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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

# UNDERSTANDING THE PHYSICAL SYSTEMS FROM THEIR UNDERLYING GEOMETRICAL AND TOPOLOGICAL PROPERTIES

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As is well known, a certain lack of theoretical understanding of the mechanisms governing various phenomena exists in several areas of physics. In particular, it concerns those which involve transport of charged particles in low dimensions. In this work, the physics of the 2-dimensional charge transport with parallel (in plane) magnetic field is analyzed from the geometrical and algebraic viewpoints with emphasis on how the physical interpretation arises from a consistent mathematical formulation of the problem. As a new result of this investigation with respect to the current literature, we explicitly show that: i) the specific form of the low-dimensional Dirac equation enforces the field solution to fulfill the Majorana condition; ii) the quantum Hall effect is successfully explained; iii) a new topological effect (as described by the Aharonov–Casher theorems) is presented, and iv) the link with supersymmetrical models is briefly commented.

Как известно, существуют пробелы в понимании механизмов различных процессов в определенных областях физики. В частности, это относится к переносу заряженных частиц в условиях небольших размерностей. В представленной работе физика двумерного переноса заряда в плоскости, параллельной магнитному полю, рассматривается с геометрической и алгебраической точек зрения с акцентом на то, как физическая интерпретация возникает из математической формулировки проблемы. Получены следующие новые результаты: 1) особая форма низкоразмерного уравнения Дирака приводит к тому, что получаемое решение обязательно удовлетворяет условию Майораны; 2) успешно объясняется квантовый эффект Холла; 3) обнаружен новый топологический эффект (описанный теоремами Ааронова–Кэшера) и 4) кратко обсуждается связь с суперсимметричными моделями.

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## 1. INTRODUCTION: WHEN MATHEMATICS ANTICIPATES PHYSICS

Every scientist throughout his life probably comes across the ideas of the mathematician Hermann Weyl and the physicist Paul Dirac. They attracted (and do attract up to now) the attention of everybody not merely as great scientists but also as great hunters for beauty. *"My work has always tried to unite the true with the beautiful and when I had to choose one or the other, I usually chose the beautiful"*, — wrote Weyl [1, 3].

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*"Physical laws should have mathematical beauty"*, — wrote Dirac on the blackboard in the Moscow University in the fall of 1955. The reason for the mysteries that most of the time truth and beauty are the same is that there need not to be conflict between them, discusses David J. Gross in his essay [1, 2] in detail: *"... the mathematical structures that mathematicians arrive at are not artificial creations of the human mind but rather have a naturalness to them as if they were as real as the structures created by physicists to describe the so-called real world. Mathematicians, in other words, are not inventing new mathematics, they are discovering it ... we might expect that physical and mathematical structures would share the characteristics that we call beauty. Our minds have surely evolved to find natural patterns pleasing"*. As is well known, in 1937 the brilliant Italian physicist Ettore Majorana proposed a new representation [4] corresponding to the celebrated Dirac equation, where the components of the bispinor solution are related by complex conjugation. However, in the middle of his personal troubles, he could not have foreseen the whirlwind of activity that would follow: not only in particle physics, which was his domain, but also in nanoscience and condensed matter physics. The particles described by these solutions (the so-called Majorana fermions) were strange objects of the physical contemporary research. The recent storm of activity in condensed matter physics has focused on the "Majorana zero modes", i.e., emergent Majorana-like states occurring at exactly zero energy that have a remarkable property of being their own antiparticles (self-conjugated). Sometimes, this property is expressed as an equality between the particle's creation and annihilation operators. As explained more fully below, there exists the general idea that any ordinary fermion can be thought of as composed of two Majorana fermions: this is only a partial picture. The real fact is that there exists a particular representation where a fermion effectively can be represented as bilinear combination of two states of fractionary spin, as was demonstrated by the author in [5] and other researchers in different contexts.

