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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

EULER–HEISENBERG–SCHWINGER LAGRANGIAN FOR NON-ADIABATICALLY VARYING FIELDS

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Non-perturbative particle production in external variable fields is important in astrophysics. Although a wide range of techniques exists for calculating production rate, none of them can handle exactly the case of a strongly inhomogeneous field. Here an electric field with an abrupt switch-on is considered. First, a standard semiclassical technique is used. Then, a scattering problem approach to this case is developed, and time-dependent corrections to the effective Euler–Heisenberg–Schwinger Lagrangian are calculated.

Непертурбативное рождение частиц во внешних полях представляет интерес для астрофизики. Несмотря на существование ряда методов расчета вероятности рождения ни один из них не описывает в точности случай сильно неоднородного поля. В данной работе рассматривается мгновенно включающееся электрическое поле. Для его описания сначала используется стандартный квазиклассический метод. Затем развивается подход в рамках теории рассеяния, что позволяет найти зависящие от времени поправки к эффективному лагранжиану Гейзенберга–Швингера–Эйлера.

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INTRODUCTION: NON-LINEAR QED IN ASTROPHYSICS

Astrophysical applications require extension of non-perturbative QED particle production rate calculation techniques beyond local field approximation to the case of strong inhomogeneous electromagnetic fields, which cannot be treated adiabatically. Strong fast time-varying (magnetic) fields are observed in stellar collapse processes and magnetic stars. Intensive space-inhomogeneous electromagnetic fields are a feature of Kerr–Newman or Reissner–Nordstrom black holes.

Ruffini and Damour [1] have argued that Euler–Heisenberg–Schwinger processes in Kerr–Newman gravitational background may account for gamma-ray bursts. According to [2], up to 50% of the energy of an extremal charged black hole may be contained in its electromagnetic field. On the other hand, field of the charged black hole will quickly dissipate and form via Euler–Heisenberg–Schwinger process a plasma of e^+e^- pairs. The pairs, in their turn, by escaping the horizon vicinity and interacting with exterior baryonic matter will produce the

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burst of γ-radiation. Therefore, to study this possible mechanism of γ-production, one should know the Euler–Heisenberg Lagrangian for inhomogeneous fields.

Extensive literature is devoted to Euler–Heisenberg Lagrangians in variable fields (for a review see [3]). At present day, exact results are available for constant field, sinusoidal standing wave $E \sim \sin(\Omega x)$ and singular pulse $E \sim \dfrac{1}{\cosh(\Omega x)^2}$ (be x either temporal or spatial coordinate), and for fields, arbitrary depending on one of the light-cone coordinates $x^{\pm} = x^0 \pm x^1$. Accurate quasiclassical results are available for fields, depending smoothly on one of their coordinates [4, 5].

A time-inhomogeneous external electric field will be considered, «switched on» in a theta-function manner

$$E_3 = E\theta(x^3) \quad \text{or} \quad E_3 = E\theta(x^0), \tag{1}$$

E_3 being the Ox_3 directed component of the field, all other components being zero. One would like to obtain an expression for particle production rate, or, equivalently, the Euler–Heisenberg Lagrangian in this case. The field is obviously non-adiabatic, so the standard «local» formula from [6] is inapplicable here. It is clear that one would hardly find a field configuration of such a shape in nature, however, it could be a kind of «toy model» to study the inhomogeneity effects in more complicated cases.

The article is organized as follows. Section 1 reminds the reader some general ideas of world-line instanton method and obtains the particle production rate quasiclassically up to 1-loop accuracy. The limited applicability of the semiclassical approach to our case is shown. In Sec. 2 the particle production rate is recalculated by scattering approach and its temporal dependence is investigated. In Sec. 3 a brief comparison of the methods is performed.

1. QUASICLASSICAL APPROACH

1.1. Instantonic Method. One can model the physical finite-time switch-on as follows:

$$E_3 = \frac{E}{2}\left(1 + \frac{\Omega x^0}{\sqrt{1 + (\Omega x^0)^2}}\right),$$

where Ω^{-1} is the characteristic switch-on duration. If $\Omega^{-1} \gg t_{\mathrm{Compt}}$, one can use the adiabatic approach, but one is interested in the reverse case. Finite switch-on time is introduced not just for physical reasonability, but also for the possibility to smoothly analytically continue it.

