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INTEGRALS OF EQUATIONS FOR COSMOLOGICAL AND STATIC REDUCTIONS IN GENERALIZED THEORIES OF GRAVITY

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We consider the dilaton gravity models derived by reductions of generalized theories of gravity and study one-dimensional dynamical systems simultaneously describing cosmological and static states in any gauge. Our approach is fully applicable to studying static and cosmological solutions in multidimensional theories and also in general one-dimensional dilaton–scalaron gravity models. We here focus on general and global properties of the models, on seeking integrals, and on analyzing the structure of the solution space. We propose some new ideas in this direction and derive new classes of integrals and new integrable models.

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INTRODUCTION

Current observational data strongly suggest that Einstein's gravity must be modified. The combination of data on dark energy and the growing evidence for inflation have generated a wide spectrum of such modifications. Superstring and supergravity ideas suggested natural modifications, but in view of the serious mathematical problems of the current string theory, strict unambiguous predictions about concrete modifications of gravity are not yet available. Moreover, the phenomenon of dark energy was not predicted by string theory, and its origin in the stringy framework proved rather difficult to uncover and understand. The problem of dark energy in string theory seems very deep and is related to many other complex issues of quantum cosmology, but it also leads to some beautiful and exciting speculations, like eternal inflation and the multiverse. On the other hand, if we first try to find a natural place for dark energy in classical cosmological models, which are almost inevitably essentially nonlinear and nonintegrable, then we best return to recalling the origin of general relativity and seek some options abandoned or not found by its creators.

Therefore, simpler modifications of gravity that affect only the gravitational sector are also popular. In essence, these modifications reduce to the standard Einstein gravity supplemented by some number of scalar bosons (the first example of such a modification was the old Jordan–Brans–Dicke theory). The main problem with this approach is that the origin of these

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scalar bosons is not clear, and there is no theoretical principle governing their coupling to gravity. Of course, there exist some phenomenological and theoretical constraints, but the spectrum of these models is generally too wide [1]. The modification proposed and studied in [2–6] satisfies some principles of geometric nature based on Einstein's idea (1923) [2] to formulate gravity theory in a non-Riemannian space with a symmetric connection determined by a special variational principle involving a *"geometric" Lagrangian*. This Lagrangian is assumed to be a function of the generalized Ricci curvature tensor and of other fundamental tensors, and is varied in the connection coefficients. A new interpretation and generalization of this approach was developed in [3, 4] for arbitrary space-time dimension D.

The connection coefficients define symmetric and antisymmetric parts (s_{ij} and a_{ij}) of the Ricci tensor and a new vector a_i. Assuming that there are no dimensional fundamental constants in the pure geometry (except the speed of light relating space to time), we choose geometric Lagrangians giving a dimensionless geometric action. The geometric variational principle puts further bounds on the geometry and, in particular, relates a_i to a_{ij}. To define a metric tensor, we must introduce a dimensional constant. We then can find a *physical Lagrangian* depending on this dimensional constant and on some dimensionless parameters. The theory thus obtained supplements standard general relativity with dark energy (the cosmological term, in the limit $a_i = a_{ij} = 0$), a neutral massive (or tachyonic) vector field proportional to a_i (a vecton), and after dimensional reductions to $D = 4$, with $D - 4$ massive (or tachyonic) scalar fields.

The most natural density of this sort in any dimension is the square root of $\det(s_{ij} + \bar{l} a_{ij})$, where $\bar{l}$ is a number [3]. The effective physical Lagrangian is the sum of the standard Einstein term, the vecton mass term, and a term proportional to $\det(g_{ij} + l f_{ij})$ to the power $\nu \equiv 1/(D-2)$, where g_{ij} and f_{ij} are the metric and the vecton field tensors *conjugate* to s_{ij} and a_{ij}, [4] and l is a parameter related to $\bar{l}$. The last term has a dimensional multiplier, which in the limit of a small field f_{ij} produces the cosmological constant. For $D = 4$, we therefore have the term first introduced by Einstein, but now usually called the Born–Infeld or *brane* Lagrangian. For $D = 3$, we have the Einstein–Proca theory, which is very interesting for studies of nontrivial space topologies.

Here, we consider the simplest geometric Lagrangian,

$$\mathcal{L}_{\rm geom} = \sqrt{-\det(s_{ij} + \bar{l} a_{ij})} \equiv \sqrt{-\Delta_s}, \tag{1}$$

where the minus sign is taken because $\det(s_{ij}) < 0$ (due to the local Lorentz invariance), and we naturally assume that the same holds for $\det(s_{ij} + \bar{l} a_{ij})$ (to reproduce Einstein's general relativity with the cosmological constant in the limit $\bar{l} \to 0$). Following the steps in [3] or using the results described in [6], we can derive the corresponding *physical Lagrangian*

$$\mathcal{L}_{\rm phys} = \sqrt{-g}\,\left[-2\Lambda\,[\det(\delta_i^j + l f_i^j)]^\nu + R(g) - m^2 g^{ij} a_i a_j\right], \quad \nu \equiv \frac{1}{D-2}, \tag{2}$$

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[1] Restricting consideration to homogeneous cosmologies, one can find that in dimensionally reduced supergravity theory there may emerge massless scalar bosons that couple to gravity only, see, e.g., [1].

[2] References to Einstein's papers as well as to many papers of other authors, which are related to the subject of this report, can be found in our publications [2–10].

[3] Einstein used Eddington's scalar density $\sqrt{|\det r_{ij}|}$, where $r_{ij} \equiv s_{ij} + a_{ij}$, as the Lagrangian.

[4] This unusual construction introduced by A. Einstein is described and generalized in [2–6].

which should be varied with respect to the metric and the vector field; m^2 is a parameter depending on the chosen model for affine geometry and on D (see [2–4]). This parameter can be positive or negative and we often use notation $m^2 \equiv \mu$. When the vecton field vanishes, we have the standard Einstein gravity with the cosmological constant. For dimensional reductions from $D \geqslant 5$ to $D = 4$, we can obtain the Lagrangian describing the vecton a_i, $f_{ij} \sim \partial_i a_j - \partial_j a_i$ and $(D-4)$ scalar fields a_k, $k = 4, \ldots, D$. We note that Lagrangian (2) is bilinear in the vecton field, for $D = 3$, and gives the three-dimensional gravity with the cosmological term in the approximation $a_i = 0$.

