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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

HARMONIC SUPERFIELD ACTION FOR $\mathcal{N} = 4$ SYM THEORY WITH CENTRAL CHARGE

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We develop a superfield formulation of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory with the rigid central charge in $USp(4)$ harmonic superspace. Component formulation of this theory was given by Sohnius, Stelle, and West [1], but its superfield formulation has not been constructed so far. We construct the superfield action, corresponding to this model, and show that it reproduces the component action from [1].

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INTRODUCTION

Maximally extended $\mathcal{N} = 4$ SYM theory with R-symmetry group $SU(4)$ possesses many remarkable properties on classical and quantum levels and is widely explored in modern theoretical and mathematical physics. Field content of this theory involves one vector, six real scalars and four Majorana spinors [2,3]. By construction, such a model is nonmanifestly supersymmetric and the supersymmetry transformations are closed only on-shell. In many cases, especially to study the quantum aspects, it would be preferable to get an off-shell formulation of $\mathcal{N} = 4$ SYM theory. However, in spite of the considerable efforts, superfield formulation of $\mathcal{N} = 4$ SYM theory in terms of unconstrained $\mathcal{N} = 4$ superfields is still unknown. The best, that has been obtained so far is its formulation in terms of $\mathcal{N} = 1$ superfields and $\mathcal{N} = 2, 3$ harmonic superfields [4].

We would like to draw attention to another $\mathcal{N} = 4$ supersymmetric model, namely, $USp(4)$ SYM theory with central charge[2]. Due to the central charge, the R-symmetry group of the corresponding superalgebra is a subgroup $USp(4)$ of the group $SU(4)$. The gauge theory, based on $\mathcal{N} = 4$ superalgebra with the central charge was constructed by Sohnius, Stelle, and West [1] in component approach. Field content of such a gauge model involves one real vector, five real scalars, four Majorana spinors and auxiliary fields — axial vector and five real scalars. Furthermore, special constraints are imposed on the auxiliary and dynamical fields. After eliminating the auxiliary fields with the help of an additional scalar field, the conventional $SU(4)$, $\mathcal{N} = 4$ SYM theory is restored. In Abelian case, the constraint is solved for the auxiliary vector field in terms of an antisymmetric second-rank field [1]. As

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[2] The structure of the extended supersymmetry theories with the central charges is discussed in [5].

a result, we get the conventional $\mathcal{N} = 4$ SYM theory, where one of the scalars is replaced by an antisymmetric tensor field. Therefore, such a model can be treated as some kind of vector–tensor multiplet theory[1].

In this paper, we develop a superspace formulation of $\mathcal{N} = 4$ SYM theory with the rigid central charge [1] in terms of $\mathcal{N} = 4$ superfields. Some superspace aspects of the theory under consideration have been discussed in the earlier papers [8, 9]. In the present paper, we prove that the constraint for the auxiliary field, introduced in [1] for non-Abelian theory, automatically follows from the $\mathcal{N} = 4$ superfield constraints stipulated by the Bianchi identities, obtained in [8]. Also, we develop a $USp(4)$, $\mathcal{N} = 4$ harmonic superspace formalism and propose a gauge-invariant, $\mathcal{N} = 4$ supersymmetric action, which exactly reproduces the component action of [1] for non-Abelian theory.

$\mathcal{N} = 4$ **SYM Model with Central Charge.** $\mathcal{N} = 4$ SYM model with the central charge has been proposed by Sohnius, Stelle, and West in the component formulation [1]. This model possesses the $USp(4)$ R-symmetry and is described by the action

$$S = \mathrm{tr} \int d^4x \left(-\frac{1}{4} F_{mn} F^{mn} - \frac{1}{2} V_m V^m + \frac{1}{2} D_m \phi_{ij} D^m \phi^{ij} + \frac{1}{2} H_{ij} H^{ij} - \right. \\ \left. - \frac{i}{4} \bar{\lambda}^i \not{D} \lambda_i - \bar{\lambda}^i [\lambda^j, \phi_{ij}] + \frac{1}{4} [\phi_{ij}, \phi_{kl}][\phi^{ij}, \phi^{kl}] \right). \quad (1)$$