On the other hand and with other motivations, Aharonov and Casher proved two theorems for the case of a 2D magnetic field [6]. The first theorem states that an electron moving in a plane under the influence of a perpendicular inhomogeneous magnetic field has $N$ ground-energy states, where $N$ is the integral part of the total flux in units of the flux quantum $\Phi_0 = 2\pi/e \equiv hc/e(m = 1)$. The corresponding Dirac equation for the Aharonov–Casher-Theorem (ACT) configuration is [1]

$$[\sigma_x(\partial_x - ieA_x) + \sigma_y(\partial_y - ieA_y)]\varphi = 0. \tag{1}$$

The interesting remark of Aharonov and Casher is that if we introduce the transformation

$$\psi = \mathrm{e}^{e\phi\sigma_z}\varphi, \tag{2}$$

this transformation (phase) permits us to eliminate explicitly the magnetic field from the Dirac equation where $\phi$ satisfies the relations

$$\partial_x\phi = A_y, \quad \partial_y\phi = -A_x \tag{3}$$

and $\varphi$ is eigenfunction of $\sigma_z(\sigma_z\varphi_s = s\varphi_s)$. Having accounted that $B(x, y) = \partial_x A_y - \partial_y A_x$, we arrive at

$$B(x, y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi. \tag{4}$$

[1] We denote the fixed reference system as $X, Y, Z$ and the coordinates in plane by $x_1$, $x_2$, $x_3$.

Asymptotically for $r \to \infty (r \equiv \sqrt{x^2 + y^2})$ we have

$$\phi(x, y) = \frac{\Phi}{2\pi} \ln\left(\frac{r}{r_0}\right), \tag{5}$$

where

$$\Phi = \int B(x, y)\, dx\, dy \tag{6}$$

is the total magnetic flux through the $(x, y)$ plane; $r_0$ is some real constant that plays the role of minimal length. Consequently, we immediately obtain

$$\varphi_s = \left(\frac{r_0}{r}\right)^{\Phi s/\Phi_0} \psi_s(w), \tag{7}$$

where $w = x + isy$ and $\psi_s(w)$ is an entire function of $w$ because after the elimination of the magnetic field from Eq. (1) it takes the simplest form

$$(\partial_x + is\partial_y)\, \psi_s(w) = 0. \tag{8}$$

To make $\varphi_s$ be a square integrable function, we should consider $\Phi s > 0$ and $\psi_s$ has to be a polynomial whose degree is not greater than $N - 1$, where $N = \{\Phi/\Phi_0\}$, obtaining $N$ independent solutions for $\psi_s$: $1, w, w^2, \ldots, w^{N-1}$. Through this paper the same procedure as for the ACT configuration will be performed; however, it will be in the interesting case of "in-plane" (parallel) magnetic field.

The plan of this paper is as follows. In Sec. 2, we obtain the conditions where the magnetic field parallel to the charge transport can be "removed" as in the case of the ACT. The conditions fulfilled by the solution: types of spinors and flux quantization are also in Sec. 2. In Sec. 3, the origin and conditions where the quantum Hall effect appears from the "in-plane" magnetic field are explicitly shown. In Sec. 4, we obtain as solution to our problem the coherent states belonging to the Metaplectic group. These solutions fulfill the symmetries and algebra of Majorana states: the relation with supersymmetry is briefly described. Finally, in Sec. 5 we give our concluding remarks and perspectives.

## 2. MAGNETIC FIELD "IN PLANE"

Now the magnetic field $B$, in contrast to the ACT configuration described before, is parallel to the plane defined by $x, y$ axis (usually denominated: "$B$ in plane") where we have the dynamics of the particle. Explicitly the Dirac equation with the magnetic field parallel takes the following form:

$$[\sigma_B \partial_B + \sigma_\perp(\partial_\perp - ieA_\perp) - ie\sigma_z A_z]\varphi = 0. \tag{9}$$

Here, the subscripts $B$, $\perp$ and $z$ denote the direction of the $B$ field in the plane, the direction of the component of the potential vector in the plane (obviously, perpendicular to the $B$ direction) and the direction of component of the potential vector coincident with the $z$ axis, respectively.

