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COSMOLOGICAL SOLUTIONS IN NONLOCAL MODELS

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A nonlocal modified gravity model with an analytic function of the d'Alembert operator that has been proposed as a possible way of resolving the singularities problems in cosmology is considered. We show that the anzats that is usually used to obtain exact solutions in this model provides a connection with $f(R)$ gravity models.

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INTRODUCTION

General relativity (GR) being a very efficient and simple theory of gravity featuring the second order equations of motion suffers from certain weaknesses. Among them is the initial singularity problem which arises in the framework of the inflationary paradigm. This is the reflection of UV incompleteness of GR. One possibility to improve the ultraviolet behavior and even to get a renormalizable theory of quantum gravity is to add higher-derivative terms to the Einstein–Hilbert action. As one of the first papers we can mention [1] where curvature squared corrections were considered. Unfortunately, this model (and models with more than two but finite number of derivatives in equations of motion in general) has ghosts. An intriguing possibility to overcome this problem is to consider a nonlocal gravity with infinitely many derivatives.

The theoretical motivation behind an introduction of infinite derivative nonlocal corrections into a local theory is the string field theory (SFT) [2]. Such corrections naturally arise in the SFT and usually consist of exponential functions of the d'Alembertian operator acting on fields. The majority of nonlocal cosmological models motivated by such structures explicitly include an analytic or meromorphic function of the d'Alembertian operator [3–9].

Both GR and the modified gravity models yield a nonintegrable system of equations of motion with only particular analytic solutions known. Having an analytic solution however

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is crucial in considering perturbations which are in turn the cornerstone of any cosmological model. Needless to say that finding analytic solutions in nonlocal nonlinear equations is an extremely hard task. Some studies of nonlocal modifications of GR that resulted in analytic solutions can be found in [5, 6, 9].

The key to finding solutions in the nonlocal gravity models of interest is to employ an ansatz which relates finite powers of the d'Alembertian operator acting on the scalar curvature. The anzats itself reduces the initial nonlocal model to an effective local model with more than two but finite number of derivatives. This does not mean that ghosts should appear as any anzats is a relation in the background while perturbations enjoy all the new properties of the full nonlocal structure. Provided there is a background configuration satisfying the anzats, one simplifies the problem of solving the equations of motion considerably.

1. ACTION AND EQUATIONS OF MOTION FOR STRING-INSPIRED NONLOCAL GRAVITY

The nonlocally modified gravity proposed in [5] is described by the following action:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R + \frac{1}{2}R\mathcal{F}\left(\frac{\Box}{M_*^2}\right)R - \Lambda + \mathcal{L}_{\rm M}\right], \tag{1}$$

where M_P is the Planck mass; Λ is the cosmological constant; M_* is the mass scale at which the higher derivative terms in the action become important; $\mathcal{L}_{\rm M}$ is the matter Lagrangian. We use the convention where the metric g has the signature $(-,+,+,+)$. M_* is the mass scale at which the higher derivative terms in the action become important. An analytic function $\mathcal{F}\left(\Box/M_*^2\right) = \sum\limits_{n\geqslant 0} f_n\Box^n$ is an ingredient inspired by the SFT. The operator $\Box$ is the covariant d'Alembertian. In the case of an infinite series we have a nonlocal action.

Introducing dimensionless coordinates $\bar{x}_\mu = M_*x_\mu$ and $\bar{M}_P = M_P/M_*$, we get $\mathcal{F}(\Box/M_*^2) = \mathcal{F}(\bar{\Box})$, where $\bar{\Box}$ is the d'Alembertian in terms of dimensionless coordinates. We shall use dimensionless coordinates only (omitting the bars).

A straightforward variation of action (1) yields the following system:

$$\frac{1}{2}[M_P^2 + 2\mathcal{F}(\Box)R]\,(2R^\mu_\nu - \delta^\mu_\nu R) = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\Big[g^{\mu\rho}\partial_\rho\Box^l R\partial_\nu\Box^{n-l-1}R +$$
$$+ g^{\mu\rho}\partial_\nu\Box^l R\partial_\rho\Box^{n-l-1}R - \delta^\mu_\nu\left(g^{\rho\sigma}\partial_\rho\Box^l R\partial_\sigma\Box^{n-l-1}R + \Box^l R\Box^{n-l}R\right)\Big] +$$
$$+ 2(g^{\mu\rho}\nabla_\rho\partial_\nu - \delta^\mu_\nu\Box)\mathcal{F}(\Box)R - \frac{1}{2}R\mathcal{F}(\Box)R\delta^\mu_\nu - \Lambda\delta^\mu_\nu + T^\mu_\nu, \tag{2}$$

where ∇_μ is the covariant derivative; T^μ_ν is the energy–momentum tensor of matter. The trace equation is useful to get exact solutions:

$$M_P^2 R - \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\left(\partial_\mu\Box^l R\partial^\mu\Box^{n-l-1}R + 2\Box^l R\Box^{n-l}R\right) - 6\Box\mathcal{F}(\Box)R = 4\Lambda - T^\mu_\mu. \tag{3}$$

2. THE ANSATZ FOR FINDING EXACT SOLUTIONS

It has been shown in [5, 6, 8, 9] that the ansatz

$$\Box R = r_1 R + r_2, \tag{4}$$

with constants $r_1 \neq 0$ and r_2, is useful in finding exact solutions.

