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HIGHER-ORDER CORRECTIONS TO THE GRIMUS–STOCKINGER FORMULA

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For the Grimus–Stockinger formula one and the same dimensionless parameter of asymptotic expansion is found by several ways of calculations. This parameter strongly depends on the width of wave packet.

Различными способами вычислений найден один и тот же безразмерный параметр асимптотического разложения для формулы Гримуса–Стокинжера. Этот параметр существенно зависит от ширины волнового пакета.

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INTRODUCTION

For the modern theory of neutrino oscillations [1,2] the main tool is the Grimus–Stockinger theorem [3], which gives the leading asymptotic behaviour with $|\mathbf{R}| = R \to \infty$ for the integral

$$\mathcal{J}(\mathbf{R}) = \int \frac{d^3q}{(2\pi)^3} \frac{\mathrm{e}^{-i(\mathbf{q}\cdot\mathbf{R})}\Phi(\mathbf{q})}{(\mathbf{q}^2 - \kappa^2 - i0)} \approx \frac{\mathrm{e}^{i\kappa R}}{4\pi R}\Phi\left(-\kappa\mathbf{n}\right)\left[1 + O(R^{-1/2})\right], \tag{1}$$

where $\mathbf{R} = R\mathbf{n}$, $\mathbf{n}^2 = 1$, and the function $\Phi(\mathbf{q}) \in C^3$ decreases at least like $1/\mathbf{q}^2$ together with its first and second derivatives. In order to understand the physical conditions necessary for this expansion, the dimensionless parameters should be determined from the higher-order corrections to this formula. Here, this parameter is defined unambiguously by the use of various asymptotic expansions allowing one to calculate the further corrections.

1. CORRECTIONS FOR THE THREE-DIMENSIONAL CASE

To obtain the higher corrections of order R^{-n} we suppose that $\Phi(\mathbf{q})$ and its first and second derivatives are represented by Fourier-transform as

$$\Phi(\mathbf{q}) = \int d^3\mathrm{x}\, \mathrm{e}^{i(\mathbf{q}\cdot\mathbf{x})}\,\varphi(\mathbf{x}), \quad \boldsymbol{\nabla}_q\Phi(\mathbf{q}) = i\int d^3\mathrm{x}\, \mathrm{e}^{i(\mathbf{q}\cdot\mathbf{x})}\,\mathbf{x}\varphi(\mathbf{x}), \tag{2}$$

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and so on. Since $1/\mathbf{q}^2$ is also Fourier-image of $1/|\mathbf{x}|$, Eqs. (2) are valid at least in the sense of distributions also for the functions defined in [3]. By using the first equality of the following well-known representations for spherical wave as a free Schrödinger three-dimensional Green function with $\kappa = 2\lambda$, $\mathbf{q} = 2\mathbf{p}$:

$$\frac{\mathrm{e}^{i\kappa|\mathbf{x}|}}{4\pi|\mathbf{x}|} = \int \frac{d^3q}{(2\pi)^3} \frac{\mathrm{e}^{\mp i(\mathbf{q}\cdot\mathbf{x})}}{(\mathbf{q}^2 - \kappa^2 - i0)} = i \int \frac{d^3p}{4\pi^3}\, \mathrm{e}^{\mp 2i(\mathbf{p}\cdot\mathbf{x})} \int_0^\infty dt\, \mathrm{e}^{it(\lambda^2 + i0 - \mathbf{p}^2)}, \tag{3}$$

and by interchanging the order of integration for integral (1) one finds

$$\mathcal{J}(\mathbf{R}) = \int d^3\mathrm{x}\, \frac{\mathrm{e}^{i\kappa|\mathbf{R}-\mathbf{x}|}}{4\pi|\mathbf{R}-\mathbf{x}|}\, \varphi(\mathbf{x}). \tag{4}$$