By defining $\omega$ the angle of the magnetic field with respect to the $x$ axis in the plane $x-y$, the transformation (2) takes in this case the following general form:

$$\psi = \mathrm{e}^{i(\alpha\sigma_x + \beta\sigma_y)}\,\varphi = \mathrm{e}^{ie\phi\cdot\sigma_B}\varphi \tag{10}$$

with

$$\alpha = \lambda\cos\omega, \quad \beta = \lambda\sin\omega, \tag{11}$$

$$|\phi|^2 = \lambda^2(\cos^2\omega + \sin^2\omega) = \lambda^2 \Rightarrow |\phi| = \pm|\lambda|. \tag{12}$$

Equation (9) explicitly written (taking into account (10)) is

$$\left[\sigma_x\partial_x + \sigma_y\partial_y - ieA_\perp(\sigma_x\sin^2\omega + \sigma_y\cos^2\omega) - ie\sigma_z A_z\right]\varphi = 0. \tag{13}$$

It is easily seen that when $\omega = 0$, $B$ coincides with $x$ axis and when $\omega = \pi/2$, $B$ coincides with the $y$ axis. The Lie algebraic relation holds:

$$\sigma_B\sigma_\perp = (\cos\omega\sigma_x + \sin\omega\sigma_y)\,(-\sin\omega\sigma_x + \cos\omega\sigma_y) = i\sigma_z \tag{14}$$

as expected.

Operating similarly as in the ACT configuration (but having account for the new transformation and physical situation), we obtain the conditions where the magnetic field can be eliminated. Precisely, using expression (10) in (9), we obtain explicitly the following nontrivial conditions:

$$-\partial_\perp\phi = iA_z, \quad \partial_B\phi = -A_\perp\sigma_\perp. \tag{15}$$

The first equation is precisely as in the ACT case, but for the second one the interpretation is more involved. The interpretation suggests, in principle, a complex structure for the field $\phi$: for example, in a doublet form. The doublet can be written as

$$\phi \equiv \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{16}$$

then the previous expressions belong to

$$-\partial_\perp\phi_1 = -\partial_\perp\phi_2 = iA_z \quad \text{and} \quad \partial_B\phi_1 = -\partial_B\phi_2 = iA_\perp. \tag{17}$$

Notice that the above condition suggests consequently the introduction of two real functions $u$ and $v$ as

$$\phi \equiv \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\phi_1\\ \phi_1^*\end{pmatrix} = \begin{pmatrix} u\,(x_\perp) + iv\,(x_B)\\ (u\,(x_\perp) + iv\,(x_B))^*\end{pmatrix} \tag{18}$$

in such a manner that the conditions to remove the magnetic field are automatically fulfilled if

$$-\partial_\perp\phi = iA_z \quad \text{and} \quad \partial_B\phi = A_\perp. \tag{19}$$

**Remark 1.** *Notice that (18) is a Majorana condition over $\phi$ that appears as a consequence of the magnetic field parallel (in a sharp contrast to that in the ACT case).*

**2.1. Structure of the Magnetic Field: Conditions over $A$ and $\phi$.** The magnetic field can be effectively generated ($B = \nabla \wedge A$) from the vector potential components of our problem, namely, $A_z$ and $A_\perp$.