Here, the vector field A_m, $USp(4)$-Majorana spinor fields $\lambda_{i\alpha} = \bar{\lambda}^{j\beta} (C^{-1})_{\beta\alpha} \Omega_{ji}$ and the antisymmetric, Ω-traceless scalar fields ϕ_{ij} are the propagating fields, whereas the pseudovector V_m and antisymmetric, Ω-traceless scalar fields H_{ij} are the auxiliary fields[2]. All fields take the values in the Lie algebra of gauge group. D_m are the conventional gauge covariant derivatives. The multiplet under consideration contains 16 bosonic and 16 fermionic components. The supersymmetry transformations are closed off-shell, up to field-dependent gauge transformations. The action (1) is invariant under these transformations if the following additional constraint:

$$0 = D^m V_m + \frac{1}{2} \{\bar{\lambda}^i, \gamma^5 \lambda_i\} - i[\phi_{ij}, H^{ij}], \quad (2)$$

is satisfied. Moreover, the action (1) is invariant under central charge transformations [1].

It is interesting to point out that in the Abelian theory, the constraint (2) can be resolved for vector V_m in terms of antisymmetric second-rank tensor field [1]. This field describes a propagating spin-0 mode, which is known as a «notoph». In addition, with respect to central charge transformations the vector transforms into the dual of the field strength of the antisymmetric tensor and the antisymmetric tensor transforms into the dual field strength of the vector.

The conventional on-shell $SU(4)$ $\mathcal{N} = 4$ SYM theory is obtained by introducing the scalar Lagrange multiplier A_5 for the constraint and eliminating the auxiliary fields V_m and H_{ij} from the equations of motion

$$V_m = -D_m A_5, \quad H_{ij} = i[A_5, \phi_{ij}]. \quad (3)$$

---

[1]The $\mathcal{N} = 2$ vector–tensor multiplet theories are discussed in [6, 7].

[2]Ω is the invariant metric of the $USp(4)$ group.

In this case, the scalar field A_5 is unified with five scalar fields ϕ_{ij} and, as a result, one gets six scalar fields of conventional $SU(4)$ $\mathcal{N}=4$ SYM theory.

The aim of the paper is to develop a formulation of the model under consideration in terms of $\mathcal{N}=4$ harmonic superfields and present the action in superfield form.

1. $\mathcal{N}=4$ CENTRAL CHARGE SUPERSPACE AND ITS GAUGING

The $\mathcal{N}=4$ central charge superspace is characterized by the coordinates $Z^M=\{x^m, z, \theta_i^\alpha, \bar\theta^i_{\dot\alpha}\}$ and the supercovariant derivatives $D_M=(\partial_m, \partial_z, D^i_\alpha, \bar D_i^{\dot\alpha})$. These derivatives are used for the definitions of the gauge covariant derivatives $\nabla_M = D_M + i\Gamma_M$ with superconnections Γ_M and gauge transformations $\nabla'_M = \mathrm{e}^{i\tau}\nabla_M\,\mathrm{e}^{-i\tau}$, where τ is a gauge superfield parameter. Then, one introduces the curvature tensors or superfield strengths defined on the $USp(4)$ $\mathcal{N}=4$ central charge superspace with the help of the algebra

$$\{\nabla_{\hat\alpha i}, \nabla_{\hat\beta j}\} = 2i\varepsilon_{\hat\alpha\hat\beta}\Omega_{ij}\nabla_z \pm 2i\varepsilon_{\hat\alpha\hat\beta}W_{ij}, \quad \{\nabla_{\alpha i}, \bar\nabla_{\dot\alpha j}\} = -2i\Omega_{ij}\nabla_{\alpha\dot\alpha},^{1} \tag{4}$$

where we impose the reality conditions under the internal symmetry:

$$\Omega^{ij}W_{ij}=0, \quad \overline{(W_{ij})} = W^{ij} = \Omega^{ik}\Omega^{jl}W_{kl} = -\frac{1}{2}\varepsilon^{ijkl}W_{kl}. \tag{5}$$

