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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

INSTANTONS AND HOLOMORPHIC SPHERES

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We discuss the relationships between instantons and complex algebraic and differential geometry, which turn out to be useful in certain physical problems.

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In this note we will focus mainly on the geometric setup when a complex sphere $\mathbb{CP}^1$ is embedded holomorphically in a complex surface (a complex manifold of dimension two). We will be mainly dealing with the case that the embedded sphere is *rigid*; i.e., its embedding cannot be deformed in a holomorphic fashion (this means that the normal bundle to the sphere is negative). In this case we shall say that the sphere has been "blown up". Blow-ups are a widespread phenomenon in geometry and are somewhat less known in physics, but we will argue here, using two examples, that there are physical situations where they are indispensable.

For an extended account of some of the topics covered here, see [1].

1. WHAT IS A BLOW-UP?

Consider a two-dimensional complex space $\mathbb{C}^2$ with coordinates (z_1, z_2). The blow-up is a replacement of one of the points in $\mathbb{C}^2$, say, the origin $(0, 0)$ by a "sphere" $\mathbb{CP}^1$. This $\mathbb{CP}^1$ encodes the angle at which we approach the point.

Formally speaking, the blown-up manifold $\mathbb{C}^2$, denoted $\widetilde{\mathbb{C}^2}$, may be defined by means of the equation

$$\widetilde{\mathbb{C}^2} = \{z_1 w_2 = z_2 w_1 \subset \mathbb{C}^2 \times \mathbb{CP}^1\}, \tag{1}$$

where $(w_1 : w_2)$ are the homogeneous coordinates on the $\mathbb{CP}^1$. Clearly, there is a projection map $\pi : \widetilde{\mathbb{C}^2} \to \mathbb{C}^2$, which is singular at the point $z_1 = z_2 = 0$ — its Jacobian is zero at this point. However, the manifold $\widetilde{\mathbb{C}^2}$ is nonsingular. It is evident from (1) that the paths in $\mathbb{C}^2$, which approach the origin at distinct angles, end up at distinct points of the $\mathbb{CP}^1$.

At first it may seem like an artificial construction, but the examples below will serve to convince the reader that this is not so.

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2. THE DEFORMED ADHM EQUATIONS

It has been an observation of [3] that blown-up spacetime $\widetilde{\mathbb{C}^2}$ appears inevitably if one considers a natural deformation of the ADHM equations [2] — the equations that describe the moduli space of instantons (gauge connections with self-dual field strength) on $\mathbb{R}^4$.

The simplest way to describe the moduli space $\mathcal{M}_{n,k}$ of instantons for gauge group $U(n)$ and instanton charge k is to regard it as a hyper-Kähler quotient $\mathbb{C}^{2k(n+k)} \underset{H}{//} U(k)$. We will view $\mathbb{C}^{2k(n+k)}$ as a collection of two $k \times k$ matrices B_0, B_1 and two matrices I, J of sizes $k \times n$ and $n \times k$, respectively. Then the action of $U(k)$ on these matrices can be described as follows:

$$B_{0,1} \to g B_{0,1} g^{\dagger}, \quad I \to gI, \quad J \to J g^{\dagger}, \quad \text{where} \quad g \in U(k). \tag{2}$$

Upon defining the moment maps for the $U(k)$ action, $\mu_{\mathbb{R}}$ and $\mu_{\mathbb{C}}$, one can write down the ADHM [2] equations:

$$\text{ADHM}: \qquad \mu_{\mathbb{R}} = \mu_{\mathbb{C}} = 0. \tag{3}$$

From a mathematical standpoint, they are not completely natural. Indeed, nothing prohibits a "central" term in the r.h.s.: $\mu_{\mathbb{R}} = \xi_1 \mathbb{1}_k$, $\mu_{\mathbb{C}} = (\xi_2 + i\xi_3)\mathbb{1}_k$, and, therefore, generically it is natural to include it. For $\xi \neq 0$ these equations, however, no longer describe the moduli space of instantons on $\mathbb{R}^4$ (the curvature is no longer self-dual). One possible physical interpretation of the deformed equations, due to [3], is that they decribe instantons on a new manifold $Y \neq \mathbb{R}^4$.