Substituting here the expansion, which in the exponential should always contain one additional order with respect to the ones in denominator

$$|\mathbf{R} - \mathbf{x}| = R\left[1 - 2\frac{(\mathbf{n}\cdot\mathbf{x})}{R} + \frac{\mathbf{x}^2}{R^2}\right]^{1/2} = R - (\mathbf{n}\cdot\mathbf{x}) + \frac{\mathbf{x}^2 - (\mathbf{n}\cdot\mathbf{x})^2}{2R} + \ldots,$$

we come to the corresponding expansion of integral (4) up to $O(R^{-2})$:

$$\mathcal{J}(\mathbf{R}) = \frac{\mathrm{e}^{i\kappa R}}{4\pi R} \int d^3\mathrm{x}\, \mathrm{e}^{-i\kappa(\mathbf{n}\cdot\mathbf{x})}\varphi(\mathbf{x}) \left[1 + \frac{(\mathbf{n}\cdot\mathbf{x})}{R} + \frac{i\kappa}{2R}\left(\mathbf{x}^2 - (\mathbf{n}\cdot\mathbf{x})^2\right) + \ldots\right],$$

that by making use of (2) transcribes as

$$\mathcal{J}(\mathbf{R}) = \frac{\mathrm{e}^{i\kappa R}}{4\pi R}\left[1 - \frac{i}{R}(\mathbf{n}\cdot\boldsymbol{\nabla}_q) + \frac{i\kappa}{2R}\left((\mathbf{n}\cdot\boldsymbol{\nabla}_q)^2 - \boldsymbol{\nabla}_q^2\right) + \ldots\right]\Phi(\mathbf{q})\Big|_{\mathbf{q}=-\kappa\mathbf{n}}, \tag{5}$$

with

$$(\mathbf{n}\cdot\boldsymbol{\nabla}_q)\Phi(\mathbf{q})\big|_{\mathbf{q}=-\kappa\mathbf{n}} = -\partial_\kappa \Phi(-\kappa\mathbf{n}), \tag{6}$$

and so on. For any positive definite quadratic form of momentum $\mathbf{q}$: $\zeta = (\mathbf{q}\mathbf{A}^{-1}\mathbf{q}) > 0$, with $\Phi(\mathbf{q}) = \mathcal{H}(\zeta)$, $\overline{\alpha}(\mathbf{n}) = \left(\mathbf{n}\mathbf{A}^{-1}\mathbf{n}\right)$, $\overline{\alpha^2}(\mathbf{n}) = \left(\mathbf{n}\mathbf{A}^{-2}\mathbf{n}\right)$, that is

$$\left[1 - \frac{i}{R}(\mathbf{n}\cdot\boldsymbol{\nabla}_q) + \frac{i\kappa}{2R}\left((\mathbf{n}\cdot\boldsymbol{\nabla}_q)^2 - \boldsymbol{\nabla}_q^2\right)\right]\Phi(\mathbf{q})\Big|_{\mathbf{q}=-\kappa\mathbf{n}} \longmapsto$$
$$\longmapsto \left[1 + \frac{i\kappa}{R}\left[3\overline{\alpha}(\mathbf{n}) - \mathrm{Tr}\left\{\mathbf{A}^{-1}\right\}\right]\partial_\zeta - \frac{i2\kappa^3}{R}\left(\overline{\alpha^2}(\mathbf{n}) - \overline{\alpha}^2(\mathbf{n})\right)\partial^2_{\zeta^2}\right]\mathcal{H}(\zeta)\Big|_{\zeta=\kappa^2\overline{\alpha}(\mathbf{n})}. \tag{7}$$

Then, for Gaussian wave packet: $\mathcal{H}(\zeta) = \mathrm{e}^{-\zeta/4}$, expression (5) reads

$$\mathcal{J}(\mathbf{R}) = \frac{\mathrm{e}^{i\kappa R - \kappa^2\overline{\alpha}(\mathbf{n})/4}}{4\pi R}\left[1 - \frac{i\kappa}{4R}\left[3\overline{\alpha}(\mathbf{n}) - \mathrm{Tr}\left\{\mathbf{A}^{-1}\right\}\right] - \frac{i\kappa^3}{8R}\left(\overline{\alpha^2}(\mathbf{n}) - \overline{\alpha}^2(\mathbf{n})\right)\right]. \tag{8}$$

Here, the square bracket evidently represents corrections only to the phase of the exponential. It may be directly obtained by the saddle-point method.