To compactify this report, we only give an overview of the main points. First, consider a rather general Lagrangian in the D-dimensional *spherically symmetric* case ($x^0 = t$, $x^1 = r$):

$$ds_D^2 = ds_2^2 + ds_{D-2}^2 = g_{ij}\, dx^i\, dx^j + \varphi^{2\nu}\, d\Omega_{D-2}^2, \tag{3}$$

where $\nu \equiv (D-2)^{-1}$. The standard spherical reduction of (2) gives the effective Lagrangian [1], the first three terms of which describe the standard spherically reduced Einstein gravity:

$$\mathcal{L}_D^{(2)} = \sqrt{-g}\, \left[\varphi R(g) + k_\nu\, \varphi^{1-2\nu} + W(\varphi)\, (\nabla\varphi)^2 + X(\varphi, \mathbf{f}^2) - m^2 \varphi\, \mathbf{a}^2\right]. \tag{4}$$

Here, $a_i(t, r)$ has only two nonvanishing components a_0, a_1, f_{ij} has just one independent component $f_{01} = a_{0,1} - a_{1,0}$; the other notations are: $\mathbf{a}^2 \equiv a_i a^i \equiv g^{ij} a_i a_j$, $\mathbf{f}^2 \equiv f_{ij} f^{ij}$, $k_\nu \equiv k(D-2)(D-3)$, $W(\varphi) = (1-\nu)/\varphi$ and, finally,

$$X(\varphi, \mathbf{f}^2) \equiv -2\Lambda\varphi\, \left[1 + \frac{1}{2}\lambda^2 \mathbf{f}^2\right]^{\nu}, \tag{5}$$

where, the parameter λ is related to dimensionless parameter l in (2), but $[\lambda] = \mathrm{L}$.

Sometimes, it is convenient to transform away the dilaton kinetic term by using the Weyl transformation, which, in our case, is the following ($w'(\varphi)/w(\varphi) = W(\varphi)$):

$$g_{ij} = \hat{g}_{ij} w^{-1}(\varphi), \quad w(\varphi) = \varphi^{1-\nu}, \quad \mathbf{f}^2 = w^2\, \hat{\mathbf{f}}^2, \quad \mathbf{a}^2 = w \hat{\mathbf{a}}^2. \tag{6}$$

Applying this transformation to (4) and omitting the hats, we find in the Weyl frame that

$$\mathcal{L}_D^{(2)} \mapsto \hat{\mathcal{L}}_D^{(2)} = \sqrt{-g}\, \left[\varphi R(g) + k_\nu \varphi^{-\nu} - 2\Lambda\varphi^\nu \left(1 + \frac{1}{2}\lambda^2 \varphi^{2(1-\nu)} \mathbf{f}^2\right)^{\nu} - m^2 \varphi \mathbf{a}^2\right]. \tag{7}$$

When $D = 3$, we have $\nu = 1$, $k_\nu = 0$, the Weyl transformation is trivial and the Lagrangian is

$$\mathcal{L}_3^{(2)} = \sqrt{-g}\varphi\left[R(g) - 2\Lambda - \lambda^2 \Lambda \mathbf{f}^2 - m^2 \mathbf{a}^2\right]. \tag{8}$$

These two-dimensional reductions are essentially simpler than their parent higher-dimensional theories. In particular, we show that the massive vecton field theory can be transformed into a dilaton–scalaron gravity model (DSG), which is easier to analyze. Unfortunately, these DSG models and their further reductions to dimension one (static and cosmological

[1] We suppose that the fields φ, a_i are dimensionless, while $[t] = [r] = \mathrm{L}$, and thus $[f_{ij}] = \mathrm{L}^{-1}$, $[R] = [k_\nu] = [X] = [m^2] = \mathrm{L}^{-2}$. For more details on our dimensions, see [6].

reductions) are also essentially nonintegrable. It is well known that the massless case, being a pure dilaton gravity, is classically integrable. Having this in mind, we will look for additional integrals of motion in similar reduced vecton theories, which we transform into dilaton–scalaron gravity models.

The structure of the two-dimensional theory allows one to find some integrable classes using simplifying assumptions about their potentials. For some multi-exponential potentials and constant ("minimal") coupling of scalars, there exist integrable systems related to the Liouville and Toda–Liouville ones (see [7–10]). The pure Liouville case was completely solved. For the Toda–Liouville one, it is difficult to find exact analytical solutions of the two-dimensional constraints, even in the simplest $u_1 \oplus su_2$ case. This problem is easily solved in the one-dimensional (static or cosmological) reduction. Unfortunately, even the one-dimensional cosmological reduction of the pure scalaron theory is not integrable, and thus we concentrate on searching for approximate potentials that allow us to find a wide enough class of analytic solutions to reconstruct the exact ones by iterations.

1. COSMOLOGICAL AND STATIC REDUCTIONS

In this report, we consider only the reductions of the two-dimensional theory to static and cosmological equations ignoring one-dimensional waves studied in our previous works [8–10]. The simplest reduced gauge fixed equations can be directly derived by supposing that in the two-dimensional light-cone equations the fields h, φ, q, ψ depend on one variable, which we denote $\tau \equiv (u + v)$. For the cosmological solutions this variable is identified with the time variable, $\tau = t$, while for the static states, including black holes, it is the space variable, $\tau = r$. The only difference between the cosmological and static solutions is in the sign of the metric, $h_{\rm c} > 0$, $h_{\rm s} < 0$.

In our study of black holes and cosmologies, we use the more general diagonal metric,

$$ds_2^2 = \mathrm{e}^{2\alpha(t,r)} dr^2 - \mathrm{e}^{2\gamma(t,r)} dt^2. \tag{9}$$

Then, the static and cosmological reductions of our two-dimensional vecton theory (7) supplemented with the standard scalar term $V(\varphi, \psi) + Z(\varphi)\nabla\psi^2$ can be presented by the Lagrangian (taking, for the moment, $Z(\varphi) = -\varphi$):

$$\epsilon \mathcal{L}_{\rm v}^{(1)} = \mathrm{e}^{\epsilon(\alpha-\gamma)} \varphi \left[\dot{\psi}^2 - 2\dot{\alpha}_\epsilon \frac{\dot{\varphi}}{\varphi} - (1-\nu)\left(\frac{\dot{\varphi}}{\varphi}\right)^2 \right] - \\ - \mathrm{e}^{\epsilon(\gamma-\alpha)} \mu\varphi a_\epsilon^2 + \epsilon\, \mathrm{e}^{\alpha+\gamma} \left[V + X(\mathbf{f}^2) \right]. \tag{10}$$

Here, we omit a possible dependence of V and X on φ and ψ, denote $Z_a \equiv -\mu\varphi \equiv m^2$, and $\epsilon = \pm$. All the fields depend on $\tau = t$ ($\epsilon = +$) or on $\tau = r$ ($\epsilon = -$). Finally,

$$a_+ = a_1(\tau), \quad a_- = a_0(\tau), \quad \alpha_+ \equiv \alpha, \quad \alpha_- \equiv \gamma, \quad \dot{\alpha}_\epsilon = \frac{d}{d\tau}\alpha_\epsilon, \quad \dot{a} = \frac{d}{d\tau} a.$$

We see that the cosmological and static Lagrangians essentially coincide, the only difference being in the sign of the potential term and of the metric exponents as well. As the kinetic term depends on $\dot{\alpha}_\epsilon$, the multiplier of the kinetic term, $l_\epsilon \equiv \exp(\alpha_{-\epsilon})$, is a Lagrange multiplier, varying of which produces the constraint equation, which is equivalent to vanishing of the Hamiltonian. In view of the implicit dependence of $\mathbf{f}^2$ on l_ϵ, it is much more convenient to first employ the canonical formulation and then identify the proper Lagrange multiplier. Omitting simple details given in [6], we only give the final result.