For $\xi \neq 0$ the problem turns out to be nontrivial even for Abelian $U(1)$ instantons. In the case $n = k = 1$ one can build, however, the gauge potential explicitly and check that the corresponding field strength is self-dual on a manifold $\widetilde{\mathbb{C}^2}$ described by (1), with the metric simply being the metric induced by the embedding (1) in $\mathbb{C}^2 \times \mathbb{CP}^1$. The physical reason for the appearance of the blow-up is that the $U(1)$ instanton carries not just the instanton charge, but also a monopole charge, whose magnetic flux "inflates" the sphere $\mathbb{CP}^1$ around itself.

For what follows we document that the Kähler potential of the induced metric is

$$K = \log x + x, \tag{4}$$

where

$$x := |z_1|^2 + |z_2|^2 \tag{5}$$

is a $U(2)$-invariant combination that we will encounter frequently.

3. EFFECTIVE ACTIONS, ADE SINGULARITIES AND ALE SPACES

Another physics realm where the blow-up appears naturally is in the framework of effective supersymmetric field theories on D-branes placed at singularities of Calabi–Yau manifolds. In this section we describe $\mathcal{N} = (1, 0)$ supersymmetric theories in 6D, which arise as effective theories for the fluctuations of six-simensional D-branes located at ADE-singularities of a transverse $K3$-surface (Calabi–Yau space) [4]. The so-called Higgs branch of such theories naturally leads to manifolds (the so-called "gravitational instantons") which include copies of $\mathbb{CP}^1$ glued in with normal bundle $\mathcal{O}(-2)$.

In six dimensions the gauge field strength superfield $(W_a)_A$ is of opposite chirality to the supercharge and may be expanded in elementary fields as follows, up to linear order in the Grassmann coordinates $(\theta_a)_A$[1]:

$$W^a_A = \lambda^a_A + F^{ab}\theta^b_A + (D_i\sigma_i)_{AB}C^{ab}\theta^b_B + \dots, \tag{6}$$

and the matrix of fields F^{ab} is "traceless": $C_{ab}F^{ab} = 0$ (C is a non-degenerate charge conjugation matrix). The reality property for the symplectic-Majorana spinor W^a_A translates into the reality properties of the component fields. In particular, λ^a_A is a symplectic-Majorana spinor as well, $(F_{ab})^*$ is linearly related to F_{ab}, hence it has only 15 real components, which can be packed into a skew-symmetric real-valued tensor $F_{\mu\nu}$, and D_i are a triplet of real auxiliary fields.

It is precisely the appearance of this triplet, in place of a singlet D, that is important for us here. Assuming that the theory includes M hypermultiplets with scalar components ${}_m\phi^a_A$, where a is an $U(N)$ gauge index and m labels the hypermultiplet ($m = 1, \dots, M$), let us now write out the part of the supersymmetric Lagrangian, where the D_i fields enter:

$$\mathfrak{L} \sim \frac{1}{2}D_i^2 + D_i \left[\sum_{m=1}^{M} ({}_m\phi^a_A)^*(\sigma_i)_{AB}\ {}_m\phi^a_B + \zeta_i\right], \tag{7}$$

where ζ_i is a triplet of Fayet–Iliopoulos terms. The fields D_i are auxiliary, in the sense that they have no kinetic terms, so they can be integrated out of (7) to produce

$$\mathfrak{L} \sim \frac{1}{2}\left[\sum_{m=1}^{M} ({}_m\phi^a_A)^*(\sigma_i)_{AB}\ {}_m\phi^a_B + \zeta_i\right]^2. \tag{8}$$

The locus of points in field space where this function reaches a (zero) minimum is given by the hyper-Kähler moment map equations $\mu_i = 0,\ i = 1, 2, 3$. Since field configurations related by gauge transformations are equivalent, we need to take the quotient with respect to the gauge group $U(N)$, hence the space of physical field configurations saturating the minimum of the potential is the hyper-Kähler quotient $\{\ \mu_i^{-1}(0),\ i = 1, 2, 3\ \}/U(N)$. These hyper-Kähler quotients are Ricci-flat asymptotically locally Euclidean (ALE) spaces, which have "blown-up" spheres embedded in them.