The instantonic method by Dunne, Schubert et al. [7, 8] is being applied to our problem. The essence of the method is that the effective action of an electromagnetic field, given as 1-particle Feynman path integral

$$S_{\mathrm{eff}}[A^{\mathrm{ext}}] = -\int\limits_0^{+\infty} \frac{dT}{T}\, \mathrm{e}^{-m^2 T} \int dx^{(0)} \int\limits_{x(0)=x(T)=x^{(0)}} \mathcal{D}x(\tau)\, \mathrm{e}^{-S_T[x(\tau), A^{\mathrm{ext}}]}$$

can be expressed via a sum of exponents of classical actions of all closed-loop Euclidean space-time (electron) paths in the external field. Here T is the world-line parameter. The integral over T becomes a sum of a discrete series of saddle points T_n, e.g., $T_n = \dfrac{\pi n}{eE}$ in case of constant field, where arbitrary natural number n has the meaning of winding number. Integration over $dx^{(0)}$ means that one has to integrate over the initial point of the loop in Euclidean space-time. $S_T[x(t), A^{\rm ext}]$ is quadratic action of a relativistic particle in the external electromagnetic field $A^{\rm ext}$. The term «world-line instanton» is used for the closed classical paths satisfying the periodic boundary conditions in Euclidean space-time. Following this analog, the sum over n in this case directly corresponds to the sum of kink–antikink pairs in ϕ^4 theory.

For details of the method see the two cited articles by Dunne, Schubert et al. For derivation of relativistic particle path integral with quadratic action see [9]. In fact, this method can be thought of as a generalization of the WKB method for multiloop trajectories.

1.2. Leading Exponent. Effective QED Lagrangian is generally expressed as a «sum over instantonic paths»

$$\operatorname{Im} \mathcal{L}_{\rm eff} = \sum f_n \, {\rm e}^{-S_n}. \tag{2}$$

Here S_n is an n-instantonic classical action; f_n is the corresponding preexponential factor. For example, in constant field case an n-instantonic trajectory yields integral of motion $a = \dfrac{2\pi m^2 n}{eE}$, action $S_n = \dfrac{m^2 \pi n}{eE}$, law of motion $x_4(\tau) = \dfrac{m}{eE} \sin(2\pi n \tau)$, preexponential factor $f_n = \dfrac{e^2E^2}{16\pi^3} \dfrac{(-1)^{n+1}}{n^2}$ (in scalar QED). Trajectories are simply circles in (x_3, x_4) plane, wound around n times by the particle.

In the discussed «θ-switch-on» case, the situation is more complicated. Generally, in variable fields simultaneous motion in the Euclidean and Minkowskian regions takes place. The contour of this motion should be chosen so that world-line parameter τ is real. Therefore, special care should be taken to make sure that the formula (2) is applicable.

In our case, the Euclidean-time vector-potential looks like

$$A_3 = \frac{E}{2\Omega} \left(i\Omega x_4 + \sqrt{1 - (\Omega x^4)^2} \right).$$

The law of motion $x_4(\tau)$, where τ is world-line parameter, is given by the following integral:

$$\tau - \tau_0 = \frac{1}{a} \int\limits_{x^4_{(0)}}^{x^4} \frac{dx'^4}{\sqrt{1 - \dfrac{e^2E^2}{4m^2} \left(x'^4 - i \dfrac{\sqrt{1 - \Omega^2 (x'^4)^2}}{\Omega} \right)^2}}.$$

The $x^3(\tau)$ can be expressed in terms of $x^4(\tau)$ via equations of motion. Here a is e.o.m. integral, $(\dot{x}^3)^2 + (\dot{x}^4)^2 = a^2$, which should be expressed via winding number n according to the condition

$$\frac{1}{2n} = \frac{1}{a} \int\limits_{x^4_{\rm min}}^{x^4_{\rm max}} \frac{dx'^4}{\sqrt{1 - \dfrac{e^2E^2}{4m^2} \left(x'^4 - i \dfrac{\sqrt{1 - \Omega^2 (x'^4)^2}}{\Omega} \right)^2}},$$

where $x^4_{\rm max}$, $x^4_{\rm min}$ — limits of the classically forbidden region.