To this end let us transcribe integral (1) for the above Gaussian wave packet by using the second representation of Eq. (3). Gaussian integration gives

$$\mathcal{J}(\mathbf{R}) = \frac{i}{4}\int\limits_0^\infty dt \left[\frac{|\mathbf{K}_t|}{\pi^3}\right]^{1/2} \mathrm{e}^{iF(t)}, \quad \mathbf{K}_t = \left(\mathbf{A}^{-1} + it\mathbf{I}\right)^{-1}, \quad \mathbf{K}_0 = \mathbf{A}, \tag{9}$$

where

$$iF(t) = it(\lambda^2 + i0) - (\mathbf{R}\mathbf{K}_t\mathbf{R}), \quad iF''(t) = 2\left(\mathbf{R}\left\{\mathbf{K}_t\right\}^3\mathbf{R}\right), \tag{10}$$

$$iF'(t) = i\left[\lambda^2 + i0 + \left(\mathbf{R}\left\{\mathbf{K}_t\right\}^2\mathbf{R}\right)\right] \mapsto 0, \quad t_0 = R/\lambda + i\overline{\alpha}(\mathbf{n}) + \epsilon, \tag{11}$$

t_0 is the saddle point as asymptotical solution of Eq. (11) for $R \to \infty$ up to $\epsilon = O(\lambda/R)$. It is obtained by diagonalization $\mathbf{A} = \mathbf{O}^\top\,\overline{\mathbf{A}}\mathbf{O}$ onto the eigenvalues $\overline{\mathbf{A}} = \mathrm{diag}\,\{\mathrm{a}_j\}$, with $0 < \mathrm{a}_j = 1/\alpha_j < \infty$ and determinant $|\mathbf{A}| \equiv \det\{\overline{\mathbf{A}}\} = \mathrm{a}_1\mathrm{a}_2\mathrm{a}_3$, by using a suitable orthogonal rotation $\boldsymbol{\varrho} = \mathbf{OR}$, $\boldsymbol{\varrho}^2 = \mathbf{R}^2$, and due to Eq. (11) defines $F(t_0)$ and $|\mathbf{K}_{t_0}|$ up to $O(\epsilon^2)$. Along the path deformed according to $\overline{\alpha}(\mathbf{n}) > 0$ we obtain

$$\mathcal{J}(\mathbf{R}) \approx \frac{i}{4}\,\mathrm{e}^{i\pi/4}\left[\frac{|\mathbf{K}_{t_0}|}{\pi^3}\right]^{1/2}\left[\frac{2\pi}{|iF''(t_0)|}\right]^{1/2}\mathrm{e}^{iF(t_0)} = \frac{\mathrm{e}^{i\Theta(\mathbf{R})}}{4\pi R}\,\mathrm{e}^{-\lambda^2\overline{\alpha}(\mathbf{n})}, \tag{12}$$

$$\Theta(\mathbf{R}) = 2\lambda R - \frac{\lambda}{2R}\left[3\overline{\alpha}(\mathbf{n}) - \mathrm{Tr}\left\{\mathbf{A}^{-1}\right\}\right] - \frac{\lambda^3}{R}\left[\overline{\alpha^2}(\mathbf{n}) - \overline{\alpha}^2(\mathbf{n})\right], \tag{13}$$

that for

$$\kappa = 2\lambda, \quad \mathrm{Tr}\left\{\mathbf{A}^{-1}\right\} = \sum_{j=1}^{3}\alpha_j, \quad \sum_{j=1}^{3}\varrho_j^2\left[\overline{\alpha^n}(\mathbf{n}) - (\alpha_j)^n\right] = 0, \tag{14}$$

exactly coincides with Eq. (8) with the same precision. The corrections in (8), (13) evidently disappear for degenerate case: $\alpha_j = \alpha_1$, for $j = 2, 3$.