Introducing the Hamiltonian formulation for general theory (10), we can apply to it the elementary canonical transformation,

$$p_a \Rightarrow -2q, \quad a \Rightarrow p/2, \quad X_{\rm eff}(p_a) \Rightarrow X_{\rm eff}(-2q), \tag{11}$$

and then derive the corresponding new Lagrangian

$$\epsilon\mathcal{L}_q^{(1)} = l_\epsilon^{-1}\left[\varphi\dot{\psi}^2 - 2\dot{\alpha}_\epsilon\dot{\varphi} - (1-\nu)\,\frac{\dot{\varphi}^2}{\varphi} + \frac{\dot{q}^2}{m^2\varphi}\right] + l_\epsilon\epsilon\,\mathrm{e}^{2\alpha_\epsilon}\,U(\varphi,\psi,q), \tag{12}$$

where $l_\epsilon \equiv \exp\left(\alpha_{-\epsilon} - \alpha_{+\epsilon}\right)$ and $U = V(\varphi,\psi) + X_{\rm eff}(-2q)$. The detailed derivation of this *transformation of the vecton into the scalaron*, explanation of notation, and discussions of the analytic approximate expressions of the potential for arbitrary D can be found in [6].

This form is more natural than (10). First, the dependence on the Lagrange multiplier l_ϵ is simple and standard, the kinetic part is quadratic in generalized velocities and can be made diagonal by a redefinition of the Lagrange multiplier and velocities. In addition, we are free to make a convenient gauge choice and to choose the Weyl frame. For example, by making the shift $\alpha_\epsilon \Rightarrow \alpha_\epsilon - (1-\nu)\ln\sqrt{\varphi}$ and redefining the potential by $U \Rightarrow \varphi^{\nu-1}U$, we remove the third term in (12) and obtain the Lagrangian in the Weyl frame. Then, we can redefine $l_\epsilon\varphi \equiv \bar{l}_\epsilon$, introduce the new field $\xi \equiv \varphi^2$ and finally rewrite (12) in a simpler form,

$$\epsilon\mathcal{L}_q^{(1)} = \bar{l}_\epsilon^{-1}\left[\xi\dot{\psi}^2 + m^{-2}\dot{q}^2 - \dot{\xi}\dot{\alpha}_\epsilon\right] + \bar{l}_\epsilon\epsilon\,\mathrm{e}^{2\alpha_\epsilon}\xi^{\nu/2-1}\,U(\sqrt{\xi},q,\psi). \tag{13}$$

Before applying it to studies of cosmologies and horizons in the scalaron theory, we discuss the effective scalaron potential, corresponding to X-potential (5) in more detail.

Note, that the scalaron kinetic term $\sim \dot{q}^2$ is independent of D, while the potential U is simple only for $D=3$, $D=4$. In the Weyl frame (see (8) and (7), respectively), it is easy to derive the effective potentials $U/w(\varphi) \equiv \hat{U}$:

$$U(\varphi,q) = \hat{U}(\varphi,q) = -2\Lambda\varphi\left[1 + q^2/4\lambda^2\Lambda^2\,\varphi^2\right], \qquad D=3, \tag{14}$$

$$U(\varphi,q) = \sqrt{\varphi}\hat{U}(\varphi,q) = -2\Lambda\varphi\left[1 + q^2/\lambda^2\Lambda^2\varphi^2\right]^{1/2} + 2k, \quad D=4. \tag{15}$$

The general effective potential in Lagrangian (13) can be written as

$$U_e(\varphi,x) \equiv \xi^{\nu/2-1}\,U(\sqrt{\xi},q,\psi) = \varphi^{\nu-2}U = -2\Lambda\varphi^{\nu-1}\,v_\nu\,(x) + k_\nu\,\varphi^{-(1+\nu)}, \tag{16}$$

where $x \equiv q/(-2\nu\lambda\Lambda\,\varphi)$, and $v_\nu(x)$ is monotonic concave function having simple expansions

$$v_\nu(x) = 1 + \nu x^2 + O(x^4), \quad v_\nu(x) = 2\nu x\left[1 + \frac{1-\nu}{2\nu}x^{-\sigma} + O(x^{-2\sigma})\right]. \tag{17}$$

With such a simple and regular potential U_e, one might expect that at least qualitative behavior of the solutions of the theory (13) could be analyzed for small and large values of x.

This is true if the theory is integrable. But it is probably not integrable, even in the simplest $D = 3$ case, when $v_1(x) = 1 + x^2$, $k_1 = 0$, and thus $U_e = -2\Lambda(1 + x^2)$ is linear in q^2/ξ. The form of the potential signals that there must exist at least one additional integral beyond the Hamiltonian constraint, and it was derived in [6]. The existence of a third integral, which should allow us to integrate the scalaron model, is doubtful. We still hope to find either a reasonable approximation for the potential or to treat the exact systems using approximate, asymptotic, and qualitative (topological) methods. In the next sections, we attempt presenting a draft panorama of old and new integrals in reasonably general dilaton gravity coupled to scalars, and briefly describe a few simple, intuitive approaches to search for new integrals.

2. INTEGRALS AND INTEGRABILITY IN SIMPLE CASES

Here, we consider a general DSG with one scalar ψ that may be a standard field or the scalaron. The general Weyl-frame Lagrangian can be written as

$$\mathcal{L}^{(2)}_{\text{dgs}} = \sqrt{-g}\left[\varphi R + Z(\varphi)(\nabla\psi)^2 + V(\varphi, \psi)\right] \tag{18}$$

(note that, in Lagrangians, we usually write $+V$ instead of the standard $-V$). For the scalaron we have $\psi = q$, $Z = Z_q = -1/(m^2\varphi)$ and the potentials are given above. For the standard scalar $Z_\psi \sim -\varphi$, but the results presented below are applicable to more general Z functions. In our notation, negative signs of Z correspond to positive kinetic energies of the scalar fields, but our classical consideration is fully applicable to both signs. The general model (18) with a general potential V is not integrable in any sense. One of the strongest obstructions to integrability is the dependence of Z on φ, and the usual simplifying assumption is that the Z functions are independent of φ. With this restriction, there exists a special class of "multi-exponential" potentials, for which the DSG theories with any number of scalar fields can be reduced to the Toda–Liouville systems and exactly solved.[1] For their static-cosmological reductions, analytic solutions were explicitly derived. Here, we try to expand this class of models. It is well known that for the constant field $\psi \equiv \psi_0$, the two-dimensional theory can be exactly solved with any potential $V(\varphi) \equiv V(\varphi, \psi_0)$. In fact, it degenerates to a one-dimensional theory, because the dilaton and the metric satisfy the D'Alembert equation, and thus depend on one variable $\tau = a(u) + b(v)$. Thus, the complete solution can be derived by solving elementary one-dimensional equations defined by Lagrangian (12) with constant ψ and q.