In this particular case, due to the convenient choice of the approximating smooth function, in fact, a wholly Euclidean finite trajectory can be found with periodical boundary conditions imposed thereupon in the range

$$1 < \Omega x_4 < \frac{1}{\gamma}. \tag{3}$$

Thus, we have the proper integration limits $x^4_{\max}$, $x^4_{\min}$ and can apply (2) straightforwardly provided $\gamma > 1$. For $\gamma < 1$ it is impossible to apply this technique. The trajectories can be seen in Fig. 1.

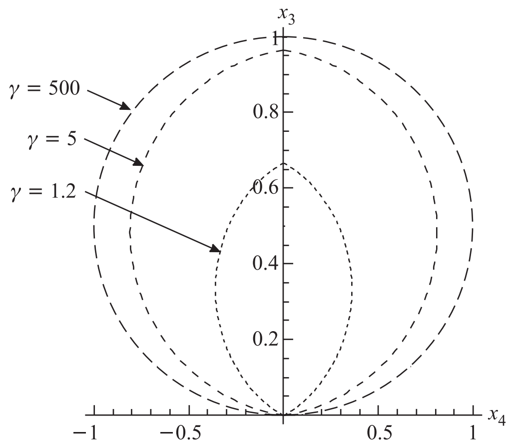

Fig. 1. «Instantonic» paths in the Euclidean plane (x_3, x_4). The faster the field is turned on, the smaller the loop is

Action on a path with winding number n is given by the formula

$$S_n = 2nm \int\limits_{x^4_{\min}}^{x^4_{\max}} dx'^4 \sqrt{1 - \frac{e^2E^2}{4m^2}\left(x'^4 - i\frac{\sqrt{1-\Omega^2(x'^4)^2}}{\Omega}\right)^2}.$$

Explicitly,

$$S = \frac{nm^2}{eE} g(\gamma),$$

where information on γ-dependence is contained in

$$g(\gamma) = \pi\left[\left(1 + \frac{1}{2\gamma^2}\right) - \frac{3}{\pi}\frac{1}{\gamma}\sqrt{1 - \frac{1}{4\gamma^2}}\right].$$

Here $\gamma = \dfrac{m\Omega}{eE}$ has the meaning of Keldysh parameter for this problem. This result contains no temporal dynamics (i.e., no dependence on time passed after field switch-on), which is an intrinsic property of the method [7]. In Fig. 2 one can see the dependence of S on γ.

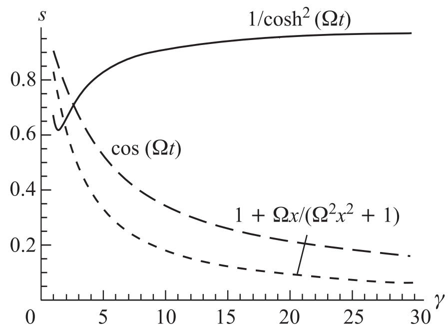

Fig. 2. Action versus Keldysh parameter. For comparison, we show γ-dependence of the 1-instantonic action for a singular pulse field $E \sim \frac{1}{\cosh^2 \Omega t}$, a sinusoidal field $E \sim \sin(\Omega t)$ and the case studied in the present paper $E \sim \left(1 + \frac{\Omega x^0}{\sqrt{1 + (\Omega x^0)^2}}\right)$

Note, the plot starts with $\gamma = 1$, because $g(\gamma)$ is undefined below it in our case. Thus, it is impossible to compare the result with constant field case at $\gamma \to 0$.

It may seem strange that asymptotically for very large Keldysh parameter γ the situation is identical to constant field case, whereas smooth field switch-on is felt in a wider range. However, one should remember that less smooth the field is, less applicable quasiclassical method becomes in general. Therefore, this unnatural behaviour of $g(\gamma)$ should be thought of as a manifestation of inapplicability of semiclassical methods to this case. In the next subsection the preexponential factor is calculated within the same approximation, however, it is clear that one should use time-dependent formalism of Sec. 2 if it is necessary to deal with singular fields.