For the neutrino oscillations problem: $\kappa = \sqrt{E_\kappa^2 - m^2} \approx E_\kappa$, and for the Gaussian wave packet with coordinate width σ_x: $\mathbf{A} \sim \sigma_x^{-2}$, so $\mathrm{Tr}\left\{\mathbf{A}^{-1}\right\} \sim \overline{\alpha}(\mathbf{n}) \sim \sigma_x^2$, whence, the true expansion parameters appear as combinations of two different dimensionless ones: $\kappa\sigma_x$ and σ_x/R, that define the application conditions of the Grimus–Stockinger formula as

$$(\kappa\sigma_x)\frac{\sigma_x}{R} \approx (E_\kappa\sigma_x)\frac{\sigma_x}{R} \ll 1, \ \text{ and } \ (\kappa\sigma_x)^3\,\frac{\sigma_x}{R} \approx (E_\kappa\sigma_x)^3\frac{\sigma_x}{R} \ll 1. \tag{15}$$

2. THE FOUR-DIMENSIONAL CASE

In fact, the above integral (1) is only the three-dimensional part of the four-dimensional one defining macroscopic Feynman diagram [2] of the problem:

$$J(R) = \int\frac{d^4q}{(2\pi)^2}\frac{\mathrm{e}^{-i(qR)}\,\Phi(q)}{(q^2 - m^2 + i0)} = \int d^4r\,\frac{m^2}{i}h\left(i0 - m^2(R + r)^2\right)\phi(r), \tag{16}$$

with

$$R^\mu = (T, \mathbf{R}) \to \infty, \ \ \sqrt{R^2} = \sqrt{R^\mu R_\mu} = \sqrt{T^2 - \mathbf{R}^2} \simeq T\frac{m}{E_\kappa} \leqslant T, \tag{17}$$

where

$$\frac{m^2}{i} h\left(m^2 a^2\right) = \int \frac{d^4 q}{(2\pi)^2} \frac{e^{-i(qx)}}{(q^2 - m^2 + i0)} \approx \frac{m^2}{i}\sqrt{\frac{\pi}{2}}\frac{e^{-ma}}{(ma)^{3/2}}, \tag{18}$$

for $a^2 = i0 - x^2 = e^{i\pi}x^2$, is the causal propagator in coordinate space, and now the four-dimensional Fourier representation is assumed for $\Phi(q)$, which for relativistic Gaussian wave packet [2] reads as

$$\Phi(q) = \int d^4 r e^{-i(qr)}\phi(r), \quad \Phi(q) \mapsto e^{-(q\mathbf{A}^{-1}q)/4}, \quad \phi(r) \mapsto \frac{|\mathbf{A}|^{1/2}}{\pi^2} e^{-(r\mathbf{A}r)}. \tag{19}$$

Here, again $\mathbf{A}^{-1} \sim \sigma_x^2$ in terms of Gaussian coordinate width for any positively defined quadratic form of momentum q in Minkowski space: $\zeta = (q\mathbf{A}^{-1}q) > 0$. Then, for $\sigma_x^2 \to 0$ one has $\mathbf{A} \to \infty$, $\Phi(q) \mapsto 1$, $\phi(r) \mapsto \delta_4(r)$, whence $iJ(R) \mapsto m^2 h\left(i0 - m^2R^2\right)$, that is reasonable from physical viewpoint.

Repeating now all the previous steps (4)–(8) for the second expression (16) of $J(R)$, with the so approximated propagator (18), for arbitrary $\Phi(q)$ (19) and $\eta^\mu = R^\mu/\sqrt{R^2}$, $l = i|l|$, $|l| = \sqrt{R^2}$, with

$$ma = ml\left[1 + \frac{2i(\eta r)}{l} - \frac{r^2}{l^2}\right]^{1/2} \approx m\left[l + i(\eta r) + \frac{(\eta r)^2 - r^2}{2l} + \ldots\right], \tag{20}$$

one obtains

$$J(R) \approx \frac{m^2}{i}\sqrt{\frac{\pi}{2}}\frac{e^{-ml}}{(ml)^{3/2}}\left\{1 + \frac{3(\eta\partial_q)}{2l} + \frac{m}{2l}\left[(\eta\partial_q)^2 - \partial_q^2\right]\right\}\Phi(q)\bigg|_{q=m\eta}, \tag{21}$$

that for the relativistic Gaussian wave packet from (19), with $\overline{\alpha}(\eta) = \left(\eta\mathbf{A}^{-1}\eta\right)$, $\overline{\alpha^2}(\eta) = \left(\eta\mathbf{A}^{-2}\eta\right)$, analogously gives

$$J(R) \approx \frac{m^2}{i}\sqrt{\frac{\pi}{2}}\frac{e^{-ml}\, e^{-m^2\overline{\alpha}(\eta)/4}}{(ml)^{3/2}} \times \\ \times \left\{1 - \frac{m}{4l}\left[4\overline{\alpha}(\eta) - \mathrm{Tr}\left\{\mathbf{A}^{-1}\right\}\right] - \frac{m^3}{8l}\left[\overline{\alpha^2}(\eta) - (\overline{\alpha}(\eta))^2\right]\right\}. \tag{22}$$