Here, Ω_{ij} is the invariant tensor of the $USp(4)$ group. The other commutators of the gauge supercovariant derivatives look like

$$[\nabla_{\hat\alpha i}, \nabla_z] = \pm iG_{\hat\alpha i}, \quad [\nabla_{\hat\alpha i}, \nabla_m] = \pm iF_{\hat\alpha im}, \quad [\nabla_m, \nabla_z] = iV_m, \quad [\nabla_m, \nabla_n] = iF_{mn}. \tag{6}$$

In this representation, let $G_{\alpha i}$ be the $USp(4)$ Majorana spinor with the reality condition $\overline{(G_{\alpha i})} = \bar G^i_{\dot\alpha} = \Omega^{ij}\bar G_{\dot\alpha i}$. As a result, the gauge theory in $USp(4)$, $\mathcal{N}=4$ central charge superspace is characterized by the superfields $W_{ij}, G_{\hat\alpha i}, F_{\hat\alpha im}, V_m, F_{mn}$. The superfield strengths satisfy some number of constraints to reduce the number of fields to an irreducible multiplet stipulated by the Bianchi identities. The solution of these relations determines the field content of the theory as well as the transformation laws of the component fields.

One can prove that the Bianchi identities are satisfied if and only if all superfield strengths are expressed in terms of a single real scalar superfield W_{ij} and its spinor derivatives. Here, we only list the results of [8] in our conventions concerning the solution to the constraints of dimension from $3/2$ to 3:

- Solution to the $\dim = 3/2$ Bianchi identities is $F_{\alpha im} = -\sigma^m_{\alpha\dot\alpha}\bar G_i^{\dot\alpha}$,

$$\nabla_{\hat\alpha k}W_{ij} = i\Omega_{ij}G_{\hat\alpha k} + 2i\Omega_{k[i}G_{\hat\alpha j]}, \quad 5iG_{\hat\alpha i} = \nabla^k_{\hat\alpha}W_{ki}, \quad \nabla_z W_{ij} \equiv H_{ij}. \tag{7}$$

- Solution to the $\dim = 2$ Bianchi identities

$$\nabla_{\hat\alpha i}G_{\hat\beta j} = -\varepsilon_{\hat\alpha\hat\beta}H_{ij} \mp \frac{1}{2}\Omega_{ij}F_{\hat\alpha\hat\beta} \pm \frac{1}{2}\varepsilon_{\hat\alpha\hat\beta}[W_{ik}, W_j{}^k], \quad \bar\nabla_{\dot\alpha i}G_{\alpha j} = i\Omega_{ij}V_{\alpha\dot\alpha} - \nabla_{\alpha\dot\alpha}W_{ij}. \tag{8}$$

[1]Here and below, we use the notation $\hat\alpha$ for the set $\{\alpha, \dot\alpha\}$. Upper sign here corresponds to α in $\hat\alpha$ and lower sign corresponds to $\dot\alpha$ in $\hat\alpha$.

• Solution to the $\dim = 5/2$ Bianchi identities

$$\nabla_z \bar{G}_{\dot{\alpha}i} = \nabla_{\alpha\dot{\alpha}} G^{\alpha}_{i} + [W_{ik}, \bar{G}^{k}_{\dot{\alpha}}], \quad \nabla_{\alpha i} V_m = \sigma^m_{\alpha\dot{\alpha}} [W_{ik}, \bar{G}^{\dot{\alpha}k}] + i(\sigma_{mn})_{\alpha}^{\beta} \nabla_n G_{\beta i},$$
$$\nabla_{\alpha i} H_{jk} = -i\Omega_{jk} \nabla_{\alpha\dot{\alpha}} \bar{G}^{\dot{\alpha}}_{i} - 2i\Omega_{i[j} \nabla_{\alpha\dot{\alpha}} \bar{G}^{\dot{\alpha}}_{k]} - i[W_{jk}, G_{\alpha i}] - \qquad (9)$$
$$- i\Omega_{jk} [W_{il}, G^{l}_{\alpha}] - 2i\Omega_{i[j} [W_{k]l}, G^{l}_{\alpha}],$$

$$\nabla_{\dot{\alpha}i} F_{\alpha\beta} = 2i \nabla_{(\alpha\dot{\alpha}} G_{\beta)i}, \quad \nabla_{\gamma i} F_{\alpha\beta} = -2i\varepsilon_{\gamma(\alpha} \nabla_{\beta)\dot{\alpha}} \bar{G}^{\dot{\alpha}}_{i}. \qquad (10)$$

