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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

CORRELATION FUNCTIONS OF CIRCULAR WILSON LOOP WITH LOCAL OPERATORS

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We discuss the correlation function of a circular Wilson loop with one or two local scalar operators in $\mathcal{N} = 4$ supersymmteric Yang–Mills theory. We show that in the case of one local operator such a correlation function is fixed by conformal invariance up to a constant, and in the case of two operators it is fixed up to a single function of two variables. These two variables have a clear geometric meaning: after conformal transformation from $\mathbf{R}^4$ to $AdS_2 \times S^2$, they become the geodesic distances in AdS_2 and S^2. We present some explicit results at weak and strong coupling. We also comment that correlators of infinite line Wilson loop with local operators are the same as those for circular loop.

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1. As is well known, conformal invariance fixes correlation functions to a large extent. The most interesting and important observables in conformal field theory (CFT) are the ones which are substantially constrained but yet nontrivial. For example, it is well known that in any CFT 2- and 3-point functions of local, scalar, gauge-invariant operators are fixed up to a constant. More precisely[2],

$$\mathcal{C}_2 = \langle \mathcal{O}(\vec{a}_1)\mathcal{O}^\dagger(\vec{a}_2)\rangle = \frac{1}{|\vec{a}_1 - \vec{a}_2|^{2\Delta}}, \quad \dim \mathcal{O} = \Delta, \tag{1}$$

and

$$\mathcal{C}_3 = \langle \mathcal{O}_1(\vec{a}_1)\mathcal{O}_2(\vec{a}_2)\mathcal{O}_3(\vec{a}_3)\rangle = \frac{C_{123}}{|\vec{a}_{12}|^{\Delta_1+\Delta_2-\Delta_3}|\vec{a}_{13}|^{\Delta_1+\Delta_3-\Delta_2}|\vec{a}_{23}|^{\Delta_2+\Delta_3-\Delta_1}}, \quad \dim \mathcal{O}_i = \Delta_i. \tag{2}$$

In general, Δ and C_{123} are functions of the coupling constant λ. Hence, we can conclude that computation of $\mathcal{C}_2$ and $\mathcal{C}_3$ is reduced to computation of scalar quantities $\Delta(\lambda)$ and $C_{123}(\lambda)$ as the dependence on the coordinates is fully fixed by conformal invariance.

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[2] In the paper, it is assumed that we performed the Wick rotation to Euclidean space, and by $\vec{a}$ we denote a point in $\mathbf{R}^4$. Strictly speaking, the results for the correlation functions are valid only in the Euclidean space where the operators cannot become null-separated.

Similarly, 4-point function is constrained as follows:

$$
\begin{gathered}
\mathcal{C}_4=\langle\mathcal{O}_1(\vec{a}_1)\mathcal{O}_2(\vec{a}_2)\mathcal{O}_3(\vec{a}_3)\mathcal{O}_4(\vec{a}_4)\rangle=\frac{1}{|\vec{a}_{12}|^{q_1}|\vec{a}_{14}|^{q_2}|\vec{a}_{24}|^{q_3}|\vec{a}_{34}|^{q_4}}G(u,v;\lambda),\\
q_1=-\Delta_1+\Delta_2+\Delta_3-\Delta_4,\quad q_2=\Delta_1-\Delta_2-\Delta_3+\Delta_4,\\
q_3=-\Delta_1+\Delta_2-\Delta_3-\Delta_4,\quad q_4=2\Delta_3.
\end{gathered}
\tag{3}
$$

Here the function $G(u,v;\lambda)$ is conformally invariant and depends on 2 conformally invariant variables (cross ratios) u and v

$$
u=\frac{|\vec{a}_{12}|^2|\vec{a}_{34}|^2}{|\vec{a}_{13}|^2|\vec{a}_{24}|^2},\quad v=\frac{|\vec{a}_{14}|^2|\vec{a}_{23}|^2}{|\vec{a}_{13}|^2|\vec{a}_{24}|^2}. \tag{4}
$$

Note that conformal symmetry does not fix the prefactor in (3) uniquely. However, the ratio of any such prefactors is conformally invariant and, hence, can be absorbed into the redefinition of the function G. Thus, computation of $\mathcal{C}_4$ at a given coupling is equivalent to computation of a single function $G(u,v)$ of 2 variables.

The above results are based only on conformal symmetry and are valid in any CFT. Below we will consider a particular CFT $-\mathcal{N}=4$ supersymmetric gauge theory in the planar limit.