A more general *static-cosmological reduction* of the two-dimensional theory (18) is

$$\epsilon\mathcal{L}^{(1)} = -l_\epsilon^{-1}\left[Z(\varphi)\,\dot{\psi}^2 + 2\dot{\alpha}_\epsilon\dot{\varphi}\right] + l_\epsilon \epsilon\, \mathrm{e}^{2\alpha_\epsilon}\, V(\varphi, \psi), \tag{19}$$

where V is the Weyl-frame potential and $h \equiv \epsilon\, \mathrm{e}^{2\alpha_\epsilon}$ can be identified with the Weyl-frame metric if we choose the gauge $l_\epsilon = 1$, which we call the LC gauge. In this gauge the equations

[1] This class includes all previously considered integrable two-dimensional DSG, which are reviewed in [8]. The first DSG of the Liouville type ("bi-Liouville"), which generalizes the so-called Jackiw and CGHS models, was proposed and solved in paper [7], the results of which were essentially generalized in [5] and [6].

of motion are most directly reduced to the parent two-dimensional dilaton gravity in the light-cone (u, v) coordinates. Also useful is the Hamilton gauge, in which $l_\epsilon Z^{-1}(\varphi) \equiv \bar{l}_\epsilon = 1$; using in addition the new variable ξ defined by $d\xi \equiv Z^{-1}(\varphi)\, d\varphi$, we have

$$\epsilon \mathcal{L}^{(1)} = -\bar{l}_\epsilon^{-1} \left[\dot{\psi}^2 + 2\dot{\alpha}_\epsilon \dot{\xi} \right] + \bar{l}_\epsilon \epsilon\, \mathrm{e}^{2\alpha_\epsilon} U(\xi, \psi), \tag{20}$$

where $U(\xi, \psi) \equiv Z(\varphi) V(\varphi, \psi)$. This gauge is especially convenient when there are many scalar fields with the same $Z(\varphi) = -\varphi$. If the potential is the sum of linear exponents of the scalar fields, there is a class of explicitly integrable models including the Toda–Liouville dilaton gravity theories (see [7–10]). Other gauge choices used in the theory of black holes and in cosmological models may also be exploited in our dynamical formulation, but they are usually less convenient in the context of our search for integrals of dynamical systems.

Let us first write the dynamical equations in the *Hamilton gauge*:

$$\ddot{\xi} + h\, U = 0, \quad 2\, \ddot{\psi} + h\, U_\psi = 0, \quad \ddot{F} + h\, U_\xi = 0, \quad \dot{\psi}^2 + \dot{F}\dot{\xi} + h\, U = 0. \tag{21}$$

Here, the last equation is the Hamiltonian constraint, $F \equiv \ln|h| \equiv 2\alpha_\epsilon$, and the lower indices φ, ψ denote the corresponding partial derivatives. Our first approach to integrability of this system was based on taking linear combinations of the equations,

$$c_1 \ddot{\xi} + c_2 \ddot{\psi} + c_3 \ddot{F} + \epsilon\, \mathrm{e}^F [c_1\, U + c_2\, U_\psi/2 + c_3\, U_\xi] = 0. \tag{22}$$

If for a given potential U the expression in brackets vanishes, we find an integral which is linear in momenta. We can also find a general solution of the partial differential equation for U giving corresponding integrals. For multi-exponential potentials we can instead try to construct, with the aid of (22), the Liouville or Toda equations choosing different c_n. This approach, first proposed in [7], was applied to constructing an integrable ("bi-Liouville") two-dimensional DSG with $U_\psi = 0$.[1] Taking $c_3 = 1$, $c_2 = 0$, $c_1 = \pm\lambda_1$, we find two Liouville equations for $F_\pm \equiv (F \pm \lambda_1 \xi)$, if the potential U satisfies two equations, $[U_\xi \pm \lambda_1\, U]\, \exp(\mp \lambda_1 \xi) = g_\pm$. These conditions are satisfied by the simplest bi-exponential potential,

$$2\lambda_1 U = g_+\, \exp(\lambda_1 \xi) - g_-\, \exp(-\lambda_1 \xi),$$

while $F_\pm$ satisfies the Liouville equations and the correspondent integrals

$$\ddot{F}_\pm + \epsilon\, g_\pm\, \exp F_\pm = 0, \quad \dot{F}_\pm^2 + 2\epsilon\, g_\pm\, \exp F_\pm = C_\pm. \tag{23}$$

This gives the complete solution of the problem: $4\lambda_1\, \dot{\psi}^2 = C$ may be taken as the third integral, and the constraint (21) is satisfied if $C + C_+ - C_- = 0$.

Note, that if we tried a more general potential, with $-\lambda_1$ replaced by λ_2, we would immediately find that the necessary condition for the existence of two integrals is $\lambda_2 = -\lambda_1$. A similar solvable model is given by one-dimensional reduction (20) of theory (18). Taking $V = g_1 \exp(2\lambda_1 \psi) + g_2 \exp(2\lambda_2 \psi)$, $Z = -\varphi \equiv -\mathrm{e}^{-\xi}$ and $\psi_1 \equiv (F - \xi)/2$, $\psi_2 \equiv (F + \xi)/2$, $\psi_3 \equiv \psi$, we first rewrite the potential term in (20) as

$$hU = \epsilon\, \mathrm{e}^F Z\, V = -(g_1\, \mathrm{e}^{q_1} + g_2\, \mathrm{e}^{q_2}), \quad q_i \equiv 2(\psi_i + \lambda_i\, \psi_3),$$

[1] In [7], Z was constant, but here we can take arbitrary $Z(\varphi)$ as it is included in U.

then find that the kinetic term is diagonal in $\dot{q}_i^2$ if $\lambda_1 \lambda_2 = 1$, and see that the Lagrangian is

$$\mathcal{L} = -\bar{l}^{-1}[-\mu_1 \dot{q}_1^2 + \mu_2 \dot{q}_2^2 + \dot{\psi}_3^2] - \bar{l}\,(g_1\, \mathrm{e}^{q_1} + g_2\, \mathrm{e}^{q_2}), \quad 4\mu_1 = -(1-\lambda_1^2)^{-1}, \quad \mu_2 = -\lambda_1^2\,\mu_1.$$

The solution is obtained as in the previous case, but the formulation can obviously be generalized to any number of scalar fields. If there are $N-2$ scalar fields with the same coupling to the dilaton, $Z(\varphi)$, the *multi-exponential* models are defined by the Lagrangian

$$\mathcal{L} = -\bar{l}^{-1} \sum_1^N \varepsilon_n \dot{\psi}_n^2 - \bar{l} \sum_1^N g_n\, \mathrm{e}^{q_n}, \quad q_n \equiv \sum_1^N \psi_m a_{mn}, \tag{24}$$

where ψ_1, ψ_2 were defined above, $\varepsilon_1 = -1$, $\varepsilon_n = 1$, $n > 1$. The properties of this multi-exponential theory depend on the symmetric matrix $\hat{A} \equiv \hat{a}^T \hat{\varepsilon}\, \hat{a}$, where $\hat{\varepsilon}_{mn} \equiv \varepsilon_m \delta_{mn}$. If $\hat{A}$ is diagonal, we have N-Liouville model, which was directly solved also in the two-dimensional case. Such models were met in simple compactifications of supergravity theories [1].

