	–0.9999379

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