• Solution to the $\dim = 3$ Bianchi identities

$$\nabla_m V_m = -\frac{i}{4}[W_{ik}, H^{ik}] + \frac{1}{2}\{G_{\alpha k}, G^{\alpha k}\} + \frac{1}{2}\{\bar{G}_{\dot{\alpha}k}, \bar{G}^{\dot{\alpha}k}\}, \qquad (11)$$

$$\nabla_z V_m = -\sigma^m_{\alpha\dot{\alpha}} \{G^{\alpha i}, \bar{G}^{\dot{\alpha}}_{i}\} - \frac{i}{4}[W_{ik}, \nabla_m W^{ik}] - \nabla_a F_{am}, \qquad (12)$$

$$\nabla_z H_{jk} = \Box W_{jk} - \frac{i}{2}\Omega_{jk}\{G_{\alpha i}, G^{\alpha i}\} - 2i\{G_{\alpha j}, G^{\alpha}_{k}\} + \frac{i}{2}\Omega_{jk}\{\bar{G}_{\dot{\alpha}i}, \bar{G}^{\dot{\alpha}i}\} +$$
$$+ 2i\{\bar{G}_{\dot{\alpha}j}, \bar{G}^{\dot{\alpha}}_{k}\} + \frac{1}{8}\Omega_{jk}[W_{il}[W^{i}{}_{p}, W^{lp}]]. \qquad (13)$$

By analogy with the case of $\mathcal{N}=2$ [6,7], Eqs. (9), (12), (13) can be called the generalized Dirac equation, the Yang–Mills equation and the Klein–Gordon equation, respectively, with central charge playing a role of "fifth coordinate". As a consequence of (7)–(13), we can construct the complete power expansion of the $W^{ij} = \sum_{k=0}^{\infty} W^{ij}_{(k)} z^k$ in superspace from the first two coefficients $W^{ij}_{(0)}, W^{ij}_{(1)}$. The set of constraints shows that the component fields in [1] are completely determined by the lowest orders in the expansion $W_{ij}(x, \theta, \bar{\theta}, z)$, in $z, \theta, \bar{\theta}$. Then, it is clear that we can completely fix the dependence of all quantities under consideration on the central charge coordinate z, as well as for theories with $\mathcal{N}=2$ rigid supersymmetry with the central charge [6, 7]. Using the covariant derivatives and solving the Bianchi identities, we can immediately write down the supersymmetry transformations of the component fields in the form $\delta\Phi| = -\epsilon\nabla\Phi|$. Rigid central charge transformations are realized on ∇_M and matter superfields Φ as $\delta_z \nabla_M = [\omega\nabla_z, \nabla_M]$, and $\delta_z\Phi = \omega\nabla_z\Phi$, with ω being a constant real gauge parameter, corresponding to transformation of central charge coordinate $\delta z = \omega$.

Now, we clarify how the conditions (3) appear in the superfield approach. It is known from [1] that Eqs. (3) allow us to reduce the $\mathcal{N}=4, USp(4)$ SYM theory to the conventional $\mathcal{N}=4, SU(4)$ SYM theory. The relations (3) are the solutions of equations of motion for auxiliary fields V_m, H_{ij} with the help of constraint (2). Therefore, if we want to get the analogous relation between the above two theories in the superfield approach, we also should use, besides the identities (7)–(13), some additional constraints. We take such restrictions in the form

$$\partial_z \Gamma_{m,z} = 0, \quad \partial_z W_{ij} = 0 \qquad (14)$$

and show that these conditions are equivalent to (3). In this case, the identities (9), (11), (12), (13) are converted into equations of motion for the conventional $\mathcal{N}=4$ SYM theory. If we do not impose the conditions (14), we have the $USp(4)$ superfield gauge theory satisfying the identities (7)–(13).