1.3. 1-Loop Corrections. The instantonic method allows us to express the 1-loop determinant by a simple integral transformation of the field. From formula (3.24) in [8] one can make sure that for scalar QED, whatever the dependence of A on x_4 is, the preexponential for n-instantonic solution is simply and universally expressed in terms of the preexponential for the 1-instantonic solution

$$f_n = \frac{(-1)^{n+1}}{n^2} f_1. \tag{4}$$

From formulae (3.44), (3.45) in [8] with integration limits modified according to (3) one obtains for the preexponential

$$f_1(\gamma) = \frac{-2\sqrt{2}\sqrt{\gamma^2 - 1}\,\gamma^3}{\sqrt{\gamma^2 - 1} - \gamma^2 \operatorname{Arcsec}(\gamma)}. \tag{5}$$

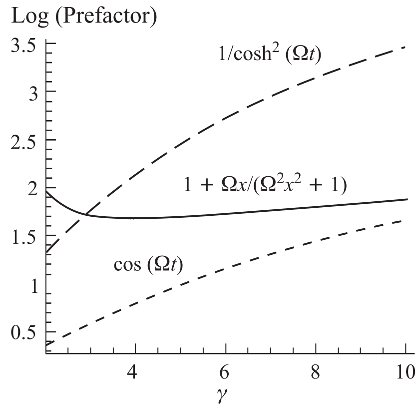

Fig. 3. Exponential prefactor versus Keldysh parameter

The plot of this function versus Keldysh parameter is depicted in Fig. 3. One can see again that this quantity tends to the constant field limit at $\gamma \to \infty$.

The final result is thus for scalar QED

$$\operatorname{Im} \mathcal{L}_{\text{eff}} = f_1(\gamma) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \exp\left(-n \frac{m^2}{eE} g(\gamma)\right)$$

with $g(\gamma) =$ and f_1 given in (5). It is easily generalized for fermionic theories. As for the action, so for the prefactor here, no x_0 dependence has been obtained due to the special feature of the semiclassical method as it yields the particle production rate already integrated over x_0. The result of this section is that for fields switched on fast enough constant field approximation will work in the long run (i.e., at $x_0 \to \infty$) better than for a slowly varying field. The latter, however, can be treated within an adiabatic approximation and is out of our interest.

2. SCATTERING APPROACH

The instantonic method has a clear physical interpretation and is easy to implement, however, it has some disadvantages. As already mentioned, the x_0 dependence of the final result is absorbed into T-integration. In fact, the quantity obtained in the previous section disregards all transition processes. What is calculated may be thought of as an average particle creation rate at sufficiently large times. Then it is obvious that the asymptotics should agree with constant field case. Below a different treatment of the same problem is presented. It will allow us to observe transition phenomena in this system. By the way, no regularization of the field in scattering approach will be necessary, i.e., one can work directly with θ-function-like field, imposing matching conditions on the boundary.

One may start with the familiar derivation, found in textbooks [10, 11]

$$-i\Delta S_{\text{eff}} = \ln\det\left(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m\right) - \ln\det\left(i\gamma^\mu\partial_\mu - m\right) = \\ = \frac{1}{2}\left[\ln\det\left((i\partial - eA)^2 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\right) - \ln\det\left((i\partial)^2 - m^2\right)\right].$$

Then, s-representation for the determinant is introduced; tr is taken over Dirac indices

$$-i\Delta S_{\text{eff}} = \frac{1}{2}\int d^4x \int \frac{ds}{s}\,\mathrm{e}^{-im^2 s}\times \\ \times\,\mathrm{tr}\left(\langle x|\exp\left(is\left((\hat{P}-eA)^2 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)\right)|x\rangle - \langle x|\mathrm{e}^{i\hat{P}^2}|x\rangle\right);$$

after inserting unity decomposition and taking Dirac matrix trace, one gets

$$-i\Delta S_{\text{eff}} = \frac{1}{2}\int d^4x \int \frac{ds}{s}\,\mathrm{e}^{-im^2 s}\frac{d^4p\,d^4p'}{(2\pi)^4}\,\mathrm{e}^{i(p-p')x}\times \\ \times\left(4\theta(x_0)\cosh\,eE_0 s\langle x|\mathrm{e}^{is(\hat{P}-eA)^2}|x\rangle - 4\langle x|\mathrm{e}^{is\hat{P}^2}|x\rangle\right),$$

