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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

INFORMATION METRIC FROM RIEMANNIAN SUPERSPACES: LAST DEVELOPMENTS

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A status report about the importance of the Fisher information metric in classical and quantum systems described by Riemannian non-degenerated superspaces is presented.

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INTRODUCTION

In the following, we review the results of [1 and references therein], where the Fisher information metric corresponding to a generalized Hitchin "on-shell" Lagrangian prescription has been calculated. As the Lagrangian of this particular supermetric brings us localized states showing a Gaussian behaviour, it is of clear interest to analyze the probabilistic and information theoretical meaning of such a geometry. The Fisher method considers a familiy of probability distributions, characterized by certain number of parameters. The metric components are then defined by considering derivatives in different "directions" in the parameters space. That is, measuring "how distant" two distinct sets of parameters put apart the corresponding probability distributions. In our case, the parameters of interest in the metric solution are $\mathbf{a}$ and $\mathbf{a}^*$, which could indicate the residual effects of supersymmetry given that they survive even when "turning off" all the fermionic fields. In the present work we will consider two different approaches to the problem. First, we will follow the "generalized Hitchin prescription", identifying our Lagrangian (calculated at the solution) with the probability distribution. Then, we will introduce a new proposal: we will take the state probability current of the emergent metric solution of the superspace as being itself the probability distribution. The results of the two approaches will be compared in order to infer the physical meaning of the $\mathbf{a}$ and $\mathbf{a}^*$ parameters appearing in the pure fermionic part of the superspace metric.

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1. GENERALIZED HITCHIN PRESCRIPTION FOR THE PROBABILITY DISTRIBUTION

Following the generalized Hitchin prescription, we identify the probability distribution with the super-Lagrangian, evaluated at solution [1], $g_{AB}(t) = \mathrm{e}^{(A(t)+\xi\varrho(t))} g_{AB}(0)$. Thus, the probability distribution density takes the form

$$P(z^A, a, a^*) := -\mathcal{L}|_{g_{AB}(t)} = \exp\left[\frac{1}{2}(A(t) + \xi\varrho(t))\right]\mathcal{L}_0, \tag{1}$$

where

$$\mathcal{L}_0 \equiv \mathcal{L}(g_{ab}(0)) = m\sqrt{\overset{\circ}{\omega}{}^{\nu}\overset{\circ}{\omega}{}^{\mu} + \mathbf{a}\dot{\theta}^{\alpha}\dot{\theta}_{\alpha} - \mathbf{a}^*\dot{\bar{\theta}}{}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}}. \tag{2}$$

Calculating the $\mathbf{a}$ and $\mathbf{a}^*$ derivatives of our probability distribution, we can now write down the Fisher metric components

$$\begin{aligned}
G_{aa} &= \int dx^4 P^{-1}\left[\frac{\partial P}{\partial \mathbf{a}}\right]^2 = \frac{\mathcal{L}_0}{16}\int dt\left(\frac{\mathbf{a}^*}{|\mathbf{a}|}\Xi(t;|a|) + \right.\\
&\qquad\qquad \left. + \frac{2m^2}{\mathcal{L}_0^2}\dot{\theta}^{\alpha}\dot{\theta}_{\alpha}\right)^2 \exp\left[\frac{1}{2}(A(t)+\xi\varrho(t))\right],\\
G_{a^*a^*} &= \int dx^4 P^{-1}\left[\frac{\partial P}{\partial \mathbf{a}^*}\right]^2 = \frac{\mathcal{L}_0}{16}\int dt\left(\frac{\mathbf{a}}{|\mathbf{a}|}\Xi(t;|a|) - \right.\\
&\qquad\qquad \left. - \frac{2m^2}{\mathcal{L}_0^2}\dot{\bar{\theta}}{}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}\right)^2 \exp\left[\frac{1}{2}(A(t)+\xi\varrho(t))\right],\\
G_{aa^*} &= g_{a^*a} = \int dx^4 P^{-1}\frac{\partial P}{\partial \mathbf{a}}\frac{\partial P}{\partial \mathbf{a}^*} = \frac{\mathcal{L}_0}{16}\int dt\Big(\Xi(t;|a|)^2 + \\
&\qquad\qquad + 4\frac{m^4}{\mathcal{L}_0^4}\dot{\theta}^{\alpha}\dot{\theta}_{\alpha}\dot{\bar{\theta}}{}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}\Big)\exp\left[\frac{1}{2}(A(t)+\xi\varrho(t))\right].
\end{aligned} \tag{3}$$

2. STATE PROBABILITY CURRENT AS DISTRIBUTION

Our second approach consists in identifying the state probability density (zero component of the probability current) of the solution as the probability density itself.

