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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

THE JACOBI IDENTITY FOR GRADED-COMMUTATIVE VARIATIONAL SCHOUTEN BRACKET REVISITED

A. V. Kiselev [1]

Johann Bernoulli Institute for Mathematics and Computer Science,
University of Groningen, Groningen, The Netherlands

This short note contains an explicit proof of the Jacobi identity for variational Schouten bracket in $\mathbb{Z}_2$-graded commutative setup; an extension of the reasoning and assertion to the noncommutative geometry of cyclic words (see [1]) is immediate. The reasoning refers to the product bundle geometry of iterated variations (see [2]); no *ad hoc* regularizations occur anywhere in this theory.

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The Jacobi identity for variational Schouten bracket $[\![\,,]\!]$ is its key property in several cohomological theories. For example, one infers that the BV-Laplacian Δ or quantum BV-operator $\mathbf{\Omega}^\hbar = \mathbf{i}\hbar\,\Delta + [\![\mathbf{S}^\hbar, \cdot\,]\!]$ are differentials in the Batalin–Vilkovisky formalism (available literature is immense; let us refer to [2] and [3]), or one deduces that $\partial_{\boldsymbol{\mathcal{P}}} = [\![\boldsymbol{\mathcal{P}}, \cdot\,]\!]$ yields the Poisson–Lichnerowicz complex for every variational Poisson bivector $\boldsymbol{\mathcal{P}}$, see [1]. Likewise, a realization of zero-curvature geometry for the inverse scattering via the classical master equation $[\![\mathbf{S}, \mathbf{S}]\!] = 0$ opens a way for deformation quantization, which is not restricted to the BV-quantization of Chern–Simons models over threefolds[2]. Therefore, it is mandatory to have a clear vision of the geometry of iterated variations and understand the mechanism for validity of the Jacobi identity.

A self-regularized calculus of variations, including the definitions of Δ and $[\![\,,]\!]$ and a rigorous proof of their interrelations, is developed in [2]. We reserved this theory's key element, the proof of Theorem 4. (iii) with Jacobi's identity for $[\![\,,]\!]$, to a separate paper which is this note. Referring to [2] for details and discussion, let us recall that — in a theory of variations for fields over the space-time — each integral functional[3] or every test shift of the fields brings its own copy of the domain of integration into the setup; the locality of couplings between (co)/vectors attached at the domains' points ensures a restriction to diagonals in the accumulated products of bundles, whereas the operational

[1] E-mail: A.V.Kiselev@rug.nl

[2] In fact, all these BV, Poisson, or IST models are examples of variational Lie algebroids [4] and their encoding by $\mathbf{Q}^2 = 0$. The construction of gauge automorphisms for the $\mathbf{Q}$-cohomology determines the next generation of such structures, with new deformation quantization parameters beyond the Planck constant.

[3] Let all functionals that take field configurations to number be *integral* in this note; formal (sums of) products of functionals such as $\exp\big((\mathbf{i}/\hbar)\mathbf{S}^\hbar\big)$ are dealt with by using the Leibniz rule, see [2, § 2.5].

definitions of Δ and $[\![\,,\,]\!]$ are on-the-diagonal reconfigurations of such couplings[1]. We expect that the reader is familiar with the concept and notation from § 1–2.4 in [2]. In particular, we let the notation for total derivatives which stem from integrations by parts keep track of the variations' arguments, so that $((\delta s)\overleftarrow{\partial}/\partial\mathbf{y})(\mathbf{y})\cdot\overrightarrow{\partial}\mathcal{L}\bigl(\mathbf{x},[\mathbf{q}],[\mathbf{q}^\dagger]\bigr)/\partial\mathbf{q}_\mathbf{x}$ at $\mathbf{y}=\mathbf{x}$ becomes $\delta s(\mathbf{y})\cdot(-\overrightarrow{d}/d\mathbf{y})(\overrightarrow{\partial}\mathcal{L}\bigl(\mathbf{x},[\mathbf{q}],[\mathbf{q}^\dagger]\bigr)/\partial\mathbf{q}_\mathbf{x})$ on that diagonal, see Example 2.4 on p. 34–36 of [2]. Similarly, the variational derivatives with respect to (anti)fields $\mathbf{q}$ or $\mathbf{q}^\dagger$ keep track of the test shifts which those variations come from: e.g., the formula above yields[2] a term in $\delta s(\mathbf{y})\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y})(\mathcal{L}(\mathbf{x},[\mathbf{q}],[\mathbf{q}^\dagger]))$ at $\mathbf{y}=\mathbf{x}$. This simplifies the reasoning[3].