where $\hat{P}_i$ are momenta operators. Hence, after taking integrals in momentum space

$$= \int d^4x \frac{1}{8\pi^2 i}\int \frac{ds}{s^2}\,\mathrm{e}^{-im^2 s}\times \\ \times\left[\theta(x_0)\cosh eE_0 s\,\frac{1}{\pi}\int dp_3 dp_0 dp_0'\,\mathrm{e}^{i(p_0-p_0')x_0}\left\langle p_0\middle|\mathrm{e}^{is\left(\hat{P}_0^2-(p_3-eE_0x_0\theta(x_0))^2\right)}\middle|p_0'\right\rangle - \frac{1}{s}\right].$$

Thus, effective Lagrangian correction at $x_0 > 0$ is given by

$$\Delta\mathcal{L}_{\text{eff}} = \frac{1}{8\pi^2}\int \frac{ds\,\mathrm{e}^{-im^2 s}}{s^2}\left[eE_0\cosh eE_0 s\, I(x_0, s) - \frac{1}{s} - \frac{1}{3}e^2E_0^2 s\right].$$

We remind here the standard Euler–Heisenberg Lagrangian

$$\Delta\mathcal{L}_{\text{eff}} = \frac{1}{8\pi^2}\int \frac{ds\,\mathrm{e}^{-im^2 s}}{s^2}\left[eE_0 \operatorname{ctanh} eE_0 s - \frac{1}{s} - \frac{1}{3}e^2E_0^2 s\right]$$

(the last term is subtracted in both expressions due to renormalization prescription). Our main goal is to calculate the term

$$I(x_0, s) = \frac{1}{\pi e E_0}\int dp_3 dp_0 dp_0{}' \exp\left(i(p_0 - p_0{}')\left(x_0 - \frac{p_3}{eE_0}\right)\right)\times$$
$$\times\left\langle p_0\right|\exp\left(is\left(\hat{P}_0^2 - e^2E_0^2x_0^2\theta\left(x_0 + \frac{p_3}{eE_0}\right) - p_3^2\theta\left(-x_0 - \frac{p_3}{eE_0}\right)\right)\right)\left|p_0{}'\right\rangle$$

and to compare it with the original $\dfrac{1}{\sinh eE_0 s}$ for constant field. This calculation is performed by solving 1-dimensional reflection problem of quantum mechanics, assuming the operator in the exponent to be the «effective Hamiltonian». We note here that pair production was first described in a similar manner in terms of 1-dimensional oscillator in [12]. The time dependence would have vanished, if the potential of this Hamiltonian had simply been $E^2x_0^2$, as it is in the standard case. The matrix element would be diagonal then, which would eliminate time-dependence at all. However, a piecewise-given potential is treated here, thus x_0 dependence persists.

This expression is being analytically continued by susbtituting $E_0 \to iE_0$, $p_3 \to ip_3$ and one considers

$$I^{\text{An}}(x_0, s) = \frac{1}{\pi e E_0}\int dp_3 dp_0 dp_0{}' \exp\left(i(p_0 - p_0{}')\left(x_0 - \frac{p_3}{eE_0}\right)\right)\times$$
$$\times\left\langle p_0\middle|\exp\left(is\left(\hat{P}_0^2 + e^2E^2x_0^2\theta\left(x_0 + \frac{p_3}{eE_0}\right) + p_3^2\theta\left(-x_0 - \frac{p_3}{eE_0}\right)\right)\right)\middle|p_0{}'\right\rangle.$$

Thus, a positive-definite Hamiltonian has been obtained, which makes reflection problem well-posed. «Reverse» analytical continuation will necessary to be performed at the end of the calculation to return to the original physical domain.

