Письма в ЭЧАЯ. 2009. Т. 6, № 7(156). С. 127–133

POSSIBLE IMPLEMENTATION OF CNOT AND CCNOT GATES

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The quantum optical gate CCNOT based on five-level atomic scheme is discussed. The qubits are encoded in polarization properties of three optical beams. The nonlinear cross Kerr effect is used for getting nonlinear phase shifts in these beams. A way of CNOT implementation using coupled double quantum wells is suggested. The device is described in the framework of the model based on the operator extensions theory.

Обсуждается реализация квантового вентиля CCNOT на атомной пятиуровневой схеме. Кубиты кодируются векторами поляризации трех пучков. Нелинейный эффект Керра используется для получения нелинейного фазового сдвига. Предложен способ реализации вентиля CNOT с помощью связанных двойных квантовых ям. Работа устройства описана в рамках модели, основанной на теории расширений операторов.

PACS: 03.67.Lx, 01.30.Cc, 03.67.-a

INTRODUCTION

The work of past several years has greatly clarified both the theoretical potential and the experimental challenges of quantum computations. The first aim of our paper is to suggest a version for CCNOT gate implementation. We chose photon as carrier of information. This way has both advantages and disadvantages: decoherentization of photon state is very weak but it is very difficult to ensure the interaction between different photons. There exist works, in which information is encoded by photonic phase (see, e.g., [2]) or squeezed light in the field quadrature. Two different ways have been proposed to circumvent the problem of the absence of significant photon–photon interactions, namely, linear optics quantum computation and nonlinear optical processes that involve few photons. While one is a probabilistic scheme implicitly based on the nonlinearity hidden in single-photon detectors, the other one is based on the enhancement of photon–photon interaction achieved either in cavity QED configurations or in dense atomic media exhibiting electromagnetically induced transparency (EIT).

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The second part of the paper deals with another implementation of qubit based on using of quantum dots and wires (see, e.g., [3]). It is preferable due to well-developed experimental methods of fabrication and investigation of such a system and also due to the fact that this system can be easily used as parts of some conventional nano- and microelectronic structures (e.g., in classical computer).

1. CCNOT GATE

The first aim of our paper is to suggest a version for CCNOT gate implementation. We chose photon as carrier of information. This way has both advantages and disadvantages: decoherentization of photon state is very weak but it is very difficult to ensure the interaction between different photons.

An idea of CNOT gate implementation using so-called M scheme was suggested in [1]. Namely, they rely on the polarization degree of freedom of two travelling single-photon wave packets and exploit the giant Kerr nonlinearities that can be observed in dense atomic media under EIT. In more detail, one has medium in strong magnetic field caused splitting of Rb electrons levels forming particularly M scheme. Note that different transitions in the scheme correspond to different circular polarization of photon. Two weak and well-stabilized probe and trigger beams pass through the medium (initially one has electrons at the ground state of the M scheme only). If the frequency and the polarization of the photon correspond to the transition parameters it causes the increase of higher levels populations. It is the background for nonlinear interaction between these two beams. It allows them to realize CNOT gate. We suggest using M scheme of levels for implementation of CCNOT gate. In this case three weak and well-stabilized beams pass through the cell with rubidium gas. Two of them are probe and the third one is trigger. When the frequency and the polarizations of photons correspond to transmission parameters it changes the conditions of transmitting of the trigger beam. We use this phenomenon to ensure nonlinear interaction between these three beams. Information is encoded by the John's vectors of polarization of the corresponding photon (we mark by «plus» the counter clockwise polarization, and by «minus» the clockwise one):

$$|\Psi_j\rangle = \begin{pmatrix} a_j^+ \, \mathrm{e}^{i\alpha_j^+} \\ a_j^- \, \mathrm{e}^{i\alpha_j^-} \end{pmatrix}, \quad j = \{p, c, t\},$$

where the notations are explained as follows: p corresponds to the first (probe) beam, c — to the second (controlling) and t — to the third (trigger) beam. The initial («IN») three-qubit state is tensor product of three one-qubit states (i.e., it is nonentangled state). The three-qubit gate based on M scheme can be implemented in magnetically confined sample of ^{87}Rb. Three weak and well-stabilized light beams exhibit a strong cross Kerr effect in the five-level structure described in Fig. 1. Two of that beams are probe and another one is trigger. A σ^+ polarized probe couples the excited state $|2\rangle$ to the ground $|1\rangle$ where all the atomic population is initially trapped. The second polarized beam couples the excited state $|2\rangle$ with other Zeeman split ground state $|3\rangle$. State $|3\rangle$ is coupled to level $|4\rangle$ by a σ^- polarized trigger beam. A fourth σ^- polarized tuner beam couples level $|4\rangle$ and a third ground-state sublevel $|5\rangle$. Owing to the tuner, the trigger group velocity can be significantly slowed down similarly to what happens to the probe ones. We anticipate that in the present M scheme the

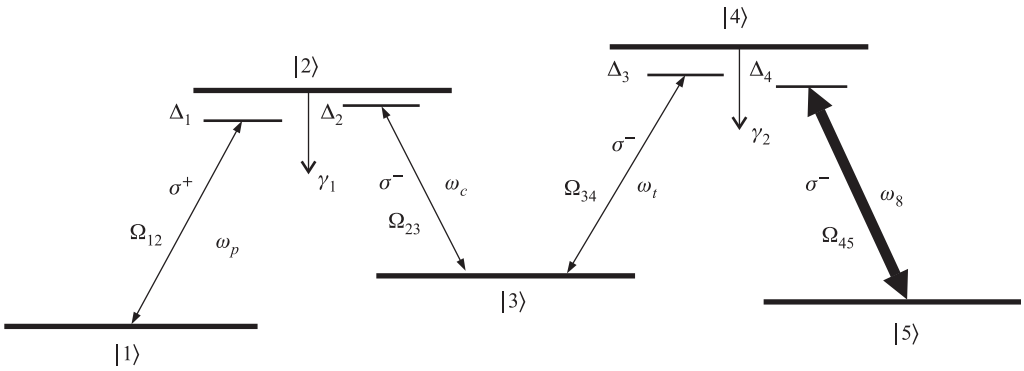

Fig. 1. M scheme

group velocity mismatch can instead be reduced to zero and the cross Kerr nonlinearity made large enough to yield cross-phase shift values of the order of π.