Theorem. *Let F, G, and H be $\mathbb{Z}_2$-parity homogeneous functionals*; *denote by $|\cdot|$ the grading so that $(-)^{|\cdot|}$ is the parity. The variational Schouten bracket $[\![\,,\,]\!]$ satisfies the shifted-graded Jacobi identity* (cf. Eq. (28) in Theorem 4. (iii) on p. 30 versus Eq. (36) on p. 37 in [2]),

$$[\![F,[\![G,H]\!]]\!]=[\![[\![F,G]\!],H]\!]+(-)^{(|F|-1)(|G|-1)}\,[\![G,[\![F,H]\!]]\!]. \tag{1}$$

The operator $[\![F,\cdot\,]\!]$ is a graded derivation of $[\![\,,\,]\!]$: identity (1) *is the Leibniz rule for it.*

Proof. The logic is straightforward[4] as soon as the matching of (co)vectors and reconfigurations of couplings are understood in [2, § 1–2]. We consider first the l.-h.s. of (1). By construction, we have that $[\![G,H]\!]=(G(\mathbf{x}_2))\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_3)(H(\mathbf{x}_3))-(G(\mathbf{x}_2))\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_3)(H(\mathbf{x}_3))$. Now expanding $[\![F,[\![G,H]\!]]\!]=(F(\mathbf{x}_1))\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_{23})([\![G,H]\!])-(F(\mathbf{x}_1))\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_{23})([\![G,H]\!])$, we obtain the sum of eight enumerated terms[5]:

$$\langle 1\rangle\quad F(\mathbf{x}_1)\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_{\mathbf{2}3})\,G(\mathbf{x}_2)\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_3)\,H(\mathbf{x}_3)+$$

$$\langle 2\rangle\quad +(-)^{|G|}\,F(\mathbf{x}_1)\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_1)\cdot G(\mathbf{x}_2)\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_{2\mathbf{3}})\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_3)\,H(\mathbf{x}_3)-$$

$$\langle 3\rangle\quad -F(\mathbf{x}_1)\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_{\mathbf{2}3})\,\bigl(G(\mathbf{x}_2)\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2)\bigr)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_3)\,H(\mathbf{x}_3)-$$

$$\langle 4\rangle\quad -(-)^{|G|-1}\,F(\mathbf{x}_1)\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_1)\cdot G(\mathbf{x}_2)\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_{2\mathbf{3}})\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_3)\,H(\mathbf{x}_3)-$$

$$\langle 5\rangle\quad -F(\mathbf{x}_1)\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_{\mathbf{2}3})\,G(\mathbf{x}_2)\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_3)\,H(\mathbf{x}_3)-$$

---

[1] It is readily seen from the proof below that composite objects such as brackets of functionals retain a kind of memory of the way how they were produced; in effect, variational derivatives detect the traces of original objects' own geometries, whence a variation within one of them does not mar any of the others.

[2] In this note we let the arrow over a variational derivative indicate the direction along which all derivatives act — but not the opposite direction along which the test shifts were transported prior to any integration by parts (cf. [2]); we thus have $\overrightarrow{\delta\mathbf{s}}(\mathbf{S})=\int d\mathbf{y}\,\Bigl\{\bigl\langle\delta s(\mathbf{y}),\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y})(\mathbf{S}(\mathbf{x}))\bigr\rangle+\bigl\langle\delta s^\dagger(\mathbf{y}),\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y})(\mathbf{S}(\mathbf{x}))\bigr\rangle\Bigr\}$ and $(\mathbf{S})\overleftarrow{\delta\mathbf{s}}=\int d\mathbf{y}\,\bigl\{\bigl\langle(\mathbf{S}(\mathbf{x}))\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}),\delta s(\mathbf{y})\bigr\rangle+\bigl\langle(\mathbf{S}(\mathbf{x}))\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}),\delta s^\dagger(\mathbf{y})\bigr\rangle\bigr\}$, where the diagonal $\mathbf{y}=\mathbf{x}$ is wrought by the coupling $\langle\,,\,\rangle$, see [2, § 2.2–3], and we display the integration variable $\mathbf{x}$ in the functional $\mathbf{S}$.