Let us introduce dimensionless variables $\tilde{x} = x\sqrt{eE_0}$, $\tilde{p}_i = \dfrac{p_i}{\sqrt{eE_0}}$, $\tilde{\epsilon} = \dfrac{\epsilon}{eE_0}$, $\tilde{s} = eE_0 s$ and write down $I^{\text{An}}(x_0, s)$ in terms of them

$$I^{\text{An}}(\tilde{x}_0, \tilde{s}) = \frac{1}{\pi}\int d\tilde{p}_3 d\tilde{p}_0 d\tilde{p}_0'\,\mathrm{e}^{i(\tilde{p}_0 - \tilde{p}_0')(\tilde{x}_0 - \tilde{p}_3)}\times$$
$$\times\left\langle \tilde{p}_0\middle|\exp\left(i\tilde{s}(\hat{P}_0^2 + \tilde{x}_0^2\theta(\tilde{x}_0 - \tilde{p}_3) + \tilde{p}_3^2\theta(-\tilde{x}_0 - \tilde{p}_3))\right)\middle|\tilde{p}_0'\right\rangle.$$

Further tilde sign is going to be omitted.

Thus, the problem has now been reduced to studying 1-dimensional Schrödinger equation with Hamiltonian $\hat{H} = \frac{1}{2}\hat{P}_0^2 + V(x_0)$, its potential being $V(x_0) = \frac{1}{2}\left(x_0^2\theta(x_0+p_3) + p_3^2\theta(-x_0-p_3)\right)$.

Now the matrix element of operator $\mathrm{e}^{-2is\hat{H}} = \mathrm{e}^{is(\hat{P}_0^2 + x_0^2\theta(x_0+p_3)+p_3^2\theta(-x_0-p_3))}$ will be calculated. Unity expansion in terms of its eigenfunctions is used $|\psi_\epsilon(x)\rangle$, defined by equation $\hat{H}|\psi_\epsilon\rangle = \epsilon|\psi_\epsilon\rangle$. The spectrum has bound states with energies $0 < \epsilon_n < p_3^2/2$, and free states with energies $\epsilon > p_3^2/2$. For free states one has to solve reflection problem, so that one can find the density of states expressed in terms of phase shift [9, p. 1120]

$$\frac{\partial n}{\partial \epsilon} = \frac{1}{\pi}\frac{\partial \delta}{\partial \epsilon},$$

and δ is expressed in terms of logarithmic derivatives

$$L(p_3, \epsilon) = \left.\frac{\partial \ln \psi_\epsilon(x)}{\partial x}\right|_{x=-p_3}$$

of wavefunctions in the matching point:

$$\delta = \tan^{-1}\frac{L}{k} - p_3.$$

Therefore, the analytically-continued function I^{An} becomes

$$I^{\mathrm{An}}(x_0, s) = \frac{1}{\pi}\int dp_3 dp_0 dp_0{}'\,\mathrm{e}^{i(p_0 - p_0{}')(x_0 - p_3)}\times$$
$$\times\left(\int d\epsilon \langle p_0|\psi_\epsilon\rangle\langle\psi_\epsilon|p_0{}'\rangle\frac{\partial n}{\partial\epsilon}\,\mathrm{e}^{2is\epsilon} + \sum_{0<\epsilon_n<p_3^2/2}\langle p_0|\psi_{\epsilon_n}\rangle\langle\psi_{\epsilon_n}|p_0{}'\rangle\,\mathrm{e}^{2is\epsilon}\right),$$

which, after inserting explicit integral representation of matrix elements $\langle p_0|\psi_\epsilon|p_0{}'\rangle$, simplifies to

$$2\int dp_3\left(\sum_{0<\epsilon_n<p_3^2/2}|\psi_{\epsilon_n}(x_0-p_3)|^2\,\mathrm{e}^{2i\epsilon_n s} + \int\limits_{p_3^2/2}^{+\infty} d\epsilon|\psi_\epsilon(x_0-p_3)|^2\frac{\partial n}{\partial \epsilon}\,\mathrm{e}^{2i\epsilon s}\right).$$