The constraints (7) allow us to get the important consequences. Specifying concrete values of the indices i, j, k and using explicit form of the matrix Ω, one gets various constraints for the superfield W_{ij}. For example, $\nabla_{\alpha 1} W_{12} = 0$, $\nabla_{\alpha 2} W_{12} = 0$, and the other analogous equations. These equations mean the very special dependence of the superfield W_{ij} on anticommuting coordinates

$$W_{12}(\theta^3, \theta^4, \bar\theta_2, \bar\theta_1), \quad W_{13}(\theta^2, \theta^4, \bar\theta_3, \bar\theta_1), \quad W_{24}(\theta^1, \theta^3, \bar\theta_4, \bar\theta_2), \quad W_{34}(\theta^1, \theta^2, \bar\theta_4, \bar\theta_3). \quad (15)$$

As a result, we see that W_{ij} with different values i, j belong to different subspaces of the full superspace. More transparent covariant solution of such constraints will take place in the framework of the harmonic superspace.

2. GAUGE THEORY IN $USp(4)$, $\mathcal{N} = 4$ HARMONIC SUPERSPACE

Following the general scheme of harmonic superspace construction [4][1], we extend the $\mathcal{N} = 4$ central charge superspace with coordinates $Z^M = (x^m, z, \theta_{\alpha i}, \bar\theta^i_{\dot\alpha})$ by the eight-dimensional coset space $USp(4)/U(1) \times U(1)$ parameterized by harmonic variables $u_i^{(\pm,0)}$, $u_i^{(0,\pm)}$, which are inert under supersymmetry and take the values in the fundamental representation of $USp(4)$ [9, 10]. Using the harmonics $u_i^{(\pm,0)}, u_i^{(0,\pm)}$, we define the harmonic derivatives $\partial^{(q_1,q_2)}$, which are left-invariant vector fields on $USp(4)$, by the rule

$$\begin{gathered}\partial^{(\pm\pm,0)} = u_i^{(\pm,0)} \frac{\partial}{\partial u_i^{(\mp,0)}}, \quad \partial^{(0,\pm\pm)} = u_i^{(0,\pm)} \frac{\partial}{\partial u_i^{(0,\mp)}}, \\ \partial^{(\pm,\pm)} = u_i^{\pm,0} \frac{\partial}{\partial u_i^{0,\mp}} + u_i^{0,\pm} \frac{\partial}{\partial u_i^{\mp,0}}, \quad \partial^{(\pm,\mp)} = u_i^{(\pm,0)} \frac{\partial}{\partial u_i^{0,\pm}} - u_i^{(0,\mp)} \frac{\partial}{\partial u_i^{\mp,0}}.\end{gathered} \quad (16)$$

There is a special involution (special complex conjugation) $\widetilde{u_i^{(\pm,0)}} = u^{(0,\pm)i}$, $\widetilde{u^{(0,\pm)i}} = -u_i^{(\pm,0)}$, and so on, allowing one to define a reality condition in harmonic superspace [9, 10]. With the help of the harmonics $u_i^{(\pm,0)}$, $u_i^{(0,\pm)}$ one can convert the spinor covariant derivatives into operators $\nabla^{(\pm,0)}_{\hat\alpha} = u_i^{(\pm,0)} \nabla^i_{\hat\alpha}$, $\nabla^{(0,\pm)}_{\hat\alpha} = u_i^{(0,\pm)} \nabla^i_{\hat\alpha}$, and reformulate the superalgebra (4) in another, more clear form. Now, we rewrite the constraints (15) as the constraints in harmonic superspace, where they will have a form of analyticity conditions. In the τ frame, we have the obvious anticommuting relations

$$\{\nabla^{(+,0)}_{\hat\alpha}, \nabla^{(+,0)}_{\hat\beta}\} = 0, \quad \{\nabla^{(\pm,0)}_{\alpha}, \bar\nabla^{(\mp,0)}_{\dot\alpha}\} = \mp 2i \nabla_{\alpha\dot\alpha}, \quad (17)$$