The "in-plane" magnetic field is consequently

$$B_B = (\partial_\perp A_z - \partial_z A_\perp), \tag{20}$$

where the simplest possibility was taken: $A \neq A(x_B)$ (e.g., the vector potential does not depend on the direction of the magnetic field, only on the plane defined by $x_\perp$ and $x_z$). From (19) we have

$$B = i\partial_\perp^2 \phi = \frac{\Phi}{x_\perp}, \tag{21}$$

where the total transversal flux to the plane per unit of longitude was used. Then $\phi$ is immediately obtained:

$$\phi \cdot \sigma_B = -i\,(\Phi\sigma_B)\,x_\perp \left[\ln\left|\frac{x_\perp}{l_0}\right| - \frac{C - x_\perp}{x_\perp}\right]. \tag{22}$$

By putting the arbitrary constant $C = 0$ for simplicity, the behaviour of the exponential function in (10) belongs to

$$\mathrm{e}^{-ie\phi\cdot\sigma_B} = \left|\frac{l_0}{x_\perp}\right|^{\frac{e\Phi}{l_0}\sigma_B x_\perp} \exp\left(-\frac{e\Phi}{l_0}\sigma_B x_\perp\right), \tag{23}$$

with $l_0$ some real constant with units of length (its physical meaning will be analyzed later). Similarly as in the ACT case, the following condition must be fulfilled in order that $\varphi$ be normalizable and square integrable:

$$\Phi s_B \geqslant 0 \tag{24}$$

($s_B$ is the spin in the $B$ direction) due to

$$\varphi = \mathrm{e}^{-ie\phi\cdot\sigma_B}\,\psi(s, w). \tag{25}$$

In the above expression, the function $\psi$ depends on the spin and on some complex variable $w$ to be determined from the Dirac–Weyl equations.

**2.2. Majorana, Dirac–Weyl States and Discrete Coordinates: Conditions over $\psi(s, z)$.** The simple Dirac–Weyl equation obtained (after the procedure to remove the magnetic field) is

$$(\mathrm{e}^{-ie\phi\cdot\sigma_B}\sigma_B\partial_B + \mathrm{e}^{ie\phi\cdot\sigma_B}\sigma_\perp\partial_\perp)\,\psi(s, z) = 0. \tag{26}$$

To solve the equation, a quantization should be imposed on the flow (strictly on the product $\phi \cdot \sigma_B$). This fact will induce an automatic discretization over the "in-plane" transverse coordinate $x_\perp$:

$$\phi \cdot \sigma_B = n\pi, \quad n = 0, 1, 2 \ldots \tag{27}$$

If the above condition holds, we obtain

$$(\sigma_B\partial_B + \sigma_\perp\partial_\perp)\psi(s, z) = 0. \tag{28}$$

This expression is very important: this is a simple 2-dimensional Dirac equation *without* $A_\mu$. The particular phase introduced as ansatz plus a quantization condition nullify the effect of the magnetic field.

**2.3. Analysis of the Solution.** When taking account of the specific form of the above equations there are two possibilities for the solution $\psi$. These possibilities are related with the spin degrees of freedom as follows:

i) $\sigma_B \psi(s,z) = s\psi(s,z)$ (eigenspinor of $\sigma_B$).

This case is compatible with the assumption that the state is eigenvector of the spin in the magnetic field direction. The Dirac equation is reduced to

$$\left(\partial_B + \frac{i\mathbb{C}}{s}\partial_\perp\right)\psi\,(s,z) = 0, \tag{29}$$

with $\mathbb{C}$ the charge conjugation operator. Then $\psi(s,z)$ (and, for instance, $\varphi(s,z)$) must fulfill the Majorana condition:

$$\mathbb{C}\varphi(s,z) = \pm c\varphi\,(s,z). \tag{30}$$

Similarly as in the AC case, $\psi(s,z)$ is an entire function of $z = x_B + \dfrac{ic}{s}x_\perp$ but the state solution is of Majorana type.

ii) $\sigma_z \psi(s,z) = s\psi(s,z)$ (eigenspinor of $\sigma_z$).

In this case the spin remains as in the ACT situation (e.g., in the $z$ direction). Now the Dirac equation is reduced to

$$(\partial_B + is\partial_\perp)\psi\,(s,z) = 0. \tag{31}$$

Similarly as in the AC case, $\psi(s,z)$ is an entire function of $z = x_B + isx_\perp$, and the state solution is Dirac–Weyl.