[3] With a bit more care taken of the order in which the factors follow each other in products, and by using the $\mathbb{Z}_2$-graded Leibniz rule for left- and right-directed derivations, we show that the claim and proof of the main theorem hold true in the setup of cyclic words and brackets of necklaces (see [1] and references therein).

[4] Obviously, the l.-h.s. of (1) does *not* contain second variational derivatives of F, whereas the r.-h.s. *does*. We show that it is precisely these terms and none others which cancel out in the r.-h.s.

[5] We denote by $\mathbf{z}_{ij}$ the integration variables which label the variations falling — in the outer brackets in (1) — on the ith or jth functional by the Leibniz rule (let F be first and so on, $1\leqslant i<j\leqslant 3$); for convenience, we highlight i in $\mathbf{z}_{ij}$, when the variation falls on the ith functional — and j otherwise.

$$
\begin{aligned}
&{}^{\langle 6\rangle} && - F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_1)\cdot G(\mathbf{x}_2)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{23}})\,\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_3)\,H(\mathbf{x}_3)+\\
&{}^{\langle 7\rangle} && + F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{23}})\,G(\mathbf{x}_2)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_3)\,H(\mathbf{x}_3)+\\
&{}^{\langle 8\rangle} && + F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_1)\cdot G(\mathbf{x}_2)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{23}})\,\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_3)\,H(\mathbf{x}_3).
\end{aligned}
$$

Arguing as above, we see that the term $[\![[\![F,G]\!],H]\!]$ in the r.-h.s. of (1) is[1]

$$
\begin{aligned}
&{}^{\langle 9\rangle} && F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{12}})\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2)\,G(\mathbf{x}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_3)\,H(\mathbf{x}_3)+\\
&{}^{\langle 1\rangle} && + F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2)\,G(\mathbf{x}_2)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{12}})\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_3)\,H(\mathbf{x}_3)-\\
&{}^{\langle 10\rangle} && - F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{12}})\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_2)\,G(\mathbf{x}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_3)\,H(\mathbf{x}_3)-\\
&{}^{\langle 5\rangle} && - F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_2)\,G(\mathbf{x}_2)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{12}})\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_3)\,H(\mathbf{x}_3)-\\
&{}^{\langle 11\rangle} && - (-)^{|G|-1}\,F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{12}})\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2)\,G(\mathbf{x}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_3)\,H(\mathbf{x}_3)-\\
&{}^{\langle 3\rangle} && - F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_1)\cdot\Big(\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2)\,G(\mathbf{x}_2)\Big)\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{12}})\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_3)\,H(\mathbf{x}_3)+\\
&{}^{\langle 12\rangle} && + (-)^{|G|}\,F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{12}})\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_2)\,G(\mathbf{x}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_3)\,H(\mathbf{x}_3)+\\
&{}^{\langle 7\rangle} && + F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_2)\,G(\mathbf{x}_2)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{12}})\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_3)\,H(\mathbf{x}_3).
\end{aligned}
$$

In the same way, we obtain the term $[\![G,[\![F,H]\!]]\!]$ not yet multiplied by the extra sign factor:

$$
\begin{aligned}
&{}^{\{1\}} && G(\mathbf{x}_2)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{13}})\,F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_3)\,H(\mathbf{x}_3)+\\
&{}^{\{2\}} && + (-)^{|F|}\,G(\mathbf{x}_2)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_2)\cdot F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{13}})\,\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_3)\,H(\mathbf{x}_3)-\\
&{}^{\{3\}} && - G(\mathbf{x}_2)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{13}})\,\Big(F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_1)\Big)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_3)\,H(\mathbf{x}_3)-\\
&{}^{\{4\}} && - (-)^{|F|-1}\,G(\mathbf{x}_2)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_2)\cdot F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{13}})\,\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_3)\,H(\mathbf{x}_3)-\\
&{}^{\{5\}} && - G(\mathbf{x}_2)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{13}})\,F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_3)\,H(\mathbf{x}_3)-\\
&{}^{\{6\}} && - G(\mathbf{x}_2)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_2)\cdot F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{13}})\,\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_3)\,H(\mathbf{x}_3)+\\
&{}^{\{7\}} && + G(\mathbf{x}_2)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_2)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{13}})\,F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_3)\,H(\mathbf{x}_3)+\\
&{}^{\{8\}} && + G(\mathbf{x}_2)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_2)\cdot F(\mathbf{x}_1)\,\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_1)\cdot\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{13}})\,\overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_3)\,H(\mathbf{x}_3).
\end{aligned}
$$