[1]The structure of harmonic variables for $\mathcal{N} = 2, 3, 4$ harmonic superspaces with various R-symmetries is discussed in [10].

and the other anticommuting relations that define five harmonic projections of the tensor W^{ij}

$$
\begin{aligned}
&\{\nabla^{(+,0)}_{\hat\alpha}, \nabla^{(-,0)}_{\hat\beta}\} = 2i\varepsilon_{\hat\alpha\hat\beta}\nabla_z \pm 2i\varepsilon_{\hat\alpha\hat\beta}W^{(0,0)},\\
&\{\nabla^{(0,+)}_{\hat\alpha}, \nabla^{(0,-)}_{\hat\beta}\} = 2i\varepsilon_{\hat\alpha\hat\beta}\nabla_z \mp 2i\varepsilon_{\hat\alpha\hat\beta}W^{(0,0)},\\
&\{\nabla^{(\pm,0)}_{\hat\alpha}, \nabla^{(0,\mp)}_{\hat\beta}\} = \pm 2i\varepsilon_{\hat\alpha\hat\beta}W^{(\pm,\mp)},\\
&\{\nabla^{(\pm,0)}_{\hat\alpha}, \nabla^{(0,\pm)}_{\hat\beta}\} = \pm 2i\varepsilon_{\hat\alpha\hat\beta}W^{(\pm,\pm)},
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
&u_i^{(+,0)}u_j^{(-,0)}W^{ij} = W^{(0,0)},\\
&u_i^{(\pm,0)}u_j^{(0,\pm)}W^{ij} = W^{(\pm,\pm)},\\
&u_i^{(\pm,0)}u_j^{(0,\mp)}W^{ij} = W^{(\pm,\mp)}.
\end{aligned}
\tag{19}
$$

These definitions simply mean that $W^{(q_1,q_2)}$ in τ basis depend linearly on harmonics $u_i^{(\pm,0)}$, $u_i^{(0,\pm)}$, and all five harmonic projections of W_{ij} transform through each other under the action of symmetry generators.

The $USp(4)$, $\mathcal{N}=4$ harmonic superspace with coordinates $\{x^m, z, \theta^{(\pm,0)}_{\hat\alpha}, \theta^{(0,\pm),}_{\hat\alpha}, u_i\}$ contains several analytic subspaces of the full superspace with eight anticommuting coordinates. It can be checked that each of the following four superfields lives in its own analytic subspace. For example,

$$
\begin{gathered}
W^{(+,+)}(\theta^{(+,0)}, \theta^{(0,+)}, \bar\theta^{(+,0)}, \bar\theta^{(0,+)}),\\
\nabla^{(+,0)}_\alpha W^{(+,+)} = \nabla^{(0,+)}_\alpha W^{(+,+)} = \bar\nabla^{(+,0)}_{\dot\alpha} W^{(+,+)} = \bar\nabla^{(0,+)}_{\dot\alpha} W^{(+,+)} = 0.
\end{gathered}
\tag{20}
$$

Acting on $W^{(q_1,q_2)}$ by one, two, three or four spinor derivatives $D^{(q_1,q_2)}_{\hat\alpha}$, one obtains a set of relations, which allow one to define the superstrength components. We do not write down all such relations. The relations that will be needed further have the form

$$
\begin{gathered}
G^{(+,0)}_{\hat\alpha} = -\frac{i}{2}\nabla^{(0,-)}_{\hat\alpha}W^{(+,+)}, \quad G^{(0,+)}_{\hat\alpha} = \frac{i}{2}\nabla^{(-,0)}_{\hat\alpha}W^{(+,+)},\\
\nabla^{(0,-)}_{\hat\alpha}G^{(+,0)}_{\hat\beta} = \varepsilon_{\hat\alpha\hat\beta}H^{(+,-)} \mp \varepsilon_{\hat\alpha\hat\beta}[W^{(0,0)}, W^{(+,-)}], \quad \bar\nabla^{(0,-)}_{\dot\alpha}G^{(+,0)}_{\alpha} = \nabla_{\alpha\dot\alpha}W^{(+,-)}.
\end{gathered}
\tag{21}
$$