This expression can be split into three parts,

$$I^{\mathrm{An}}(x_0, s) = I_1^{\mathrm{An}}(x_0, s) + I_2^{\mathrm{An}}(x_0, s) + I_3^{\mathrm{An}}(x_0, s), \tag{6}$$

the first one being the contribution of bound states, the two other ones coming from free spectrum. The contribution of free states has been separated into two parts because of the special feature of the potential: for $p_3 > 0$ free spectrum has $p_3^2/2$ as the lowest energy level,

$$I_2^{\mathrm{An}}(x_0, s) = \frac{4}{\pi}\int\limits_0^{+\infty} dp_3\int\limits_{p_3^2}^{+\infty}\frac{d\epsilon}{\sqrt{\epsilon}}\times$$
$$\times\exp\left(2is\left(\epsilon + \frac{p_3^2}{2}\right)\right)\frac{-L\left(p_3, \epsilon + \frac{p_3^2}{2}\right) + 2\epsilon L'_\epsilon\left(p_3, \epsilon + \frac{p_3^2}{2}\right)}{L\left(p_3, \epsilon + \frac{p_3^2}{2}\right)^2 + 2\epsilon}\left|\psi_{\epsilon + \frac{p_3^2}{2}}(x_0 - p_3)\right|^2,$$

whereas for $p_3 < 0$ free spectrum starts already with zero

$$I_3^{\mathrm{An}}(x_0, s) = \frac{4}{\pi} \int_{-\infty}^{0} dp_3 \int_{0}^{+\infty} \frac{d\epsilon}{\sqrt{\epsilon}}\, \mathrm{e}^{2is\epsilon}\, |\psi_\epsilon(x_0 - p_3)|^2\, \frac{-L(p_3, \epsilon) + 2\epsilon L'_\epsilon(p_3, \epsilon)}{L(p_3, \epsilon)^2 + 2\epsilon}.$$

Later, we are going to make sure that the contributions of I_2 and I_3 are negligible. The leading contribution, i.e., sum over bound states in the I_1 is

$$I_1^{\mathrm{An}}(x_0, s) = 2 \int_{0}^{+\infty} dp_3 \sum_{0<\epsilon_n<p_3^2/2} |\psi_{\epsilon_n}(x_0 - p_3)|^2\, \mathrm{e}^{2is\epsilon_n},$$

which can be regrouped as

$$I^{\mathrm{An}}(x_0, s) = 2 \sum_{n=0}^{+\infty} \exp\left(2is\left(n + \frac{1}{2}\right)\right) \int_{\sqrt{2n+1}}^{+\infty} dp_3 |\psi_{\epsilon_n}(x_0 - p_3)|^2.$$

Here one makes use of the fact that bound states eigenfunctions and eigenvalues of this problem are very close to those of harmonic oscillator. The difference is essential only between asymptotics of wavefunctions[1]. But this region does not give any important contributions to $I^{\mathrm{An}}(x_0, s)$. Thus, further simplification arises, as the complicated wavefunctions $\psi_\epsilon(x_0)$ can be traded for simple Hermitian polynomials

$$I_1^{\mathrm{An}}(x_0, s) = 2 \sum_{n=0}^{+\infty} \exp\left(2is\left(n + \frac{1}{2}\right)\right) \int_{\sqrt{2n+1}}^{+\infty} \frac{H_n(p_3 - x_0)^2\, \mathrm{e}^{-(p_3 - x_0)^2}}{\sqrt{\pi} 2^n n!},$$

which becomes

$$I_1^{\mathrm{An}}(x_0, s) = 2 \sum_{n=0}^{+\infty} \exp\left(2is\left(n + \frac{1}{2}\right)\right) \left(1 - \int_{x_0 - \sqrt{2n+1}}^{+\infty} \frac{H_n(p_3)^2\, \mathrm{e}^{-p_3^2}}{\sqrt{\pi} 2^n n!}\right) =$$

$$= -\frac{1}{i \sin s} - 2 \sum_{n=0}^{+\infty} \exp\left(2is\left(n + \frac{1}{2}\right)\right) \frac{H_n(p_3)^2\, \mathrm{e}^{-p_3^2}}{\sqrt{\pi} 2^n n!}.$$

One can interprete this formula intuitively in the following way: in constant field all energy levels contribute to the trace of the operator, whereas, if the field is turned on in a moment, the levels are also «switched on» consequently, dependent on the value of transversal momentum of the wavefunction. This is, to our understanding, the difference between constant field and switched-on field case. By doing the reverse analytic continuation one obtains

$$I_1 = \frac{1}{\sinh s} - 2 \sum_{n=0}^{+\infty} \exp\left(-2s\left(n + \frac{1}{2}\right)\right) \int_{x_0 - \sqrt{2n+1}}^{\infty} dp_3 \frac{H_n(p_3)^2\, \mathrm{e}^{-p_3^2}}{\sqrt{\pi} 2^n n!}. \qquad (7)$$