Let us now use the $\mathbb{Z}_2$-graded commutativity assumption for the setup. Transporting the variations of F leftmost, we restore the lexicographic order $F \prec G \prec H$. Finally, we

[1] The labelling of terms by superscripts $\langle 1\rangle$ – $\langle 8\rangle$ shows their matching with summands in the l.-h.s. of (1) or, for the index running from $\langle 9\rangle$ to $\langle 12\rangle$, points at the four second-order variations of F which cancel out in the two r.-h.s. summands in Jacobi's identity.

multiply $\llbracket G, \llbracket F, H, \rrbracket\rrbracket$, reordered as above, by the sign factor $(-)^{(|F|-1)(|G|-1)}$; this yields[1]

$$
\begin{aligned}
&{\scriptstyle\langle 10\rangle}\; (-)^{|F|-1}\, \overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{13}})\, F(\mathbf{x}_1) \overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_1) \cdot G(\mathbf{x}_2) \overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_2) \cdot \overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_3)\, H(\mathbf{x}_3) + \\
&{\scriptstyle\langle 2\rangle}\; + (-)^{|G|-1}\, F(\mathbf{x}_1) \overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_1) \cdot G(\mathbf{x}_2) \overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_2) \cdot \overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{13}}) \overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_3)\, H(\mathbf{x}_3) + \\
&{\scriptstyle\langle 12\rangle}\; + (-)^{|F|+|G|}\, \overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{13}}) \Big(F(\mathbf{x}_1) \overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_1)\Big) \cdot G(\mathbf{x}_2) \overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_2) \cdot \overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_3)\, H(\mathbf{x}_3) - \\
&{\scriptstyle\langle 6\rangle}\; - F(\mathbf{x}_1) \overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_1) \cdot G(\mathbf{x}_2) \overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_2) \cdot \overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{13}}) \overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_3)\, H(\mathbf{x}_3) + \\
&{\scriptstyle\langle 9\rangle}\; + (-)^{|G|}\, \overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{13}})\, F(\mathbf{x}_1) \overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_1) \cdot G(\mathbf{x}_2) \overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_2) \cdot \overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_3)\, H(\mathbf{x}_3) + \\
&{\scriptstyle\langle 4\rangle}\; + (-)^{|G|}\, F(\mathbf{x}_1) \overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_1) \cdot G(\mathbf{x}_2) \overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_2) \cdot \overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{13}}) \overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_3)\, H(\mathbf{x}_3) + \\
&{\scriptstyle\langle 11\rangle}\; + \overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{13}})\, F(\mathbf{x}_1) \overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_1) \cdot G(\mathbf{x}_2) \overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_2) \cdot \overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_3)\, H(\mathbf{x}_3) + \\
&{\scriptstyle\langle 8\rangle}\; + F(\mathbf{x}_1) \overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_1) \cdot G(\mathbf{x}_2) \overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z}_2) \cdot \overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{13}}) \overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_3)\, H(\mathbf{x}_3).
\end{aligned}
$$

Terms $\langle 1\rangle$–$\langle 8\rangle$ are present in the r.-h.s. of (1) and terms $\langle 9\rangle$–$\langle 12\rangle$ cancel out; it is only the indices $\langle 3\rangle$ and $\langle 12\rangle$ which require special attention. Consider $\langle 3\rangle$ in $\llbracket\llbracket F, G\rrbracket, H\rrbracket$; by relabelling the integration variables, $\mathbf{y} \rightleftarrows \mathbf{z}$ (i.e., by swapping the test shifts), we obtain

$$-F(\mathbf{x}_1) \overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{z}_1) \cdot \Big(\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{12}})\, G(\mathbf{x}_2)\Big) \overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2) \cdot \overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_3)\, H(\mathbf{x}_3).$$