Such, at first sight, rough calculations are exactly confirmed again by saddle-point method. Indeed, by exponentiating like (3) the denominator of the first expression (16) with $m = 2\mathrm{m}$ and representation-dependent $\boldsymbol{g} \mapsto g_{\mu\nu}$ or δ^μ_ν, by means of Gaussian integration clarified in Appendix, one has instead of (9)–(11):

$$J(R) = \frac{1}{i}\int_0^\infty dt |\mathbf{K}(t)|^{1/2} \exp\left\{-i\mathcal{F}(t)\right\}, \quad \mathbf{K}(t) = \left[\mathbf{A}^{-1} - it\boldsymbol{g}\right]^{-1}, \tag{23}$$

$$-i\mathcal{F}(t) = -it\left(\mathrm{m}^2 - i0\right) - (R\mathbf{K}(t)R), \quad -i\mathcal{F}''(t) = 2\left(R\mathbf{K}^3(t)R\right), \tag{24}$$

$$-i\mathcal{F}'(t) = -i\left[\mathrm{m}^2 - i0 + \left(R\mathbf{K}^2(t)R\right)\right] \mapsto 0, \quad t_0 = |l|/\mathrm{m} - i\overline{\alpha}(\eta) + \epsilon, \tag{25}$$

t_0 is again the saddle point as asymptotical solution of Eq. (25) for $|l| \to \infty$ up to $\epsilon = O(\mathrm{m}/|l|)$. It is obtained now by diagonalization in Minkowski space as $\mathbf{A} = \boldsymbol{\xi}^{(j)}(\overline{\mathbf{A}})_{jn}\boldsymbol{\xi}^{(n)}$

onto the eigenvalues $(\overline{\mathbf{A}})_{jn} = g_{jn}a_{(j)}$ ordered [4] as $0 < g_{jj}a_{(j)} = g_{jj}/\alpha_j < \infty$, with determinant $|\mathbf{A}| = |A_{\mu\nu}| = |(\overline{\mathbf{A}})_{jn}|$, by using a suitable Lorentz transformation as $\varrho^j = \xi^{(j)}_\mu R^\mu$ with $\varrho_j = g_{jj}\varrho^j = g_{jk}\varrho^k$, $\varrho^2 = R^2$, and due to Eq. (25) defines again $\mathcal{F}(t_0)$ and $|\mathbf{K}(t_0)|$ up to $O(\epsilon^2)$. Along the path deformed according to $\overline{\alpha}(\eta) > 0$, instead of (12) one finds

$$J(R) \approx \frac{1}{i}|\mathbf{K}(t_0)|^{1/2}\left[\frac{\mathrm{e}^{-i\pi/2}\,2\pi}{|-i\mathcal{F}''(t_0)|}\right]^{1/2}\mathrm{e}^{-i\mathcal{F}(t_0)} = \frac{\sqrt{\mathrm{m}\pi}}{i}\,\frac{\mathrm{e}^{-iB(R)}\mathrm{e}^{-\mathrm{m}^2\overline{\alpha}(\eta)}}{l^{3/2}}, \tag{26}$$

$$-iB(R) = -2\mathrm{m}l - \frac{\mathrm{m}}{2l}\left[4\overline{\alpha}(\eta) - \mathrm{Tr}\,\{\mathbf{A}^{-1}\}\right] - \frac{\mathrm{m}^3}{l}\left[\overline{\alpha^2}(\eta) - (\overline{\alpha}(\eta))^2\right], \tag{27}$$

that for

$$\mathrm{Tr}\,\{\mathbf{A}^{-1}\} = (\mathbf{A}^{-1})^\lambda{}_\lambda = \sum_{j=0}^{3}\alpha_j, \quad |A_{\mu\nu}| = \prod_{j=0}^{3} g_{jj}a_{(j)} > 0, \tag{28}$$

$$\overline{\alpha^n}(\eta) = \sum_{j=0}^{3}(\alpha_j)^n g_{jj}\frac{\varrho_j^2}{\varrho^2} = \left(\eta\mathbf{A}^{-n}\eta\right), \quad g_{\mu\nu}\,, g_{jk} = \mathrm{diag}\,\{1,-1,-1,-1\}, \tag{29}$$

exactly coincides with Eq. (22) with the same precision.