The zero component of the probablity current can be obtained by making

$$j_0(t) = 2E^2 g_{ab}(t) g^{ab}(t). \tag{4}$$

Then, putting $\mathcal{K}_0 \equiv 32E^2|\alpha|^2$, we have $j_0(t) = \frac{1}{16}\mathcal{K}_0 \exp\left[-2\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + 2c_2 + 2\xi\varrho(t)\right]$, $\dfrac{\partial j_0}{\partial|\mathbf{a}|} = \dfrac{1}{8}\mathcal{K}_0\Xi(t;|a|)\exp\left[-2\left(\dfrac{m}{|\mathbf{a}|}\right)^2 t^2 + 2c_2 + 2\xi\varrho(t)\right]$.

Therefore, taking $P \equiv j_0(t)$, we get

$$G_{aa} = \int dx^4 P^{-1} \left[\frac{\partial P}{\partial \mathbf{a}}\right]^2 = \left(\frac{\mathbf{a}^*}{|\mathbf{a}|}\right)^2 \mathcal{I}_\Xi, \tag{5}$$

$$G_{a^*a^*} = \int dx^4 P^{-1} \left[\frac{\partial P}{\partial \mathbf{a}^*}\right]^2 = \left(\frac{\mathbf{a}}{|\mathbf{a}|}\right)^2 \mathcal{I}_\Xi, \tag{6}$$

$$G_{aa^*} = G_{a^*a} = \int dx^4 P^{-1} \frac{\partial P}{\partial \mathbf{a}} \frac{\partial P}{\partial \mathbf{a}^*} = \mathcal{I}_\Xi, \tag{7}$$

where $\mathcal{I}_\Xi$ corresponds to the integral of the temporal part:

$$\mathcal{I}_\Xi \equiv \frac{1}{16}\mathcal{K}_0 \int dt\, \Xi(t;|a|)^2 \exp\left[-2\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + 2c_2 + 2\xi\varrho(t)\right].$$

3. GENERALIZED HITCHIN PRESCRIPTION WITH ZERO FERMIONS

When putting all fermion to zero, the derivative of the time-dependent exponential and the $\mathcal{L}_0$ initial value "on-shell" Lagrangian reduce, respectively, to $\Xi(t;|a|)|_{\theta=\chi=0} = \frac{2m^2}{|\mathbf{a}|^3}t^2$ and $\mathcal{L}_0|_{\theta=\chi=0} = m\sqrt{\dot{x}^\mu \dot{x}_\mu} = m$, where the last equality makes explicit the relativistic constraint $v^2 \equiv \dot{x}^\mu \dot{x}_\mu = 1$. In that case, and writing the complex parameters as $\mathbf{a} = |\mathbf{a}|\, \mathrm{e}^{i\phi}$, the metric components take the simple form

$$G_{ab} = \mathcal{I}(m, c_1, c_2, |\mathbf{a}|) \begin{pmatrix} \mathrm{e}^{-i2\phi} & 1 \\ 1 & \mathrm{e}^{i2\phi} \end{pmatrix}, \tag{8}$$

where indices a, b take values in $\{\mathbf{a}, \mathbf{a}^*\}$, and the prefactor is the integral of the time varying factor, that can be easily performed to obtain

$$\mathcal{I}(m, c_1, c_2, |\mathbf{a}|) \equiv \frac{\sqrt{\pi}}{64}\left[\left(\frac{c_1}{m}\right)^4 |\mathbf{a}|^3 + 12\left(\frac{c_1}{m}\right)^2 |\mathbf{a}| + 12|\mathbf{a}|^{-1}\right] \exp\left[\frac{1}{4}\left(\frac{c_1}{m}\right)^2 |\mathbf{a}|^2 + c_2\right].$$

4. STATE PROBABILITY CURRENT WITH ZERO FERMIONS

Notice that one of the main properties of the superfield solution of the non-degenerate supermetric under consideration is that, although the fermions are turning out, the Gaussian (localized) behaviour of the solution remains due to the complex fermionic coefficients a. This is very important for the phenomenological point of view, because the localized behaviour of the solution remains also after the susy breaking.

Then, only for simplicity, we can turn off all the fermions in the solution; therefore, the metric components in this case take the simple form

$$G_{ab} = \mathcal{J}(m, E, |\alpha|, c_2, |\mathbf{a}|) \begin{pmatrix} \mathrm{e}^{-i2\phi} & 1 \\ 1 & \mathrm{e}^{i2\phi} \end{pmatrix}. \tag{9}$$

Again, a, b take values in $\{\mathbf{a}, \mathbf{a}^*\}$, and now the prefactor is obtained performing the integral: $\mathcal{J}(m, E, |\alpha|, c_2, |\mathbf{a}|) \equiv \dfrac{3\sqrt{2\pi}}{4}\dfrac{E^2|\alpha|^2}{m}\,\mathrm{e}^{2c_2}|\mathbf{a}|^{-1}$, where in last equality we have made explicit the value of $\mathcal{K}_0$.