The physical interpretation of the situation elaborated in this section is quite remarkable: the smooth ALE metrics provided by the hyper-Kähler quotients are, in fact, metrics on the resolutions of the ADE-singularities at which we place our D-branes!

The Eguchi–Hanson space is a special case when $N = 1, M = 2$. It is described by a Kähler potential that, in principle, is known explicitly, but for us the only important thing will be its expansion around the origin:

$$K = \log x + x^2 + \dots, \qquad \text{as} \quad x \to 0. \tag{9}$$

The important difference between (4) and (9) is the x vs. x^2 terms in the expansions of the Kähler potentials. Our point is that they are different because the $\mathbb{CP}^1$'s in the two cases are embedded with different normal bundles: $\mathcal{O}(-1)$ and $\mathcal{O}(-2)$, respectively.

[1]Here $a = 1, 2, 3, 4$ is the $SU(4)$ index, $A = 1, 2$ is the $SU(2)$ R-symmetry index.

4. EINSTEIN METRICS ON BLOWN-UP SPACES

A natural question, which arises from the above analysis, is what happens in the case of a sphere $\mathbb{CP}^1$ embedded with a normal bundle $\mathcal{O}(-m)$ for higher m, i.e., $m \geqslant 3$. From the adjunction formula it follows that this can only happen for a $\mathbb{CP}^1$ in a surface Y of "negative curvature", i.e., $c_1(Y) < 0$, — this is in contrast to the $\mathcal{O}(-1)$ case (in a surface of "positive curvature") and the $\mathcal{O}(-2)$ case (Calabi–Yau).

It is interesting to note that in the negative-curvature case it is possible to build Kähler–Einstein metrics on the total spaces of the $\mathcal{O}(-m),\ m \geqslant 3$ line bundles over $\mathbb{CP}^1$. Indeed, we look for the metrics $g_{i\bar{j}}$ satisfying

$$R_{i\bar{j}} = -g_{i\bar{j}}. \tag{10}$$

The metric is assumed to be $U(2)$-invariant and originating from a Kähler potential: $g_{i\bar{j}} = \dfrac{\partial^2 K}{\partial z^i \partial \bar{z}^j}$ with $K = K(x)$. For such an ansatz Eq. (10) with the boundary condition (chosen by analogy with (4) and (9))

$$K(x) = \log x + a\, x^m + \dots \qquad \text{as} \quad x \to 0 \tag{11}$$

may be solved explicitly. Introducing a new function $Q := xK'$, we can write the solution as follows:

$$Y_m : \quad \begin{cases} m > 3: & x = \prod\limits_{i=1}^{3} (Q - y_i)^{\frac{1}{2+y_i}}, \\ & \text{where } \ y_i^3 + 3y_i^2 - (m-2)^2(m+1) = 0, \\ m = 3: & x = \exp\left(-\dfrac{2}{Q+2}\right)\left(\dfrac{Q-1}{Q+2}\right)^{1/3}. \end{cases}$$

The interesting fact is that, for $m \geqslant 3$, the Kähler potential tends to infinity as $|z_1|^2 + |z_2|^2 \to 1$. Moreover, asymptotically near $|z_1|^2 + |z_2|^2 \simeq 1$ the metric behaves as the Lobachevsky space H_4 metric near the boundary. However, the requirement that the topological characteristics of this space — the Euler characteristic and signature — are integers, implies[1] that the boundary cannot be $S^3 = \partial \mathrm{H}_4$, but it rather has to be a quotient thereof, more precisely the lens space $L(m, 1) = S^3/\mathbb{Z}_m$. Requiring that the boundary is the appropriate lens space, we find that Y_m has the topological numbers of a line bundle over $\mathbb{CP}^1$.

To summarize, we have described the neighborhood of a sphere $\mathbb{CP}^1$ embedded with an arbitrary negative normal bundle $\mathcal{O}(-m)$, $m > 0$. The $m = 1$ case corresponds to the classical "blow-up" (1).

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[1] For more details, including the calculation of the topological characteristics of the above manifolds Y_m using Chern–Weil formulas, see our paper [1].

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