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ON INTEGRABLE ISOSPIN PARTICLE SYSTEM ON HIGH DIMENSIONAL QUATERNIONIC SYSTEMS

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We explicitly construct the projection map of a fibration of odd-dimensional complex projective space over quaternionic projective one. Performing a Hamiltonian reduction by $U(1)$ subset of the isometries of both total space and bundle, we construct an integrable system of free particle on $\mathbb{HP}^n$ with the presence of Yang's monopole.

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INTRODUCTION

Integrable systems (IS) play a crucial role in all branches of the modern theoretical physics. In fact, any perturbation theory is a small deformation of a known integrable system. However, there are no many known ISs around. Therefore, the introduction of any new IS can be considered a remarkable step. By far most of the known ISs are based on the flat space. On the other hand, many interesting physical systems give rise to quantum mechanical problems on curved spaces. Say, the slow motion of a particle in the gravitational field of a Black hole is an example of mechanical system on curved space. Thus, the construction of the integrable mechanical systems on curved spaces is desirable both from the viewpoint of applications and for its own interest.

A common way of constructing new integrable systems is by performing a Hamiltonian reduction from a higher dimensional more simple system. Due to the large number symmetries of the high dimensional system, after fixing some set of integrals of motion we obtain still integrable lower dimensional more complicated one.

Using the first and the second Hopf maps, performing a Hamiltonian reduction, from the most simple free particle system in the 4D and 8D Euclidean spaces one obtains more complicated 3- and 5-dimensional Kepler-like systems with the presence of Dirac and Yang monopoles, respectively.

On the other hand, each Hopf map is the first members in the family of higher dimensional spheres over the projective spaces, recall that $\mathbb{CP}^1 = S^2$ and $\mathbb{HP}^1 = S^4$:

$$S^{2k+1}/S^1 \simeq \mathbb{CP}^k, \quad S^{4k+3}/S^3 \simeq \mathbb{HP}^k, \tag{1}$$

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or, combining them (it is possible!):

$$\mathbb{CP}^{2n+1}/S^2 \simeq \mathbb{HP}^n. \tag{2}$$

We will perform the Hamiltonian reduction taking as intergrals of motion the Nöther constants corresponding to the isometries of the total space, which act by isometries also in the bundle.

The resulting system represents a free particle system on $\mathbb{HP}^n$ moving in the field of (BPST) instanton. Besides the academic value this system is interesting also in another context. The simplest, one-dimensional quaternionic Landau problem on $\mathbb{H}P^1 = S^4$ [1] has been used previously for developing the model of the "four-dimensional Hall effect" [2], and, for this reason, it attracted much attention (see, e.g., [3] and the brief review [4]). Nevertheless, all these studies were restricted to the systems on $\mathbb{H}P^1 = S^4$, and there were no attempts to consider even the higher dimensional quaternionic Landau problem.

The paper entirely relies on [5].

1. THE GEOMETRIC CONSTRUCTION

Now, let us pass to the explicit construction of these fibrations. We will follow the method developed in [6]. We start from the $(2n+2)$-dimensional complex plane $\mathbb{C}^{2n+2} \simeq \mathbb{H}^{n+1}$ with complex coordinates λ or quaternionic ones: $v_i = \lambda_{2i-1} + j\lambda_{2i}$ $(i = 1, \ldots, n+1)$. By definition the coordinates

$$q_\alpha = v_\alpha v_{n+1}^{-1} \equiv v_\alpha \frac{\bar{v}_n}{||v_n||^2}, \quad \alpha = 1, \ldots, n, \tag{3}$$

define a chart on the quaternionic projective space $\mathbb{H}P^n$.

The inverse formulas look as follows:

$$v_\alpha = q_\alpha v_{n+1} = q_\alpha(\lambda_{2n+1} + j\lambda_{2n+2}) = \lambda_{2\alpha-1} + j\lambda_{2\alpha}. \tag{4}$$

Multiplying the last equation by λ_{2n+2}^{-1}, one finds

$$q_\alpha(z_{2n+1} + j) = z_{2\alpha-1} + jz_{2\alpha}, \tag{5}$$

where the quantities $z_r = \lambda_r/\lambda_{2n+2}$ $(r = 1, \ldots, 2n+1)$ define a chart on the complex projective space $\mathbb{C}P^{2n+1}$. It is clear that any coordinate of $\mathbb{C}P^{2n+1}$ by itself defines a chart on a $\mathbb{C}P^1 \simeq S^2$. In particular, one can consider as such the last coordinate z_{2n+1}.

We can rewrite (5) in the following form:

$$z_{2\alpha-1} + jz_{2\alpha} = q_\alpha(u + j), \quad z_{2n+1} = u. \tag{6}$$

The form of transition functions can be easily found from the construction described above.

For our further consideration it is convenient, instead of the quaternionic coordinates q, to use complex coordinates w which we introduce by the following formula:

$$q_\alpha = w_{2\alpha-1} + jw_{2\alpha}. \tag{7}$$

In these coordinates (6) takes the following form:

$$z_{2\alpha-1} = w_{2\alpha-1}u - \bar{w}_{2\alpha}, \quad z_{2_\alpha} = w_{2\alpha}u + \bar{w}_{2\alpha-1}, \quad z_{2n+1} = u. \tag{8}$$

The transition functions are obvious from the construction.