**Remark 2.** *The specific form of Eq. (29) shows that the result is not accidental: the states are Majorana. The inclusion of the charge conjugation operator $\mathbb{C}$, due to the symmetry of the physical scenario, enforces the Majorana condition over the state solution.*

## 3. QUANTUM HALL EFFECT AND THE "IN-PLANE" MAGNETIC FIELD

We can expect that if the plane where the charges are moving is finite, an "in-plane" current transversal to the magnetic field $B$ must appear (e.g., in the $x_\perp$ direction). This current will be quantized due to the condition (27). This condition explicitly can be written as

$$\phi \cdot \sigma_B = (\Phi\sigma_z)\widetilde{x}_\perp \left[\ln\left|\frac{x_\perp}{l_0}\right| - 1\right] = n\pi, \quad n = 0, 1, 2\ldots, \tag{32}$$

where $\widetilde{x}_\perp = \sigma_\perp x_\perp$ is a new matrix valuated coordinate whose meaning will be analyzed later.

The explicit formula for the Hall current comes from the expression for the surface current

$$n \times B = K_{\text{surface}} \tag{33}$$

($n$: versor normal to the interface surface). This current is obviously perpendicular to the magnetic field "in plane" (e.g., $x_\perp$ direction). Due to the quantization condition, the Hall current is also quantized leading to the Quantum Hall Effect (QHE):

$$\frac{\Phi}{x_\perp}\overset{v}{x}_\perp = \frac{2\pi N\hbar c}{ex_\perp}\overset{v}{x}_\perp = K_{\text{surface}}, \tag{34}$$

where $\overset{v}{x}_\perp$ is a unitary vector in the $x_\perp$ direction.

**Generalized Momentum Operator and Majorana Conditions.** We can elucidate the interpretation of the nonstandard Dirac equation:

$$[\sigma_B \partial_B + \sigma_\perp (\partial_\perp - ieA_\perp) - ie\sigma_z A_z]\varphi = 0 \tag{35}$$

by rewritting it as

$$\Big[\sigma_B \underbrace{(\partial_B - ie\sigma_B\sigma_z A_z)}_{\widetilde{\Pi}_B} + \sigma_\perp \underbrace{(\partial_\perp - ieA_\perp)}_{\Pi_\perp}\Big]\varphi = 0 \Rightarrow \Big[\sigma_B \widetilde{\Pi}_B + \sigma_\perp \Pi_\perp\Big]\varphi = 0. \tag{36}$$

The question that immediately arises from (36) is then: what is the operator $\widetilde{\Pi}_B$? The answer is obvious if we use the algebra (14) and the definition of the charge conjugation operator as a function of the sigma matrices. Consequently,

$$(\partial_B - ie\sigma_B\sigma_z A_z) = (\partial_B + ie\mathbb{C}A_z). \tag{37}$$

**Remark 3.** *As in ordinary non-Abelian gauge theories, the operator* $\widetilde{\Pi}_B$ *in (37) seems to be equipped with a* ***non-Abelian*** *vector potential* $\widetilde{A_B} \equiv -\mathbb{C}A_z$.

## 4. DIRAC–MAJORANA OSCILLATOR: SUSY, ALGEBRA AND PARASTATISTICS

A relativistic fermion under the action of a linear vector potential is usually called the Dirac oscillator [7]. The standard Dirac oscillator can be exactly solved in one, two and three dimensions. It has in the nonrelativistic limit the associated Klein–Gordon equations describing a harmonic oscillator in the presence of a strong spin-orbit coupling, and the first experimental realization of this system has been reported recently [8]. Motivated by these important reasons plus the possibility to analyze the (super) symmetries in the obtained spectrum, our goal in this section is to rewrite conveniently the Dirac equation corresponding to the "in-plane" magnetic field configuration in the form of the Dirac oscillator.