More complex compactifications may lead to integrable models that are related to the Toda systems. In fact, any matrix A_{mn}, which is a direct sum of a diagonal $(L \times L)$ matrix $\gamma_n^{-1}\delta_{mn}$ and a symmetric matrix $\bar{A}_{mn}$, can be represented in the form $\hat{A} \equiv \hat{a}^T \hat{\varepsilon}\, \hat{a}$, if the sum of γ_n^{-1} is a certain function of the matrix elements $\bar{A}_{mn}$, see [10]. Then, if $\bar{A}_{mn}$ is a direct sum of Cartan matrices, the differential equations for the ψ functions can be reduced to L-independent Liouville (Toda $\mathcal{A}_1$) equations and the higher-rank Toda system. We call it the *Toda–Liouville* system, application of which to the dilaton gravity was discussed in [10]. The class of multi-exponential models is the main source of completely integrable dilaton gravity theories with many degrees of freedom. Most of them can be analytically solved in one-dimensional case, while the two-dimensional N-Liouville and the simplest Toda–Liouville theories were classically solved in two dimensions.

The one-dimensional exact solutions can easily be quantized (at least, formally) as their Hamiltonians split into the sum of the Liouville Hamiltonians $p^2 + 2g\,\mathrm{e}^q$, where we suppress the indices of p_n, q_n and factors ε_n in (24). Introducing the new canonical variables

$$P \equiv \sqrt{p^2 + 2g\,\mathrm{e}^q}, \quad Q \equiv -\,\mathrm{arccosh}\,(1 + p^2 g^{-1} \mathrm{e}^{-q}),$$

we find the complete solution of the N-Liouville theory:

$$H = \sum \varepsilon_n P_n^2, \quad Q_n = P_n(\tau - \tau_n), \quad \exp\,(-q_n) = g_n P_n^{-2} \cosh^2 \left[\frac{P_n(\tau - \tau_n)}{2} \right], \tag{25}$$

where we "recovered" sign factors enabling to satisfy the constraint $H = 0$. In fact, we were a bit sloppy in writing these simple formulae, and one must be more accurate, especially when quantizing this apparently trivial theory. Anyway, this is a constrained (gauge) theory and should be quantized as such. In more complex, especially, in nonintegrable gauge models discussed below quantizing is more tricky and will not be discussed here.

With small number of scalar fields, there may exist other integrals, not related to the Toda–Liouville class. The simplest integrals emerge when $U(\xi, \varphi)$ is independent either of ξ or of ψ. Remember that $U \equiv Z(\varphi) V(\xi, \psi)/w(\varphi)$, where w is the Weyl transition function defined above, and thus V can be nontrivial, even when $U_\xi = U_\psi = 0$. If the potential is a constant, the system is obviously explicitly integrable. The integrals can exist also when the potential satisfies some weaker restriction; we first discuss the potential $U = g\,\xi + v(\psi)$. In the simplest case, when $g = 0$, the theory can be explicitly solved if, in addition, $v(\psi)$ is a simple function. The main simplification is that $F = a\tau$; assuming that $v = \bar{g}\,\psi^p$, we find the following rather tractable equations to be discussed in a moment:

$$\psi''(F) = -\gamma p\, \mathrm{e}^F \psi^{p-1}, \quad \xi''(F) = -\gamma\, \mathrm{e}^F \psi^p, \quad \gamma \equiv \epsilon \bar{g}\, a^{-2}. \tag{26}$$

The simplest model is $U = g$. Then, with $N-2$ scalar fields ψ_n ($3 \leqslant n \leqslant N$), we have N integrals. Supposing that all the scalars have the same $|Z(\varphi)|$, we find

$$\dot{F} = C_1 \equiv a, \quad \dot{\xi} + (g/a)\, h = C_2, \quad \dot{\psi}_n = C_n, \quad C_1 C_2 = \sum \epsilon_n\, C_n^2,$$

where $\epsilon_n = +1$ for the normal field and $\epsilon_n = -1$ for the phantom (ghost) field ψ_n, and thus

$$\xi - \xi_0 = -\hat{h} + \ln|\hat{h}|^{\,\delta}, \quad \hat{h} \equiv \frac{h}{h_0}, \quad h_0 \equiv \frac{C_1^2}{g}, \quad \delta \equiv \frac{C_2}{C_1} = \sum \epsilon_n \left(\frac{C_n}{C_1}\right)^2. \tag{27}$$

The first equation is what we call the *portrait* of the integrable DSG system. We consider only the cosmological part of it, which looks like the portrait of a more realistic cosmological system. It is instructive to draw it in the $(\varphi, \hat{h})$-plane supposing that $\xi - \xi_0 = \ln\varphi$. One then can see that the *separatrix* $\delta = 0$ describes the solution with one *horizon*, while the other two correspond to $|\delta| = 1$. In addition, the portrait has singularities. To get the physics portrait in the $(\varphi, \hat{h})$-plane, one should first identify all possible singularities in this plane: the saddle point of the horizon, $(1, 0)$, the nodes $(0, 0), (0, 1), (\infty, 0), (\infty, 1)$ and the most interesting node $(1/e, 1)$ — the cross-point of all cosmological solutions. This portrait topologically resembles the part of that derived in [7]. However, even these parts are globally inequivalent, because the structure of their characteristic nodes is not equivalent under differentiable topological mappings.

For the linear and quadratic potentials $v(\psi)$, we can derive explicit analytic solutions. If $p = 1$, we find a very simple solution of Eqs. (26) and of the constraint:

$$\psi = -\gamma\,(\mathrm{e}^F + C_1 F + C_2), \quad \xi - \xi_0 = \gamma^2 \left[\frac{\mathrm{e}^{2F}}{4} + (C_2 - 2C_1)\,\mathrm{e}^F + C_1 F\,\mathrm{e}^F - \frac{C_1^2\,F}{2}\right].$$

These expressions are similar to the simplest solutions discussed above and we do not discuss them in detail. One can see that the horizon emerges when $C_1 = 0$ and $F \to -\infty$: then, $h \equiv \epsilon\,\mathrm{e}^F \to 0$, while ξ is finite, $\xi \to \xi_0$. The topological portrait, $\xi(h)$, of this system is somewhat more complex than the previous one and we cannot discuss it here.