If the scalar curvature R satisfies (4), then equations (2) are

$$\frac{1}{2}\left[M_P^2 + 2\left(\mathcal{F}(r_1)R + \frac{r_2}{r_1}(\mathcal{F}(r_1) - f_0)\right)\right](2R^\mu_\nu - \delta^\mu_\nu R) = T^\mu_\nu +$$
$$+ \mathcal{F}'(r_1)\left[\partial^\mu R \partial_\nu R - \frac{\delta^\mu_\nu}{2}\left(g^{\sigma\rho}\partial_\sigma R \partial_\rho R + r_1\left(R + \frac{r_2}{r_1}\right)^2\right)\right] - \Lambda\delta^\mu_\nu +$$
$$+ 2\mathcal{F}(r_1)\left[\nabla^\mu\partial_\nu R - \delta^\mu_\nu(r_1 R + r_2)\right] - \frac{\delta^\mu_\nu}{2}\left[\mathcal{F}(r_1)R^2 - \frac{r_2^2}{r_1^2}(\mathcal{F}(r_1) - f_0)\right], \tag{5}$$

where $\mathcal{F}'$ is the first derivative of $\mathcal{F}$ with respect to the argument.

We proceed to consider a traceless radiation along with a cosmological constant. Under condition (4) the trace equation with $T^\mu_\mu = 0$ becomes especially simple:

$$AR + \mathcal{F}'(r_1)\left(2r_1 R^2 + \partial_\mu R \partial^\mu R\right) + B = 0, \tag{6}$$

where the constants A and B are defined as follows:

$$A = 4\mathcal{F}'(r_1)r_2 - M_P^2 - 2\frac{r_2}{r_1}(\mathcal{F}(r_1) - f_0) + 6\mathcal{F}(r_1)r_1, \quad B = 4\Lambda + \frac{r_2}{r_1}M_P^2 + \frac{r_2}{r_1}A.$$

The simplest way to get a solution to Eq. (6) is to impose $\mathcal{F}'(r_1) = 0$ and to put $A = B = 0$. These relations fix values of r_1, r_2 and the cosmological constant:

$$r_2 = -\frac{r_1[M_P^2 - 6\mathcal{F}(r_1)r_1]}{2[\mathcal{F}(r_1) - f_0]}, \quad \Lambda = -\frac{r_2 M_P^2}{4r_1} = M_P^2\frac{[M_P^2 - 6\mathcal{F}(r_1)r_1]}{8[\mathcal{F}(r_1) - f_0]}. \tag{7}$$

Under conditions $A = B = \mathcal{F}'(r_1) = 0$ the complete set of equations (5) simplifies to

$$2\mathcal{F}(r_1)(R + 3r_1)G^\mu_\nu = T^\mu_\nu + 2\mathcal{F}(r_1)\left[g^{\mu\rho}\nabla_\rho\partial_\nu R - \frac{1}{4}\delta^\mu_\nu\left(R^2 + 4r_1 R + r_2\right)\right]. \tag{8}$$

In general, one is required to include radiative sources to get exact solutions of all equations [6]. Let us emphasize that equations are general and we do not take into account the properties of the metric.

3. RELATION BETWEEN THE ANZATS AND R^2 MODIFIED GRAVITY

It is well known [10] that the $f(R)$ gravity model, described by the action

$$S_f = \int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}f(R) + \mathcal{L}_{\rm M}\right), \tag{9}$$

has the following equations:

$$M_P^2\left(f^{(1)}(R)R^\mu_\nu - \frac{1}{2}f(R)\delta^\mu_\nu - (g^{\mu\rho}\nabla_\rho\partial_\nu - \delta^\mu_\nu\Box)f^{(1)}(R)\right) = T^\mu_\nu, \tag{10}$$

where $f^{(1)}(R)$ is the first derivative of $f(R)$ with respect to R.

From (10) for traceless matter, we get the following trace equation:

$$f^{(1)}(R)R - 2f(R) + 3\Box f^{(1)}(R) = 0. \tag{11}$$

One can see that at

$$f(R) = \frac{\mathcal{F}(r_1)}{M_P^2}\left[R^2 + 6r_1R + 3r_2\right], \tag{12}$$

Eq. (11) coincides with the ansatz (4). Moreover, Eqs. (8) are equivalent to (10). Note that condition (7) gives the following connection:

$$M_P^2 = \frac{2}{r_1}\left[3\mathcal{F}(r_1)r_1^2 - (\mathcal{F}(r_1) - f_0)r_2\right].$$

We proved that any solution of the modified gravity model (9) with $f(R)$ given by (12) and traceless matter is a solution of the initial system of Eqs. (2) on condition that $A = B = \mathcal{F}'(r_1) = 0$ and the anzats (4) is satisfied.

CONCLUSIONS

We have shown that any solution of the R^2 modified gravity model (9), either without matter, or with the radiation, is a solution of the corresponding SFT inspired nonlocal gravity models (1). We do not assume some special form of the metric to prove this result. Note that the initial nonlocal model is not totally equivalent to an R^2 model. Full consideration of the model includes both the background solution and perturbations, which may not satisfy the ansatz (4) in general. The analysis of the cosmological perturbations in the considered nonlocal models is given in [8]. Note that the R^2 modified gravity is very well studied [10] and looks as a realistic modification of gravity. In particular, the modern cosmological and astrophysical data [11] confirm the predictions of the Starobinsky inflationary model [12].

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