Our starting point is as follows: in 2 dimensions we have

$$[c\sigma_\perp p_\perp + eB\sigma_z X_\perp + mc^2]\varphi = E\varphi \equiv H_{2\mathrm{D}}\varphi. \tag{38}$$

Introducing the corresponding creation and annihilation operators as

$$H_{2\mathrm{D}} = i\left(\frac{eBc\hbar}{2}\right)^{1/2} \sigma_\perp (a^+ - a) + \left(\frac{eBc\hbar}{2}\right)^{1/2} \sigma_z (a^+ + a) + mc^2, \tag{39}$$

we can redefine and rearrange the operators in order to put the Hamiltonian in the simpler form:

$$H_{2\mathrm{D}} = \frac{i}{\sqrt{2}}[a^+(\sigma_\perp - i\sigma_z) - a(\sigma_\perp + i\sigma_z)] + \mu, \tag{40}$$

where the energy is given in $(eBc\hbar)^{1/2}$ units and we have defined $\mu = mc\sqrt{\dfrac{c}{eB\hbar}}$. Explicitly

$$H_{2D} = H_{2D} = \begin{pmatrix} \dfrac{(a^+ + a)}{\sqrt{2}} + (\mu - E) & \dfrac{(a^+ - a)}{\sqrt{2}}\mathrm{e}^{-i\omega} \\ \dfrac{(a^+ - a)}{\sqrt{2}}\mathrm{e}^{+i\omega} & -\dfrac{(a^+ + a)}{\sqrt{2}} + (\mu - E) \end{pmatrix} \varphi = 0. \tag{41}$$

The first important observation is that the Hamiltonian (41) has the suggestive fashion of the BHZ phenomenological model [7]. This BHZ model was a "by-hand" attempt to explain the topological insulator mechanism. Then we are able to bring a natural explanation to the topological insulators described in [7] from a pure phenomenological viewpoint. Expanding the state $\varphi$ in the $n$ basis and taking into account that it must be invariant under $i\mathbb{C}(\equiv -\sigma_2)$, we obtain the following expression:

$$\varphi = \begin{pmatrix} 1 \\ \mathrm{e}^{i\pi/2} \end{pmatrix} \sum_{k=0}^{\infty} [A_{2k}|2k\rangle + A_{2k+1}|2k+1\rangle]. \tag{42}$$

However, the coefficients $A_n$ are not independent. $A_{2k}$ and $A_{2k+1}$ are related to the two first coefficients $A_0$ and $A_1$ corresponding to the states $|0\rangle$ and $|1\rangle$, respectively, provided again that the following quantization condition over the $\omega$ necessarily arises:

$$\omega = \pi(k+1), \quad k = 0, 1, 2 \ldots \tag{43}$$

Consequently, the normalized state solution takes the following form:

$$|\varphi\rangle = \begin{pmatrix} 1 \\ \mathrm{e}^{i\pi/2} \end{pmatrix} \sum_{k=0}^{\infty} \left[ A_0 \underbrace{\frac{\sqrt{(2k-1)!!}}{\mathrm{e}^{1/4}} \frac{|2k\rangle}{\sqrt{2k!}}}_{|\Psi_{1/4}\rangle} + A_1 \underbrace{\frac{\sqrt{(2k)!!}}{(\sqrt{e\pi/2}\mathrm{Erf}(1/2))^{1/2}} \frac{|2k+1\rangle}{\sqrt{(2k+1)!}}}_{|\Psi_{3/4}\rangle} \right] \equiv$$

$$\equiv \begin{pmatrix} 1 \\ \mathrm{e}^{i\pi/2} \end{pmatrix} (A_0|\Psi_{1/4}\rangle + A_1|\Psi_{3/4}\rangle) \tag{44}$$