In the quadratic case, $p = 2$, the ψ equation is linear and can be solved in terms of the Bessel functions, $\psi = Z_0(2\sqrt{2\gamma\,\mathrm{e}^F})$. The $\xi(F)$ for the horizon separatrix,

$$\xi - \xi_0 = \psi_0^2 \sum_{n=0}^{\infty} d_n\,(n+1)^{-2}\,(-2\gamma\,\mathrm{e}^F)^{n+1}, \quad d_n \equiv \sum_{m=0}^{n} [m!\,(n-m)!],$$

generalizes the expressions above, while the complete portrait contains similar entire function multiplied by powers of F and depending on an additional parameter δ. It follows that the qualitative properties of this model do not radically differ from the previous cases.

When $v = \bar{g}^p$ and $p \neq 0, 1, 2$, we may expect much richer picture. If p is real, $p \neq 0, 1, 2$, Eq. (26) is called the *Emden–Fowler equation*. It was studied in great detail, especially, for integer and rational values of p, see, e.g., [11]. As is well known, it has a very important but simple *enveloping* solution,

$$\psi_p = c_p\, \mathrm{e}^{\lambda_p F}, \quad \lambda_p = -(p-2)^{-1}, \quad c_p = [-\gamma p\,(p-2)^2]^{\lambda_p}, \quad -\gamma p > 0, \quad p \neq 0, 2, \quad (28)$$

that suggests the transformation $\psi = \psi_p\, z(\lambda_p\, F)$. Then, we can replace Eq. (18) for ψ by the autonomous equation for z or, equivalently, by the first-order system for z and $y \equiv z'$:

$$z'' + z' + z - z^{p-1} = 0, \quad (z^{p-1} - z - 2\,y)\, dy = y\, dz. \quad (29)$$

Although, in general, these equations cannot be solved analytically, we may regard them almost integrable, at least for rational p. Indeed, the behavior of their solutions was analyzed in great detail, including exact asymptotic behavior. More recently, a few significant results were obtained on classical integrability of Eq. (29), see [12, 13].

Before turning to a more general approach for searching additional integrals, we mention an interesting integrable potential $U = g\xi + \bar{g}\psi^2$. Then, F satisfies Eqs. (23), and thus the potential e^F is given by Eq. (25); the linear equation for ψ (see (26)) is related to the Legendre equation and can be explicitly solved. Moreover, for certain discrete values of the parameters, the potential in the equations for ψ and ξ, which is proportional to $\cosh^{-2}(c\tau)$, becomes "transparent". Then, the solution can be expressed in terms of elementary functions [14].

3. ON A MORE SYSTEMATIC APPROACH TO SEARCH OF INTEGRALS

In our general approach to the DG dynamical systems, we try to take care for nice features both of the Hamiltonian and LC formulations. In [7] and [15], we proposed a first-order dynamical system, which is related to the Hamilton dynamics, but does not coincide with it. This allowed us to find rather nontrivial integrals, to study some global properties of the solutions, and to construct convergent analytic expansions near horizons and singularities. Our method can be applied to many scalars, but here we mostly consider the models with one scalar having arbitrary coupling $Z(\varphi)$ and arbitrary potential $V(\varphi, \psi)$.

Introducing the new momentum-like variables χ, η, ρ, [1]

$$\dot{\varphi} = \chi, \quad Z(\varphi)\, \dot{\psi} = \eta, \quad Z(\varphi)\, \dot{F} = \rho, \quad (30)$$

we rewrite the main dynamical equations and the constraint (20) in the form

$$\dot{\chi} + hV = 0, \quad 2\dot{\eta} + hV_\psi = 0, \quad \dot{\rho} + h(ZV)_\varphi = 0, \quad \chi\rho + hZV + \eta^2 = 0. \quad (31)$$

Equations (30) and the first three equations of (31) are equivalent to the canonical system with the Hamiltonian, which is equal to the constraint divided by Z. To completely solve this system, we must search for two additional constraints canonically commuting with the

[1] It is easy to check $\chi = -p_F$, $\eta = -p_\psi/2$, $\rho = -Zp_\varphi$, where the momenta are derived for the Lagrangian (19) with LC gauge $l_\epsilon = 1$.

Hamiltonian and with each other. As far as we are interested in the classical theory, we usually will look for integrability in the Liouville sense. Moreover, we are mostly interested in explicit analytic expressions for the solutions or, at least, in exact analytic relations between the "physical" variables h, φ, ψ.

Looking at the system (31), we immediately see that there exist integrals $\eta = \eta_0$, if $V_\psi = 0$, and $\rho = \rho_0$, if $(ZV)_\varphi = 0$. The best studied is the first case of the ψ-independent potential. When $\eta_0 = 0$, the dilaton gravity can be explicitly solved with arbitrary potentials; the simplest case $(ZV)_\varphi = (ZV)_\psi = 0$ was solved above. In [7], we derived three dilaton gravity models with $\eta = \eta_0$, for which there exists one more integral, and demonstrated how they can be explicitly solved. Two of them are closely related to the Liouville theory, but the third one requires a generalization of the Liouville integral that will be discussed in a moment.

Recalling the previous section, we change the variable τ to ξ, what is defined by the relations $\chi d\tau = d\varphi \equiv Z d\xi$. Then, we rewrite Eqs. (30), (31) denoting the derivative $d/d\xi$ by the prime and introducing useful notation: $H(\xi) \equiv h/\chi$, $G(\xi) \equiv \eta/\chi$, and $U \equiv ZV$ as above. The main independent equations for χ, η, ψ, H now have a very compact form:

$$\psi' = G, \quad H' = -G^2 H, \quad \chi' + UH = 0, \quad 2\eta' + U_\psi H = 0, \tag{32}$$

where the second equation is in fact equivalent to the constraint. The *extended system* contains two equations with ρ (see (30), (31)):

$$\chi F' - \rho = 0, \quad \rho' + U_\xi H = 0, \quad \chi G' = UH(G - U_\psi/2U), \tag{33}$$

where we add the explicit equation for G, which can replace the last equation in (32).

The system (32) is most convenient for deriving the *solutions near horizons* and in asymptotic regions as well as for studying their general properties. For example, a very important property of its solutions is that $(\ln H)' = -G^2 < 0$. This property does not depend on the potential and is true for any number of scalar fields if their Z functions are negative, as was first shown in [5]. Indeed, in this case, the constraint equation can be written as

$$\Phi' \equiv (\ln H)' = -Z_0 \sum_{n=0}^{N} Z_n^{-1}(\xi) \left(\frac{\eta_n}{\chi}\right)^2, \quad \eta_n \equiv Z_n \dot{\psi}_n \equiv \chi Z_n Z_0^{-1} \psi_n'. \tag{34}$$

For normal fields $Z_n < 0$ for all n, and thus $\Phi' < 0$. For anomalous fields, like the scalaron corresponding to the tachyonic vecton, Z may be positive, and then the sign of Φ' may be negative or positive, depending on concrete solutions. In case of the same signs, Eq. (34) resembles the second law of thermodynamics and defines an *"arrow of time"* for our dynamical system. If this is not true, the theorem is violated in a very specific way. It may be an interesting point for cosmological modeling.