2. Local operators are not the only interesting observables in CFT. Among non local operators the most popular ones are Wilson loops. In $\mathcal{N}=4$ gauge theory they are defined as [1, 2]

$$
W=\frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\left[\int d\tau(iA_\mu\dot{x}^\mu(\tau)+\Phi^I\theta^I(\tau)|\dot{x}(\tau)|)\right],\quad\sum_{I=1}^{6}\theta^I(\tau)\theta^I(\tau)=1. \tag{5}
$$

Here A_μ and Φ^I are the bosonic fields in $\mathcal{N}=4$ gauge theory; $(x^\mu(\tau),\theta^I(\tau))$ is a contour in $\mathbf{R}^4\times S^5$, and N is the rank of the gauge group.

In this paper, we will consider Wilson loops in the form of a circle [3] with $\theta^1=1$, $\theta^2=\ldots\theta^6=0$. Let us comment that a circle is related by a conformal transformation to infinite straight line. However their expectation values are not equal, in fact [4, 5][1],

$$
\langle W_L\rangle=1,\quad\langle W_c\rangle=\frac{2}{\sqrt{\lambda}}I_1(\sqrt{\lambda}). \tag{6}
$$

This is due to anomaly in the special conformal transformation which is not well-defined at infinity in $\mathbf{R}^4$ [6]. However, one should expect that

$$
\frac{\langle W_c\,\mathcal{O}\ldots\rangle}{\langle W_c\rangle}=\frac{\langle W_L\,\mathcal{O}\ldots\rangle}{\langle W_L\rangle}, \tag{7}
$$

that is, the anomaly cancels in the ratio. In [3], this was checked in a number of examples. In this paper, we will concentrate only on the case of circular Wilson loop.

[1]These results are exact to all orders in the ’t Hooft constant λ, I_1 is the Bessel function.

3. Let us consider $\frac{\langle W_c\, \mathcal{O} \ldots \rangle}{\langle W_c \rangle}$, where we inserted n local scalar operators. Such a correlator depends on

$$d_n = 4n - (\Gamma - \Gamma_n) \tag{8}$$

independent variables. Here $4n$ is the total number of coordinates, Γ is the number of generators preserving the circle and Γ_n is the number of generators preserving the circle and n points[1]. One can show that [3] $\Gamma = 6$, $\Gamma_1 = 2$, $\Gamma_2 = \Gamma_3 = \ldots = 0$. Hence, $d_1 = 0$, $d_2 = 2$, $d_3 = 6, \ldots$ This means that correlation function of a circle with just one local operator is fixed by conformal symmetry up to a constant [7, 8] (like 3-point function) and a similar correlator with two local operators is fixed up to a function of 2 variables (like 4-point function).

To construct the general form of the correlators and the 2 conformally invariant variables, we will use the fact that $\mathbf{R}^4$ is conformally equivalent to $AdS_2 \times S^2$. Let us write the metric in $\mathbf{R}^4$ in polar coordinates in 2 different planes

$$ds^2 = dx_1^2 + \ldots + dx_4^2 = dr^2 + r^2 d\psi^2 + dh^2 + h^2 d\phi^2. \tag{9}$$

We will put the circle in the (x_1, x_2) plane

$$x_1^2 + x_2^2 = R^2, \quad x_3 = x_4 = 0. \tag{10}$$

Now we change variables $(r, \psi, h, \phi) \to (\rho, \psi, \theta, \phi)$ as

$$r = \ell \sinh \rho, \quad h = \ell \sin \theta, \quad \ell = \frac{R}{\cosh \rho - \cos \theta} = \frac{\sqrt{(r^2 + h^2 - R^2)^2 + 4R^2 h^2}}{2R}. \tag{11}$$

In these new variables we obtain

$$ds^2 = \ell^2 [d\rho^2 + \sinh^2 \rho d\psi^2 + d\theta^2 + \sin^2 \theta d\phi^2] = \ell^2 ds^2_{AdS_2 \times S^2}. \tag{12}$$

The circle (10) is now parameterized by the angle ψ and is mapped to the boundary of AdS_2. Since the boundary is invariant under isometries of $AdS_2 \times S^2$, we conclude that the 6 conformal transformations preserving the circle become now 6 isometries of $AdS_2 \times S^2$. Let us consider the correlator $\langle W_c\, \mathcal{O} \rangle / \langle W_c \rangle$ in $AdS_2 \times S^2$. Since it is invariant under isometries, it is a constant which we will denote $C_c(\lambda)$. Transforming it back to $\mathbf{R}^4$ and using the fact that the Weyl factor is given by ℓ in (11), we obtain

$$\frac{\langle W_c\, \mathcal{O}(\vec{a}) \rangle}{\langle W_c \rangle} = \frac{C_c(\lambda)}{[\ell(\vec{a})]^{\Delta}} = C_c(\lambda) \left(\frac{4R^2}{(r^2 + h^2 - R^2)^2 + 4R^2 h^2} \right)^{\Delta/2}, \quad \dim \mathcal{O} = \Delta. \tag{13}$$

Here $\vec{a} = (a_1, a_2, a_3, a_4)$, $r^2 = a_1^2 + a_2^2$, $h^2 = a_3^2 + a_4^2$. Computation of this correlator is, hence, equivalent to computation of a singe constant $C_c(\lambda)$.