Further, we take the superfield $W^{(+,+)}$ as the basic superfield strength and construct superfield action in its terms. This superfield is a function on analytic subspace of the harmonic superspace parameterized by

$$
\{\zeta^M, u\} = \{x^m_A, z_A, \theta^{(+,0)}_\alpha, \theta^{(0,+)}_\alpha, \bar\theta^{(+,0)}_{\dot\alpha}, \bar\theta^{(0,+)}_{\dot\alpha}, u_i^{(\pm,0)}, u_i^{(0,\pm)}\},
\tag{22}
$$

where

$$
\begin{gathered}
x^m_A = x^m - i\theta^{(-,0)}\sigma^m\bar\theta^{(+,0)} - i\theta^{(+,0)}\sigma^m\bar\theta^{(-,0)} - i\theta^{(0,-)}\sigma^m\bar\theta^{(0,+)} - i\theta^{(0,+)}\sigma^m\bar\theta^{(0,-)},\\
z_A = z + i\theta^{(-,0)\alpha}\theta^{(+,0)}_\alpha + i\theta^{(0,-)\alpha}\theta^{(0,+)}_\alpha - i\bar\theta^{(+,0)}_{\dot\alpha}\bar\theta^{(-,0)\dot\alpha} - i\bar\theta^{(0,+)}_{\dot\alpha}\bar\theta^{(0,-)\dot\alpha}.
\end{gathered}
\tag{23}
$$

One can prove that the above analytic subspace is closed under the supersymmetry transformations and is real with respect to "tilde"-conjugation. Hence, the analytic superfield $W^{(+,+)}$ also can be chosen real. In the λ frame, the covariant spinor derivatives are short

$$D^{(+,0)}_{\hat{\alpha}} = \frac{\partial}{\partial\theta^{(-,0)\hat{\alpha}}}, \quad D^{(0,+)}_{\hat{\alpha}} = \frac{\partial}{\partial\theta^{(0,-)\hat{\alpha}}}. \tag{24}$$

The harmonic derivatives $D^{(q_1,q_2)}$ in the analytic basis are presented in [9]. The fundamental property of the operators $D^{(++,0)}, D^{(\pm,\mp)}$, and $D^{(0,++)}$ is that if $W^{(+,+)}$ is a covariantly analytic superfield, i.e., if $D^{(+,0)}_{\hat{\alpha}}W^{(+,+)} = 0, D^{(0,+)}_{\hat{\alpha}}W^{(+,+)} = 0$, then it is also a harmonic analytic superfield, i.e., $D^{(++,0)}W^{(+,+)} = 0, D^{(\pm,\mp)}W^{(+,+)} = 0, D^{(0,++)}W^{(+,+)} = 0$.

Similarly, we can consider the solution of the Grassmann constraints for other superfields $W^{(q_1,q_2)}$ by passing to the corresponding analytic coordinates.

Superfield Action. Here, we formulate the superfield action. We show that such a superfield action is written in terms of superfield $W^{(+,+)}$ and exactly reproduces the component action (1). The action under consideration must be gauge-invariant, $\mathcal{N} = 4$ supersymmetric and invariant under central charge transformation. To find such an action, one uses a prescription, which was formulated in $\mathcal{N} = 2$ central charge harmonic superspace [7]. We will see that this prescription perfectly works in $USp(4)$, $\mathcal{N} = 4$ harmonic superspace.

We propose the superfield action for $USp(4)$, $\mathcal{N} = 4$ SYM theory in the form

$$S \sim \mathrm{tr} \int d\zeta^{(-4,-4)}\, du((\theta^{(+,0)})^2 - (\bar{\theta}^{(+,0)})^2)((\theta^{(0,+)})^2 - (\bar{\theta}^{(0,+)})^2)\mathcal{L}^{(2,2)}. \tag{25}$$