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[1] $O(x^\alpha\, \mathrm{e}^{-x^2/2})$, $x \to \pm\infty$-type behaviour for harmonic oscillator, whereas for exact solution of our potential asymptotics of type $O(x^\alpha\, \mathrm{e}^{-x})$, $x \to -\infty$ and $O(x^\alpha\, \mathrm{e}^{-x^2})$, $x \to +\infty$.

The last equation makes it possiblle to separate the $\frac{1}{\sinh s}$ term, which is already present in the constant field case, and the second term, which represents the non-trivial contribution of abrupt field switch-on to Euler–Heisenberg Lagrangian. Here we stress that this sum is an exact expression, which is not a perturbation series in E or in x_0, but incorporates all possible non-linear effects in 1-loop vacuum QED. It is obvious that the sum in (7) tends to zero, as $x_0 \to \infty$, i.e., effects of field switch-on gradually die out and one is left with the standard expression.

2.1. Numerical Results. Evaluating (7) is easy, as it contains just 1-dimensional numerical integration. One easily obtains the following asymptotic behaviour, dependent on s

$$\Delta I_1(x_0, s) = \begin{cases} \dfrac{\eta(x_0)}{s}, & 0.3 < s < 1, \\ \mathrm{e}^{-\alpha_2(x_0)s - \beta_2(x_0)}, & 1 < s < \infty. \end{cases}$$

The functions $\alpha_i(x_0), \beta_i(x_0), \eta(x_0)$ have the following simple approximation, obtained numerically in the region $0 < x_0 < 4$:

$$\begin{aligned} &\eta(x_0) = 0.06 - 0.01x_0, \\ &\alpha_2(x_0) = 0.44x_0 + 0.4, \\ &\beta_2 = 0.40x_0 - 1.57. \end{aligned}$$

Thus, it follows that the typical dimensionless time x_0, during which nonstationary effects are seen, is of the order of magnitude $x_0 \sim 1$. Restoring dimensionful time, the typical «nonstationarity time»

$$\tau_{\mathrm{NS}} = \sqrt{eE_0}$$

is obtained.

3. DISCUSSION

Our main result is a simple formula for a time-dependent non-perturbative correction to Euler–Heisenberg Lagrangian, which is valid in case of θ-function external field time-dependence

$$\delta\mathcal{L}_{\mathrm{eff}} = -2\theta(x_0) \int \frac{ds}{s^2} \exp\left(-i\frac{m^2}{eE_0}s\right) \times \\ \times \sum_{n=0}^{+\infty} \exp\left(-2s\left(n+\frac{1}{2}\right)\right) \int\limits_{x_0-\sqrt{2n+1}}^{\infty} dp_3 \frac{H_n(p_3)^2\,\mathrm{e}^{-p_3^2}}{\sqrt{\pi}2^n n!}. \quad (8)$$

Free spectrum contributions have been neglected here for simple reasons. It is stressed once more that (8) has a very simple physical interpretation: abrupt switch-on of the field excites the oscillators of the Schrödinger operator in a non-uniform way. This makes the effective Lagrangian time-dependent. In fact, what has been calculated here can be thought of (in a somewhat loose language) as «vacuum polarizability rate». Here one should agree that, in fact, the most self-consistent treatment of such a system would have been performed in the framework of non-equilibrium thermal QED. So, the result produced here should be thought of reference point for the true thermal QFT [9] treatment.

Another important result of this paper is giving the 1-loop semiclassical result for particle production rate in case of smooth step-like field switch-on. Its limited validity in case of very inhomogeneous field has already been discussed above. It can be believed that both the semiclassical result and the time-dependent $\ln\det$ computation will help to understand the complicated astrophysical processes, especially those taking place around charged black holes. In particular, it can be possible that the two different physical processes — particle production from vacuum by a strong EM field and Hawking radiation — can be treated within the same formalism and we are going to extend our studies towards this more complicated case of two competing processes.

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