Note that this last expression presents only the $|\mathbf{a}|^{-1}$ singular term. This is precisely due to the lack of the "free wave" (linear in t) term in the exponential factor, which leads to a complete departure of the Gaussian behaviour remaining in common just the singular term.

5. P AS THE CURRENT OF PROBABILITY: THE QUANTUM CORRESPONDENCE

Consider a Hilbert space $\mathcal{H}$ with a symmetric inner product G_{ij}. For instance, we can have in mind the case $\mathcal{H} = \mathbf{L}_2(\mathcal{R})$, where $\mathcal{R} = \mathbb{R}^{2m}$ (e.g., the phase space of a classical dynamical system, the configuration space spin systems, etc.). The equation $G_{ab}(\theta) = 4\int d^D x\ \partial_a P^{1/2}(x;\theta)\,\partial_b P^{1/2}(x;\theta)$, as was first observed by Caianiello et al. [1], puts in evidence the clear possibility of mapping the probability density function $P(x;\theta)$ on $\mathcal{R}$ to $\mathcal{H}$ by forming the square root. As we propose in the present paper, using directly (and precisely) the probability current j_0, of the metric state solution $g_{AB}(t)$, as the probability density $P(x;\theta)$, $j_0 = \dfrac{1}{16}\mathcal{K}_0 \exp\left[-2\left(\dfrac{m}{|\mathbf{a}|}\right)^2 t^2 + 2c_2 + 2\xi\varrho(t)\right] \equiv P(x;\theta)$, leads to an identification (intuited by Caianiello) that can be immediately implemented and effectively realized. The metric components take then the form $G_{ab}(\theta) = 4\int d^D x\, \partial_a P^{1/2}(x;\theta)\partial_b P^{1/2}(x;\theta)$. Then, the quantum "crossover" is $G_{ab}(\theta) = 4\partial_a g_{AB}(x;\mathbf{a},a^*)\partial_b g_{CD}(x;\mathbf{a},a^*)\eta^{(AB)(CD)} \equiv \langle\partial_a g_{AB}(x;\mathbf{a},a^*)\partial_b g_{CD}(x;\mathbf{a},a^*)\rangle$. The resemblance with a Sigma model is quite evident. Even more, the "natural" geometrical normalization condition $g_{AB}g^{AB} = D$ (regarding the genuine emergent origin of the spin-2 state $g_{AB}(x;\mathbf{a},a^*)$) suggests a pseudospherical constraint in $\mathcal{H}$. However, the above relation is not the more general possibility of quantum extension of the Fisher metric. We can also introduce the following Hermitian metric tensor:

$$\widetilde{G}_{ab}(\theta) = \langle\partial_a g_{AB}(x;\mathbf{a},a^*)\partial_b g_{CD}(x;\mathbf{a},a^*)\rangle - \tag{10}$$

$$- \langle\partial_a g_{AB}(x;\mathbf{a},a^*) g_{CD}(x;\mathbf{a},a^*)\rangle \langle g_{AB}(x;\mathbf{a},a^*)\partial_b g_{CD}(x;\mathbf{a},a^*)\rangle\,, \tag{11}$$

since its real part can be *exactly* rewritten as [1]

$$\mathrm{Re}\widetilde{G}_{ab}(\theta) = \left\langle \partial_a L_{AB}^{1/2}(x;\mathbf{a},a^*)\partial_b L_{CD}^{1/2}(x;\mathbf{a},a^*)\right\rangle_{\mathrm{HS}}, \tag{12}$$

where L_{AB} (non-diagonal representation) is explicitly given, and $\langle , \rangle_{\mathrm{GS}}$ stands for the customary inner product in the Banach space of the Gram–Schmidt operators in $\mathcal{H}$. The complex numbers α and α^* in the exponential factor are the *eigenvalues* of the coherent states. From the above expression, the corresponding Gram–Schmidt operator reads:

$$G = \int \frac{d^2\alpha}{\pi}\left[\int \frac{d^2 w}{\pi}\exp\left(\frac{|w|^2}{4}\right) \times \right.$$
$$\left. \times \exp\left\{\frac{i}{2}\left[(\alpha - \alpha')\,w^* + (\alpha^* - \alpha^{*\prime})\,w\right]\right\}\right] |\Psi_m(\alpha)\rangle\langle\Psi_n(\alpha')|\,,$$

numbers α and α^* in the exponential factor are the *eigenvalues* of the coherent states. In summary, our results come to realize the connexion conjectured by Caianiello.

REFERENCES

1. *Cirilo-Lombardo D. J., Afonso V. I.* Information Metric from Riemannian Superspaces // Phys. Lett. A. 2012. V. 376. P. 3599–3603.