The true parameters of expansion appear again as the following products of the two dimensionless parameters, that are now $m\sigma_x$ and $\sigma_x/|l|$:

$$(m\sigma_x)\frac{\sigma_x}{|l|} \ll 1, \quad \text{and} \quad (m\sigma_x)^3\,\frac{\sigma_x}{|l|} \ll 1, \tag{30}$$

and they have the same order for $(m\sigma_x) \leqslant 1$.

It is easy to see that both conditions for the three- and four-dimensional cases are practically the same. Indeed, Eq. (17) implies that

$$\mathbf{v} = \frac{\boldsymbol{\kappa}}{E_\kappa} = \frac{\mathbf{R}}{T}, \quad \text{whence} \quad R^2 = |l|^2 = \frac{\mathbf{R}^2}{\kappa^2}m^2 = \frac{T^2}{E_\kappa^2}m^2. \tag{31}$$

Since for ultrarelativistic neutrino $T \approx |\mathbf{R}|$ and $E_\kappa \approx \kappa = |\boldsymbol{\kappa}|$, conditions (30) may be rewritten as

$$(\kappa\sigma_x)\frac{\sigma_x}{|\mathbf{R}|} = (E_\kappa\sigma_x)\frac{\sigma_x}{T} \approx (E_\kappa\sigma_x)\frac{\sigma_x}{|\mathbf{R}|} \ll 1, \tag{32}$$

and

$$(m\sigma_x)^2(\kappa\sigma_x)\frac{\sigma_x}{|\mathbf{R}|} \approx (m\sigma_x)^2(E_\kappa\sigma_x)\frac{\sigma_x}{|\mathbf{R}|} \ll 1. \tag{33}$$

Thus, for $(m\sigma_x) \leqslant 1$ these both conditions are the same as the first one in the three-dimensional case (15). Moreover, the same dimensionless parameter (32) defines in fact the asymptotical solutions of both saddle-point equations (11) and (25). Note, that exact values of the first and second square brackets in (13) and/or (27), respectively, may be different, and their determination in terms of σ_x for the four-dimensional case [2] (27)–(29) is different from that for the three-dimensional case (13), (14).

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APPENDIX

In order to strictly calculate a standard Gaussian integral over the Minowski space [2]:

$$\int d^4 y\, \mathrm{e}^{-(y\mathbf{A}y)+2(By)}, \tag{34}$$

where the quadratic form $(y\mathbf{A}y) = y_\mu A^{\mu\nu} y_\nu$ is symmetric and positive definite, the following solution of the eigenvalue problem may be used:

$$\mathbf{A}\boldsymbol{\xi}^{(n)} = a_{(n)}\boldsymbol{\xi}^{(n)},\ \ (\boldsymbol{\xi}^{(n)})^2 = g^{nn},\ \ A_{\mu\nu} = \xi^{(l)}_\mu (\overline{\mathbf{A}})_{ln} \xi^{(n)}_\nu,\ \ (\overline{\mathbf{A}})_{ln} = g_{ln} a_{(n)}. \tag{35}$$

In spite of ambiguity ([5, § 94]) of diagonalization procedure for symmetric tensor in Minkowski space, the positive definiteness of $\mathbf{A}$ leaves the used type of its diagonalization only, leading to eigenvectors $\boldsymbol{\xi}^{(n)}$, $n = 0–3$ (35), whose components $\xi^{(n)}_\nu$ define Lorentz transformation diagonalizing the form. Then, with the substitutions $Y^n = \xi^{(n)}_\nu y^\nu$ transforming $(y\mathbf{A}y) = (Y\overline{\mathbf{A}}Y)$, $b^m = B^\mu \xi^{(m)}_\mu$, $(By) = (bY) = b^m g_{mn} Y^n$, integration (34) factorizes to