2. THE REDUCTION

The projection map (8) enables us to perform the Hamiltonian reduction from the free particle system on $\mathbb{CP}^{2k+1}$ to a system on $\mathbb{HP}^n$ containing the Yang monopole.

We start from the natural Lagrangian/Hamiltonian on $\mathbb{CP}^{2n+1}$:

$$L_0 = \frac{\dot{\bar{z}} \cdot \dot{z}}{1 + z\bar{z}} - \frac{(\dot{\bar{z}}z)(\dot{z}\bar{z})}{(1 + z\bar{z})^2}, \quad H = (1 - z\bar{z})\left(p\bar{p} + (zp)(\bar{z}\bar{p})\right). \tag{9}$$

Using (8) it reads

$$L = \frac{\dot{q}\dot{\bar{q}}}{1 + q\bar{q}} - \frac{(\bar{q}\dot{q})(\dot{\bar{q}}q)}{(1 + q\bar{q})^2} + \frac{(\dot{u} + A)(\dot{\bar{u}} + \bar{A})}{(1 + u\bar{u})^2}, \tag{10}$$

where A is the complex part of the quaternionic vector potential:

$$A = j\left.\frac{(\bar{u} - j)\bar{q}dq(u + j)}{1 + w\bar{w}}\right|_{\mathbb{C}}. \tag{11}$$

A for $n = 1$ coincides with the one for Yang monopole. This is the reason why we consider it a natural generalization to higher dimensions.

For simplicity, we will split the quaternionic coordinates into the sum of two complex ones:

$$q_\alpha = w_{2\alpha-1} + jw_{2\alpha}, \tag{12}$$

and define antisymmetric matrix

$$\Omega = \mathrm{diag}\,(\epsilon, \epsilon \ldots), \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{13}$$

We can extend the transformation (8) to the canonical one by adding the following transformation rule for the conjugated momenta:

$$p_\mu = \frac{\bar{u}}{1 + u\bar{u}}\pi_\mu + \frac{1}{1 + u\bar{u}}\Omega_{\mu\nu}\bar{\pi}_\nu, \quad p_{2n+1} = p_u - \frac{1}{1 + u\bar{u}}\left(\bar{u}w^\mu\pi_\mu - \Omega_{\mu\nu}w^\mu\bar{\pi}_\nu\right). \tag{14}$$

It is an exercise to check that the canonical transformation (8), (14) leads to the following form of the Hamiltonian:

$$H_0 = \bar{P}P + (1 + u\bar{u})^2 p_u\bar{p}_u, \tag{15}$$

where we have introduced the covariant momenta

$$P_\mu = \pi_\mu - \imath\bar{w}_\mu\frac{I_3}{1 + w\bar{w}} - \Omega_{\mu\nu}w^\nu\frac{I_+}{1 + w\bar{w}}, \tag{16}$$

with the $su(2)$ generators $I_\pm, I_3$ defining the isometries of S^2:

$$I_3 = -\imath(up_u - \bar{u}\bar{p}_u), \quad I_+ = \bar{p}_u + u^2 p_u, \quad I_- = p_u + \bar{u}^2\bar{p}_u, \tag{17}$$

$$\{I_3, I_\pm\} = \pm\imath I_\pm, \quad \{I_+, I_-\} = 2\imath I_3. \tag{18}$$

The $P\bar{P}$ is understood with respect to the standard metrics on $\mathbb{HP}^n$ rewritten in complex basis (ω).

For the most interesting Poisson brackets we have

$$\{w^\mu, P_\nu\} = \delta^\mu_\nu, \quad \{P_\mu, P_\nu\} = -2\frac{\Omega_{\mu\nu}}{1+w\bar{w}} I_+, \quad \{P_\mu, \bar{P}_\nu\} = \imath \frac{\Omega_{\mu\nu} I_3}{(1+w\bar{w})^2}. \tag{19}$$

It is clear that the operator $I^2 = I_+ I_- + I_3^2$ acts by isometries of both the total space and the bundle of the fibration (2). Thus, to perform the Hamiltonian reduction we should just impose a condition $I^2 = s^2 = \mathrm{const}$. The variables I will play the role of isospin variables.

CONCLUSIONS

Using explicitly constructed fibrations of odd-dimensional projective spaces over quaternionic projective ones, we have constructed a new integrable system of a free particle on $\mathbb{HP}^n$ moving in the field of Yang's monopole. The resulting system possesses a $Sp(2n)$ algebra of Nöther constants of motion of $\mathbb{HP}^n$ as well as a cubic algebra of hidden symmetries, thus remaining a superintegrable one. Such a large number of symmetries allow us to modify the system manually by adding a quadratic potential $w\bar{w}$ keeping it integrable. In contrast to the spherical (Higgs) [8] and $\mathbb{C}P^n$ [7] oscillators, whose hidden symmetries are of the second order in momenta, our model has additional constants of motion, which are of the third order in momenta.

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