Here, $\mathcal{L}^{(2,2)} = W^{(+,+)}W^{(+,+)}$ is an analytic (20) and harmonically "short" superfield. Analytic superspace dimensionless integration measure looks like

$$d\zeta^{(-4,-4)}du = d^4x_A\, d^2\theta^{(+,0)}\, d^2\theta^{(0,+)}\, d^2\bar{\theta}^{(+,0)}\, d^2\bar{\theta}^{(0,+)}\, du,$$

where du denotes the left-right invariant measure of $USp(4)$. The action (25) is obviously gauge-invariant. Also, this action is $\mathcal{N} = 4$ supersymmetric. The proof of this statement is analogous to that in $\mathcal{N} = 2$ theory with intrinsic central charges [7]. Though the action (25) does not contain integration over z, actually it is z-independent due to the identities $D^{(++,0)}\mathcal{L}^{(2,2)} = D^{(0,++)}\mathcal{L}^{(2,2)} = 0$. Therefore, this action is invariant under central charge transformation $\delta x^m_A = 0$, $\delta z_A = \omega$ with constant parameter ω. Now, we rewrite the action (25) in component form. To do that, we should integrate over harmonic and over all anticommuting coordinates. We take into account that on a gauge-invariant quantities $D^2 = \nabla^2$. Also, one uses the integration rule

$$\int d\zeta^{(-4,-4)}du = \frac{1}{256}\int d^4x\, du^{(\pm,0)}du^{(0,\pm)}(\nabla^{(-,0)})^2(\bar{\nabla}^{(-,0)})^2(\nabla^{(0,-)})^2(\bar{\nabla}^{(0,-)})^2. \tag{26}$$

After all these integrations, we get

$$S \sim -\frac{1}{20}\mathrm{tr}\int d^4x(\nabla^{\alpha(i}\nabla^{j)}_{\alpha} - \bar{\nabla}^{(i}_{\dot{\alpha}}\bar{\nabla}^{\dot{\alpha}j)})\mathcal{L}_{ij}\Big|_{\theta=0} = \mathrm{tr}\int d^4x\mathcal{L}, \tag{27}$$

where the integrand is as follows: $\mathcal{L}_{ij} = -\frac{1}{2}(G^{\alpha}_i G_{\alpha j} + \bar{G}^{\dot{\alpha}}_i \bar{G}_{\dot{\alpha}j} + \frac{i}{2}H_i{}^k W_{jk})$. This expression contains all the necessary terms corresponding to the component action (1). We can show, after some rather cumbersome calculations using the identities (7)–(9), that the action (27) with a coefficient $1/4$ in the definition (25) is rewritten in the form

$$\mathcal{L} = -\frac{1}{4}F^{mn}F_{mn} - \frac{1}{2}V^m V_m + \frac{1}{8}H^{ij}H_{ij} + \frac{1}{8}\nabla^m W_{ij}\nabla_m W^{ij} + \\ + \frac{1}{16}[W_{ik}, W_j{}^k][W^i{}_l, W^{jl}] + iG^{\alpha i}\nabla_{\alpha\dot\alpha}\bar{G}_i^{\dot\alpha} + \frac{i}{2}[W_{ik}, G^{\alpha k}]G^i_\alpha + \frac{i}{2}[W_{ik}, \bar{G}^k_{\dot\alpha}]\bar{G}^{\dot\alpha i}. \quad (28)$$

We can see that each term in the action (28) has the corresponding analogous term in the action (1), that is Eq. (28) coincides with Eq. (1) up to the coefficients. As a result, we finally derive the action (1) from the superfield action (25).

3. SUMMARY

We have developed the harmonic superspace formulation of $\mathcal{N}=4$ SYM theory with the rigid central charge. Component formulation of this theory was given in [1]. We studied the gauge theory in $USp(4)$, $\mathcal{N}=4$ superspace and showed that all the constraints in the component theory [1], the supersymmetry transformations and the central charge transformation are the consequences of the Bianchi identities for the superfield strengths. Also, it was proved that the Lagrange multiple A_5 and the expressions of auxiliary fields V_m and H_{ij} in terms of A_5, used in [1], have a natural origin as the conditions of central charge independence of gauge superconnections $\Gamma_{m,z}$ and superstrength W_{ij}. Gauge-invariant, $\mathcal{N}=4$ supersymmetric action, invariant under the central charge transformations is proposed and it is proven that this action reproduces the component action given in [1].

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