$$\int d^4Y \mathrm{e}^{-(Y\overline{\mathbf{A}}Y)+2(bY)} \equiv \prod_{n=0}^{3} \left\{ \int_{-\infty}^{\infty} dY^n \mathrm{e}^{-(Y^n)^2 g_{nn} a_{(n)} + 2b^n g_{nn} Y^n} \right\} =$$

$$= \sqrt{\frac{\pi^4}{|(\overline{\mathbf{A}})_{ln}|}}\, \exp\left(b\left(\overline{\mathbf{A}}\right)^{-1} b\right) = \sqrt{\frac{\pi^4}{|A_{\mu\nu}|}}\, \exp\left(B\mathbf{A}^{-1}B\right), \tag{36}$$

where $A_{\mu\nu}\left(\mathbf{A}^{-1}\right)^{\nu\lambda} = \delta_\mu{}^\lambda$, and $|A_{\mu\nu}| = |(\overline{\mathbf{A}})_{ln}|$ is defined by (28).

Nevertheless, it is instructive to obtain the same result without reference to diagonalization by using the direct integration over space and time variables separately. Since for the n-dimensional Minkowski space with signature metric $g_{\mu\nu} = \mathrm{diag}\{1, -1, -1, \ldots, -1\}$ in any given orthogonal basis the symmetric tensor $\mathbf{A}$ is represented by the block matrix $A_{\mu\nu} = A_{ij}$ for $i, j, \mu, \nu = 0 \div n-1$, with the rightmost bottom block $\mathcal{A}_{ij} = \mathcal{A}^{ij} = \mathcal{A}_{ji}$, for $i, j = 1 \div n-1$:

$$A_{\mu\nu} = \begin{pmatrix} A_{00} & A_{0j} \\ A_{i0} & \mathcal{A}_{ij} \end{pmatrix}, \tag{37}$$

integral (34) with $d^4y \mapsto d^ny$ may be rewritten as

$$\int_{-\infty}^{\infty} dy^0 \mathrm{e}^{-y^0 A_{00} y^0 + 2(B_0 y^0)} \int d^{n-1}\mathrm{y} \exp\left[-\mathrm{y}^l \mathcal{A}_{lk} \mathrm{y}^k - 2\left(y^0 A_{0k} - B_k\right)\mathrm{y}^k\right].$$

The both integrals are over Euclidian space now, so they are evaluated to

$$\sqrt{\frac{\pi^n}{\alpha|\mathcal{A}_{ij}|}}\, \exp\left\{\left[B_l \left(\boldsymbol{\mathcal{A}}^{-1}\right)^{lk} B_k\right] + \frac{1}{\alpha}\left[B_0 - B_l\left(\boldsymbol{\mathcal{A}}^{-1}\right)^{lk} A_{k0}\right]^2\right\}, \tag{38}$$

if $\alpha \equiv A_{00} - A_{0l}\left(\boldsymbol{\mathcal{A}}^{-1}\right)^{lk} A_{k0} > 0$. The expression is simplified by Laplace expansion of the determinant $|A_{\mu\nu}|$, where $M\left(\begin{smallmatrix} i_1 & i_2 & \cdots \\ j_1 & j_2 & \cdots \end{smallmatrix}\right)$ means the minor of the matrix $A_{\mu\nu}$, whose rows

$i_1, i_2, \ldots$ and columns $j_1, j_2, \ldots$ are deleted:

$$|A_{\mu\nu}| \equiv \sum_{k=0}^{n-1} (-1)^k A_{0k} M \binom{0}{k} = A_{00} M \binom{0}{0} - \sum_{k=1}^{n-1} (-1)^{k+1} A_{0k} M \binom{0}{k} =$$
$$= A_{00} M \binom{0}{0} - \sum_{k=1}^{n-1} (-1)^{k+1} A_{0k} \sum_{i=1}^{n-1} A_{i0} (-1)^{i+1} M \binom{0\ i}{0\ k} =$$
$$= |\mathcal{A}_{lj}| \left[A_{00} - A_{0k} \left(\boldsymbol{\mathcal{A}}^{-1}\right)^{ki} A_{i0} \right] = \alpha |\mathcal{A}_{lj}|. \quad (39)$$