Our system of equations (32) has other interesting global properties. The general solution of the first two equations can be written in terms of integrals of $G(\xi)$ and $G^2(\xi)$:

$$\psi(\xi) = \psi_0 + \int_{\xi_0}^{\xi} G, \quad H(\xi) = H_0 \exp\left(-\int_{\xi_0}^{\xi} G^2\right). \tag{35}$$

Then, inserting these "solutions" into the third and the forth equations and integrating them, we can write one integral equation for $G(\xi)$ instead of system (32):

$$G(\xi) \equiv \frac{\eta}{\chi} = \left(\eta_0 - \frac{1}{2} \int\limits_{\xi_0}^{\xi} U_\psi H \right) \left(\chi_0 - \int\limits_{\xi_0}^{\xi} UH \right)^{-1}, \tag{36}$$

where $\psi(\xi)$ and $H(\xi)$ are given by Eq. (35). As was discussed in [7] and [15], the standard (regular and nondegenerate) horizon appears when $\chi_0 = \eta_0 = 0$. Then, $h(\xi_0) = 0$, while $G(\xi_0) = U_\psi(\xi_0)/\,2U(\xi_0)$ is finite, if $U_\psi \neq 0$. It follows that G, ψ, H are finite and can be expanded in convergent series around ξ_0, if the potential $U(\xi, \psi)$ is analytic in a neighborhood of (ξ_0, ψ_0). [1] When $U_\psi \equiv 0$, there is the obvious integral of motion $\eta = \eta_0$. As can be seen from the above equations and was proved in [7], there is no horizon, if $\eta_0 \neq 0$; when $\eta_0 = 0$, we have $G \equiv 0$ and return to pure dilaton gravity with horizons.

In simple cases, the integral equation can easily be reduced to a differential one. For example, if $U = u(\xi)\, v(\psi)$ and $U_\psi = 2gU$, the integral equation can be reduced to the second-order differential equation, which is not integrable for arbitrary $u(\xi)$, but is explicitly integrable, if $U_\xi = g_1 U$. This result is quite natural as, in this case, there exist two additional integrals, $\eta = g\chi + \eta_0$, $\rho = g_1\chi + \rho_0$, and therefore the most direct approach is to use the extended differential system. In nonintegrable cases or when there is only one additional integral, this *"master" integral equation* still can be a quite useful analytical tool.

Above, we mostly supposed that the potential U is known and tried to find integrals or directly integrate some equations. Now, we consider a different approach supposing that we do not fix the potential and try to find equations for the potentials allowing some integrals. To get a feeling of the approach, look at Eq. (22). If we take potentials, for which the expression in brackets vanishes, we immediately find that the solution of the homogeneous equation for $\ln U$ depends on an arbitrary function of one variable, $f(c_2\xi - 2c_3\psi)$, and on arbitrary parameter defining solutions of inhomogeneous equation. In general, we thus obtain one integral. However, for linear f we can derive two independent integrals. This approach becomes much more powerful if we use extended system (32), (33). Here, we only briefly outline a generalization of the approach of [7] to finding potentials U, for which the extended differential system has additional integrals (see also [16]).

Generalizing the approach of [7] and the above remarks about possible integrals of motion, let us collect those equations, which can generate such integrals:

$$0 = \rho + UH + \frac{\eta^2}{\chi} = \chi' + UH = \eta' + \frac{U_\psi H}{2} = \rho' + U_\xi H = 0, \tag{37}$$

$$0 = \psi\eta' + \frac{\psi U_\psi H}{2} = \eta\,\psi' - \frac{\eta^2}{\chi} = \xi\rho' + \xi\, U_\xi\, H = \xi'\rho - \rho = 0. \tag{38}$$

The first equation in (37) is the energy constraint, which we denote E_0 and the next items in this chain of equations are denoted by E_i, $i = 1, \ldots, 7$. Now, taking the sum $\sum\limits_0^7 c_i E_i$ with

[1] This is shown in [15]. One can find a detailed discussion of regular solution with horizons, including a generalization of the Szekeres–Kruskal coordinates and examples of singular horizons, in [5].

$c_4 = c_5 = c_6 = c_7 = c_0$, we find that the solutions of Eqs. (37) satisfy the identity

$$[c_1\chi + c_2\,\eta + c_3\,\rho + c_4(\psi\,\eta + \xi\,\rho)]' = \\ = -H\left[(c_1 + c_4)\,U + \frac{c_2 U_\psi}{2} + c_3 U_\xi + c_4\left(\frac{\psi\,U_\psi}{2} + \xi\,U_\xi\right)\right]. \quad (39)$$

Therefore, if the r.h.s. identically vanishes, the l.h.s generates the integral of motion,

$$c_1\chi + c_2\eta + c_3\rho + c_4(\psi\eta + \xi\rho) = I. \quad (40)$$

This means that for the potentials $U(\xi, \psi)$, satisfying the partial differential equation

$$(c_1 + c_4)\,U + \frac{c_2 U_\psi}{2} + c_3 U_\xi + c_4\left(\frac{\psi U_\psi}{2} + \xi\,U_\xi\right) = 0, \quad (41)$$

there exists the corresponding integral of Eqs. (37). Many of the above integrals can be obtained by applying this theorem. The solution of Eq. (41) depends on an arbitrary function of one variable. Using this fact, it is possible, in some simple cases, to derive one more integral. This is true, for example, if the potential is exponential.

In the above example of the potential $U = U_0 \exp{(2g\psi + g_1\xi)}$ with two additional integrals, we immediately find the equation for χ,

$$\chi' = (g^2 + g_1)\chi + (\rho_0 + 2\eta_0 g) + \frac{\eta_0^2}{\chi}, \quad (42)$$

by solving of which we explicitly express ξ, η, ρ, h and ψ as functions of χ. This is sufficient for finding the portrait of this physically interesting system.

To demonstrate the problems, which remain even for apparently simpler systems, we consider the potential $U(\psi)$ also having two additional integrals. The obvious linear integral is $\rho = \rho_0$. To obtain one more integral, suppose that $\psi U_\psi = 2gU$. Then, we have the additional integral of the three differential equations:

$$\psi\eta - (g+1)\chi + \rho_0\xi = I, \quad \chi' = \rho_0 + \frac{\eta^2}{\chi}, \quad \eta\psi' = \frac{\eta^2}{\chi}, \quad \psi\eta' = g\chi'. \quad (43)$$

We can exclude ψ (or, η), and thus get two equations for χ and η (or ψ). But this is not an integrable dynamical system because of its explicit dependence of I on ξ. If we take $\rho_0 = 0$ in the expression for the above integral, the system can be explicitly integrated, but this is only a *"partial" solution.*[1] Unfortunately, here we do not know the transformation to an autonomous system, which helped us in the Emden–Fowler case, see (28), (29).