The variation's argument in parentheses has grading $|G| - 1$, which yields the sign factor $(-)^{(|G|-1)-1}$, when the left-acting parity-odd variation $\overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2)$ is brought to the other side of its argument, becoming $\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2)$. Hence $(-)^{|G|-2} \overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2)\big(\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{23}})\,(G(\mathbf{x}_2))\big) \overset{(i)}{=} (-)^{|G|-1}\, \overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{23}})\big(\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2)\,(G(\mathbf{x}_2))\big) \overset{(ii)}{=} (-)^{|G|-1}\,(-)^{|G|-1}\, \overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{23}})\big((G(\mathbf{x}_2))\times \overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_2)\big)$, where (i) the parity-odd variations are swapped and (ii) the inner variational derivative is transported around G of grading $|G|$. The two sign factors cancel out, and the overall minus matches that near $\langle 3\rangle$ in the l.-h.s. of (1).

We do the same with $\langle 12\rangle$. Consider such a term in $(-)^{(|F|-1)(|G|-1)}\llbracket G, \llbracket F, H\rrbracket\rrbracket$; clearly, the factor $(-)^{|G|}$ is irrelevant because it is present also near $\langle 12\rangle$ in $\llbracket\llbracket F, G\rrbracket, H\rrbracket$. Transporting the parity-odd variation $\overrightarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{13}})$ around the object of grading $|F| - 1$ in parentheses, we gain the factor $(-)^{|F|-2}$, which cancels out with $(-)^{|F|}$. Next, relabel $\mathbf{y} \rightleftarrows \mathbf{z}$, which gives

$$F(\mathbf{x}_1) \overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{z_{13}}) \overleftarrow{\delta}/\delta\mathbf{q}^\dagger(\mathbf{y}_1) \cdot G(\mathbf{x}_2) \overleftarrow{\delta}/\delta\mathbf{q}(\mathbf{y}_2) \cdot \overrightarrow{\delta}/\delta\mathbf{q}(\mathbf{z_{13}})\, H(\mathbf{x}_3).$$

The parity-odd variations follow in the order which is reverse with respect to that in $\langle 12\rangle$ in $\llbracket\llbracket F, G\rrbracket, H\rrbracket$, hence these terms cancel out. The proof is complete. □

Variations $\delta\mathbf{s}$ act via graded Leibniz rule on products of integral functionals, e.g., $F \cdot \llbracket G, H\rrbracket$; within composite objects like $\llbracket G, H\rrbracket$, they act also by derivation w.r.t. own

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[1]For each term labelled by $\{1\}$–$\{8\}$ in $\llbracket G, \llbracket F, H, \rrbracket\rrbracket$, let us calculate the product of three signs: one, which was written near the respective summand, the other, which comes from the reorderings to $F \prec G$, and the third, $(-)^{(|F|-1)(|G|-1)}$; here is the list: $\{1\}$: $(-)^{(|F|-1)\cdot|G|}(-)^{(|F|-1)(|G|-1)} = (-)^{|F|-1}$, $\{2\}$: $(-)^{|F|}(-)^{|F|\cdot|G|}(-)^{(|F|-1)(|G|-1)} = (-)^{|G|-1}$, $\{3\}$: $-(-)^{(|F|-2)\cdot|G|}(-)^{(|F|-1)(|G|-1)} = (-)^{|F|+|G|}$, $\{4\}$: $-(-)^{|F|-1}(-)^{(|F|-1)\cdot|G|}(-)^{(|F|-1)(|G|-1)} = -1$, $\{5\}$, $\{6\}$: $-(-)^{|F|\cdot(|G|-1)}(-)^{(|F|-1)(|G|-1)} = (-)^{|G|}$, $\{7\}$, $\{8\}$: $(-)^{(|F|-1)\cdot(|G|-1)}(-)^{(|F|-1)(|G|-1)} = +1$.

geometries of the blocks G, H; variations are graded-permutable in each block. Neither Δ nor $[\![\, , \,]\!]$ depend on a choice of normalized test shift $\delta \mathbf{s}$. This yields (1) and $\Delta^2(F \cdot G \cdot H) = 0$.

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