Thus, $\alpha > 0$ due to positivity condition of form (37) implying that $|A_{\mu\nu}|, |\mathcal{A}_{lj}| > 0$. Furthermore, if for any symmetric block matrix $\mathbf{A}$:

$$\mathbf{A}\mathbf{A}^{-1} \equiv \mathbf{A}\mathbf{B} = \begin{pmatrix} \mathbf{P}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{12}^{\top} & \boldsymbol{\mathcal{A}}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{H}_{11} & \mathbf{b}_{12} \\ \mathbf{b}_{12}^{\top} & \boldsymbol{\mathcal{B}}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{11} & \mathbf{0}_{12} \\ \mathbf{0}_{12}^{\top} & \boldsymbol{\mathcal{I}}_{22} \end{pmatrix} \equiv \mathbf{I},$$

then

$$\boldsymbol{\mathcal{B}}_{22} = \left(\boldsymbol{\mathcal{A}}_{22} - \mathbf{a}_{12}^{\top}\mathbf{P}_{11}^{-1}\mathbf{a}_{12}\right)^{-1} = \boldsymbol{\mathcal{A}}_{22}^{-1} + \boldsymbol{\mathcal{A}}_{22}^{-1}\mathbf{a}_{12}^{\top}\left(\mathbf{P}_{11} - \mathbf{a}_{12}\boldsymbol{\mathcal{A}}_{22}^{-1}\mathbf{a}_{12}^{\top}\right)^{-1}\mathbf{a}_{12}\boldsymbol{\mathcal{A}}_{22}^{-1},$$

whence the rightmost bottom block of the inverse to (37) is expressed for $i, k, l, j = 1 \div n-1$ as

$$\left(\mathbf{A}^{-1}\right)^{ik} = \left(\boldsymbol{\mathcal{B}}_{22}\right)^{ik} = \left(\boldsymbol{\mathcal{A}}^{-1}\right)^{ik} + \frac{1}{\alpha}\left(\boldsymbol{\mathcal{A}}^{-1}\right)^{il} A_{l0} A_{0j} \left(\boldsymbol{\mathcal{A}}^{-1}\right)^{jk}, \quad (40)$$

and since

$$\left(\boldsymbol{\mathcal{A}}^{-1}\right)^{ik} = \frac{(-1)^{i+k} M \binom{0\ i}{0\ k}}{|\mathcal{A}_{lj}|}, \quad \left(\mathbf{A}^{-1}\right)^{l0} = \frac{(-1)^l}{|A_{\mu\nu}|} M \binom{0}{l},$$

the argument of the exponential in (38) is also reduced to expression (36):

$$\frac{1}{\alpha}B_0^2 - \frac{2}{\alpha}B_0 B_l \left(\boldsymbol{\mathcal{A}}^{-1}\right)^{li} A_i + B_l \left[\left(\boldsymbol{\mathcal{A}}^{-1}\right)^{lk} + \frac{1}{\alpha}\left(\boldsymbol{\mathcal{A}}^{-1}\right)^{li} A_{i0} A_{0j} \left(\boldsymbol{\mathcal{A}}^{-1}\right)^{jk}\right] B_k =$$
$$= B_\mu \left(\mathbf{A}^{-1}\right)^{\mu\nu} B_\nu = \left(B\mathbf{A}^{-1}B\right).$$

A generalization of integral (34) with arbitrary polynomial or smooth function similar to the well-known approximations for Euclidian case [6] here is also straightforward.

REFERENCES

1. *Beuthe M.* Towards a Unique Formula for Neutrino Oscillation in Vacuum // Phys. Rev. D. 2002. V. 66. P. 013003; arXiv:hep-ph/0202068.
2. *Naumov V., Naumov D.* A Diagrammatic Treatment of Neutrino Oscillations // J. Phys. G: Nucl. Part. Phys. 2010. V. 37. P. 105014.
3. *Grimus W., Stockinger P.* Real Oscillations of Virtual Neutrinos // Phys. Rev. D. 1996. V. 54. P. 3414.
4. *Synge J. L.* Relativity: General Theory. Amsterdam: North-Holl. Publ. Company, 1960.
5. *Landau L. D., Lifshitz E. M.* The Classical Theory of Fields. M.: Nauka, 1988 (in Russian).
6. *Fedosov B.* Deformation Quantization and Index Theory. Berlin: Akad. Verl., 1996.

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