$(A_1^2,\ A_0^2 = \pm 1)$. As is easily seen, $|\varphi\rangle$ is a coherent state of Klauder–Perelomov/Barut–Girardello type. It can be generated by a displacement operator $D$ and under normalization, it is eigenstate of the annihilation operator $a$. The coefficients $A_0$ and $A_1$ are arbitrary, in principle, with the property $A_1^2$, $A_0^2 = \pm 1$. This fact permits us to have two eigenstates of the annihilation operator $a$ with different parity behaviour under such an operator: $A_0 = \pm A_1 \rightarrow |\varphi_\pm\rangle = A_0 \begin{pmatrix} 1 \\ \mathrm{e}^{i\pi/2} \end{pmatrix} (|\Psi_{1/4}\rangle \pm |\Psi_{3/4}\rangle)$, then

$$a|\varphi_\pm\rangle = \pm|\varphi\rangle. \tag{45}$$

**Remark 4.** *Notice the important fact that the states solution $|\varphi\rangle$ is independent of the energy. It is a characteristic of the Majorana states that commonly appear in quantum transport in nanostructures.*

**Relation with Supersymmetric Models.** The dynamics of the $|\Psi\rangle$ fields were extensively studied in supersymmetric models. In previous references [5], it was demonstrated that the analysis of the particular representation that we are interested in can be simplified considering these fields as coherent states in the sense that they are eigenstates of $a^2$ [5]:

$$\begin{aligned}|\Psi_{1/4}(0,\xi,q)\rangle &= \sum_{k=0}^{+\infty} f_{2k}(0,\xi)|2k\rangle = \sum_{k=0}^{+\infty} f_{2k}(0,\xi)\frac{(a^\dagger)^{2k}}{\sqrt{(2k)!}}|0\rangle,\\ |\Psi_{3/4}(0,\xi,q)\rangle &= \sum_{k=0}^{+\infty} f_{2k+1}(0,\xi)|2k+1\rangle = \sum_{k=0}^{+\infty} f_{2k+1}(0,\xi)\frac{(a^\dagger)^{2k+1}}{\sqrt{(2k+1)!}}|0\rangle.\end{aligned} \tag{46}$$

From a technical point of view, these states are one-mode squeezed states constructed by the action of the generators of the $SU(1,1)$ group over the vacuum. For simplicity, we will take all normalization and fermionic dependence into the functions $f(\xi)$. Explicitly (supposing in principle no time dependence, e.g., $t=0$),

$$\begin{aligned}|\Psi_{1/4}(0,\xi,q)\rangle &= f(\xi)|\alpha_+\rangle,\\ |\Psi_{3/4}(0,\xi,q)\rangle &= f(\xi)|\alpha_-\rangle,\end{aligned} \tag{47}$$

where $|\alpha_\pm\rangle$ are the CS basic states in the subspaces $\lambda = 1/4$ and $\lambda = 3/4$ of the full Hilbert space [5]. In the case of the physical state spanning the full Hilbert space, the Heisenberg–Weyl (HW) realization for the states $\Psi$ must be used:

$$|\varphi\rangle = \frac{f(\xi)}{2}(|\alpha_+\rangle + |\alpha_-\rangle) = f(\xi)|\alpha\rangle. \tag{48}$$

In (48) the linear combination of the states $|\alpha_+\rangle$ and $|\alpha_-\rangle$ corresponding to the $\lambda = 1/2$ CS basis spans now the full Hilbert space (dense). As we will see in a future paper, this particular representation describes perfectly the Majorana fermion behaviour that is phenomenologically obtained [9].

## 5. CONCLUDING REMARKS

In this article we have shown how mathematics can predict physical effects and describe various phenomena with great precision and reliability. Through this letter we have given examples accompanied with new results using as the physical scenario to describe the quantum transport of charged particles a two-dimensional space with a parallel magnetic field. With the consistent mathematical description of the problem, quantum effects that have been inconsistently explained through empirical/phenomenological methods ("by hand") are now easily explained as the quantum Hall effect and the rise of Majorana states in low-dimensional structures with particular field conditions.

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