It should be emphasized that Eqs. (39)–(41) only generate the integrals that are linear in momenta (like the variables χ, η, ρ), but do not allow derivation of possible bilinear integrals. The simplest example of an additional *bilinear integral* can be obtained, if we multiply the last equation in (37) by 2ρ,

$$0 = 2\rho\,(\rho' + U_\xi H) \equiv (\rho^2)' + 2h' U_\xi, \quad (44)$$

[1] This is a typical problem — integrals often have ξ-depending terms that describe a sort of a "back-reaction" of gravity on matter.

and then suppose that U_ξ is a constant, $U_\xi = g$. This gives a new integral, $\rho^2 + 2gh = I$, generalizing the Liouville one. This integral exists, if $U = g\,\xi + v(\psi)$, and we can find $v(\psi)$, for which there exists one more integral, with the aid of Eq. (41). Inserting in it the expression for U, we find that $c_1 + 2c_4 = 0$, and thus find the equation for v:

$$(c_2 + c_4\psi)\, v'(\psi) - 2\, c_4 v(\psi) + 2\, g c_3 = 0. \tag{45}$$

If $c_4 = 0$, the solutions are linear in ψ and we have the additional integral $c_2\eta + c_3\rho = I$. If $c_4 \neq 0$, we find that $v = g\, c_3/c_4 + c_0(\psi + c_2/c_4)^2$ and the integral is

$$-2c_4\chi + c_2\eta + c_3\rho + c_4(\psi\eta + \rho\xi) = I. \tag{46}$$

Some integrals of this sort were first discovered in [7]. To derive them using the present approach, we somewhat generalize this process and find the following nontrivial integral:

$$(a + b\xi)\, \rho^2 + ch + b\eta_0^2 \ln|h| = I, \tag{47}$$

where we used the constraint multiplied by ρ, the identity $\rho H = h'U$, and supposed that $U_\psi = 0$, which gives the integral $\eta = \eta_0$. The solution of the equation for the potential is

$$U = \frac{c}{b} + \frac{c_0}{\sqrt{a + b\xi}},$$

where c_0 is arbitrary. The integral and potential do not coincide with those of [7] and look rather exotic. To reproduce the most interesting integral of [7], we note that here we work exclusively in the Weyl frame $W = 0$ (but omitting hats), while there was used the original frame $W = (1 - \nu)/\varphi$, see Eqs. (4), (6), (7) above.

To return to the general W frame with the arbitrary dilaton kinetic potential $W(\varphi) \equiv W(\xi)$, we thus apply the Weyl transformation $h \mapsto hw$, $V \mapsto V/w$ and, correspondingly,

$$H \mapsto Hw, \quad U \mapsto U/w, \quad F \mapsto F + \ln w, \quad \rho \mapsto \rho + \chi(\ln w)'. \tag{48}$$

This gives us more freedom in search for integrals. For example, if we apply (48) to (44), replace h, U, ρ by $\hat{h} \equiv hw$, $\hat{U} \equiv U/w$, $\hat{\rho}$, transform $\hat{h}'\, \hat{U}_\xi$ into

$$h'\, [w\, \hat{U}_\xi] \,+\, hU\, [U^{-1} w'\, \hat{U}_\xi] \,=\, h'\, [w\, \hat{U}_\xi] - \chi\chi'\, [U^{-1} w'\, \hat{U}_\xi],$$

and make the expressions in the square brackets constant, we will find a new bilinear integral. With this approach, we can recover the complex bilinear integral of [7] and find the new ones. This subject requires more careful investigation and will be discussed elsewhere.

Finally, let us return to our main goal formulated in Introduction and Sec. 1 of the present report — to integrate the equations or, at least, to find a global portrait of the simplest or approximate scalaron cosmologies. Here, we present a realistic example of one additional integral. Consider the integral and the corresponding equation for the potential

$$c_1\chi + (\psi\eta + \rho\xi) = I, \quad 2\beta U + 2\xi U_\xi + \psi U_\psi = 0, \quad \beta \equiv c_1 + 1. \tag{49}$$

The general solution for this equation is

$$U(\xi, \psi) = \psi^{2\alpha}\, \xi^{\beta - \alpha} \mathcal{F}(1 + \psi^2 \xi^{-1}), \tag{50}$$

where $\mathcal{F}(x)$ is an arbitrary function of one variable, α — arbitrary parameter. Taking

$$\mathcal{F} \equiv \mathcal{F}_1 = \left(1 + \frac{\lambda_0^2 \psi^2}{\xi}\right), \quad \alpha = 0, \quad \beta = 0,$$

we get the effective potential for the scalaron model in $D = 3$, see (14), (16). For other dimensions, neglecting the curvature term (i.e., $k_\nu = 0$), Eq. (16) gives

$$\mathcal{F} \equiv \mathcal{F}_\nu = v_\nu(\lambda_0 \psi/\varphi), \quad \alpha = 0, \quad \beta = \frac{\nu - 1}{2}.$$

We still hope to find one more integral for $D = 3$. However, for other dimensions we must look for some approximations.

4. SUMMARY AND OUTLOOK

In conclusion, we summarize the main points of the report. Dilaton gravity with scalars is, in general, not integrable even with formally sufficient number of integrals of motion. The models with massless scalars qualitatively differ from the scalaron models (DSG) that inevitably include nonintegrability. Fortunately, in some physically important cases, the non-integrable systems are partially integrable, and therefore can be effectively studied, at least qualitatively. The solutions near horizons and singularities can be derived analytically — by using exact series expansions or, alternatively, by iterations of the master integral equation. On the other hand, our approach to constructing systems with additional integrals may help to find integrable or partially integrable systems that are qualitatively close to the realistic ones. We demonstrated that there are two kinds of additional integrals: 1) linear in momenta and described by a sufficiently general approach; 2) quadratic in momenta, similar to the Liouville integrals. In addition, we argued that in the dilaton gravity the scalar field equations resemble the well-known generalized Emden–Fowler equations. The rich technology developed for understanding the global structure and asymptotic properties of these equations may prove helpful in the context of scalaron models in cosmology.

Understanding global properties of classical solutions is also desirable for their quantization. The simplest approach was attempted some time ago for classically integrable gravitational systems with minimal number of degrees of freedom (see, e.g., [17]).

We hope that the above panoramic presentation of several new ideas on finding integrals of nonlinear equations of modern cosmological models, which are met in various generalized theories of gravity, may be of interest in studies of their global properties.

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