Now we will consider $\frac{\langle W_c\, \mathcal{O}_1(\vec{a}_1) \mathcal{O}_2(\vec{a}_2) \rangle}{\langle W_c \rangle}$. Again, first, we will consider it in $AdS_2 \times S^2$. Since it is invariant under isometries, it has to be a function F_c of two geodesic distances s_1 and s_2 in AdS_2 and S^2, respectively. Transforming it back to $\mathbf{R}^4$, we obtain

$$\frac{\langle W_c\, \mathcal{O}_1(\vec{a}_1) \mathcal{O}_2(\vec{a}_2) \rangle}{\langle W_c \rangle} = \frac{1}{[\ell(\vec{a}_1]^{\Delta_1} [\ell(\vec{a}_2]^{\Delta_2}} F_c(\mathrm{u}, \mathrm{v}; \lambda), \tag{14}$$

[1]In this counting, we view the circle as a fixed object and consider only those conformal transformations which preserve it. As we will see below, these conformal transformations have a clear geometric meaning.

where Δ_1 and Δ_2 are dimensions of the two operators, and u and v are functions of the geodesic distances in AdS_2 and S^2. It is convenient to choose

$$\begin{aligned} \mathrm{u} &= \cosh s_1 = \cosh \rho_1 \cosh \rho_2 - \sinh \rho_1 \sinh \rho_2 \cos(\psi_2 - \psi_1), \\ \mathrm{v} &= \cos s_2 = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_2 - \phi_1). \end{aligned} \tag{15}$$

Equation (14) is the general form of the correlator based only on symmetry considerations. As an example, let us consider the case of two chiral primary operators in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory[1]

$$\mathcal{O}_1(\vec{a}_1) = c_2 \mathrm{Tr}\,[Z^2(\vec{a}_1)], \quad \mathcal{O}_2(\vec{a}_2) = c_2 \mathrm{Tr}\,[\bar{Z}^2(\vec{a}_2)], \quad Z = \Phi^1 + i\Phi^2, \quad c_2 = \frac{4\pi^2}{\sqrt{2}N}. \tag{16}$$

At week coupling there is a disconnected contribution proportional to the 2-point function

$$\langle \mathcal{O}_1(\vec{a}_1)\mathcal{O}_2(\vec{a}_2)\rangle = \frac{1}{|a_1 - a_2|^4} \Longrightarrow F_{c,0}(\mathrm{u},\mathrm{v}) = \frac{1}{4(\mathrm{u}-\mathrm{v})^2}. \tag{17}$$

Note that from (15) it follows that $\mathrm{u} \geqslant 1$, $\mathrm{v} \leqslant 1$, and $\mathrm{u} = \mathrm{v} \Leftrightarrow \mathrm{u} = \mathrm{v} = 1$, which implies $s_1 = s_2 = 0$. That is, the limit $\mathrm{u} = \mathrm{v}$ is the limit when the two points coincide. The first nontrivial connected contribution at weak coupling can be computed to be [3]

$$F_{c,1}(\mathrm{u},\mathrm{v}) = \frac{\lambda}{16N^2}\frac{1}{\mathrm{u}-\mathrm{v}}. \tag{18}$$

At strong coupling it can be shown that the correlator factorizes to the leading order [3]

$$\frac{\langle W_c\, \mathcal{O}_1(\vec{a}_1)\mathcal{O}_2(\vec{a}_2)\rangle}{\langle W_c\rangle} = \frac{\langle W_c\, \mathcal{O}_1(\vec{a}_1)\rangle}{\langle W_c\rangle}\frac{\langle W_c\, \mathcal{O}_2(\vec{a}_2)\rangle}{\langle W_c\rangle}. \tag{19}$$

Recalling (13), we conclude that the function F in this limit is independent of u and v and is given by $F_c(\mathrm{u},\mathrm{v}) = C_{c,1}(\lambda)C_{c,2}(\lambda)$. In fact, the factorization (19) is universal and is independent of the choice of the operators as long as their dimensions are much less than $\sqrt{\lambda}$ in the large λ limit. For the above operators we get

$$C_{c,1}(\lambda) = C_{c,2}(\lambda) = \frac{\sqrt{2}\sqrt{\lambda}}{8N}. \tag{20}$$

Additional examples can be found in [3].

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[1]The normalization is chosen in the standard way so that their 2-point function is canonically normalized.

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