Phase gating is realized when only one of the eight possible probes and trigger polarization configurations exhibits a strong nonlinear cross-phase shift. When all beams are σ^- polarized, it can be seen, in fact, that for not too large detuning there is no sufficiently close excited state to which level $|1\rangle$ couples and no population in $|2\rangle$ and $|3\rangle$ to drive the relevant trigger transition. All beams only acquire the trivial vacuum phase shift. Likewise for other cases when first probe beam is σ^- polarized. In other cases, when the first probe is σ^+ polarized, the electrons from ground state $|1\rangle$ are driven to second level and population there begins to increase. But if the second probe beam is σ^+ polarized there is no population in $|3\rangle$ for trigger transmission. The first probe acquires the nontrivial phase shift, but for other beams we have the previous result. If both of two probe beams have proper polarizations the nonlinear phase shift between them takes place, in other words, there is an addition to trivial phase shift. When the trigger beam is σ^+ polarized the ground state $|3\rangle$ is populated but there is no relevant trigger transmission. At last, the case when the trigger beam is σ^- polarized is more interesting for us because there is nonlinear phase shift between three of input beams. This is the strongest phase shift. Our next target is to find the parameters which ensure the greatest differences between the phase shifts. The Table contains summary phase shifts $(\varphi_p + \varphi_c + \varphi_t)$ for different initial qubit states and indication of performing of the corresponding transition («yes»/«no»).

3-qubit	$1 \to 2$	$2 \to 3$	$3 \to 4$	Summary phase
000	n	n	n	0
001	n	n	n	0
010	n	n	n	0
011	n	n	n	0
100	y	n	n	0
101	y	n	n	0
110	y	y	n	0
111	y	y	y	π

In other words, the M scheme action may be described as an operator of the following transformation:

$$\widehat{M}\begin{pmatrix} a_p^+ a_c^+ a_t^+ \ \mathrm{e}^{i\left(\alpha_p^+ + \alpha_c^+ + \alpha_t^+\right)} \\ a_p^+ a_c^+ a_t^- \ \mathrm{e}^{i\left(\alpha_p^+ + \alpha_c^+ + \alpha_t^-\right)} \\ a_p^+ a_c^- a_t^+ \ \mathrm{e}^{i\left(\alpha_p^+ + \alpha_c^- + \alpha_t^+\right)} \\ a_p^+ a_c^- a_t^- \ \mathrm{e}^{i\left(\alpha_p^+ + \alpha_c^- + \alpha_t^-\right)} \\ a_p^- a_c^+ a_t^+ \ \mathrm{e}^{i\left(\alpha_p^- + \alpha_c^+ + \alpha_t^+\right)} \\ a_p^- a_c^+ a_t^- \ \mathrm{e}^{i\left(\alpha_p^- + \alpha_c^+ + \alpha_t^-\right)} \\ a_p^- a_c^- a_t^+ \ \mathrm{e}^{i\left(\alpha_p^- + \alpha_c^- + \alpha_t^+\right)} \\ a_p^- a_c^- a_t^- \ \mathrm{e}^{i\left(\alpha_p^- + \alpha_c^- + \alpha_t^-\right)} \end{pmatrix} = \begin{pmatrix} a_p^+ a_c^+ a_t^+ \ \mathrm{e}^{i\left(\alpha_p^+ + \alpha_c^+ + \alpha_t^+\right)} \\ a_p^+ a_c^+ a_t^- \ \mathrm{e}^{i\left(\alpha_p^+ + \alpha_c^+ + \alpha_t^-\right)} \\ a_p^+ a_c^- a_t^+ \ \mathrm{e}^{i\left(\alpha_p^+ + \alpha_c^- + \alpha_t^+\right)} \\ a_p^+ a_c^- a_t^- \ \mathrm{e}^{i\left(\alpha_p^+ + \alpha_c^- + \alpha_t^-\right)} \\ a_p^- a_c^+ a_t^+ \ \mathrm{e}^{i\left(\alpha_p^- + \alpha_c^+ + \alpha_t^+\right)} \\ a_p^- a_c^+ a_t^- \ \mathrm{e}^{i\left(\alpha_p^- + \alpha_c^+ + \alpha_t^-\right)} \\ a_p^- a_c^- a_t^+ \ \mathrm{e}^{i\left(\alpha_p^- + \alpha_c^- + \alpha_t^+\right)} \\ -a_p^- a_c^- a_t^- \ \mathrm{e}^{i\left(\alpha_p^- + \alpha_c^- + \alpha_t^-\right)} \end{pmatrix}.$$

The matrix in left hand is a tensor product of three one-qubit states and the matrix in right hand is the result of M transformation which couldn't be represented as a direct product. It allows us to write the M operator in an explicit form:

$$\widehat{M} = \mathrm{diag}\left\{1 - 2\delta_{i,8}\right\}_{i=1}^{8}.$$

To get the matrix of CCNOT operation one should do the following transformations:

$$Q^0 = (I \otimes I \otimes H) \cdot \widehat{M} \cdot (I \otimes I \otimes H),$$

where Q^0 is CCNOT operator; H — the one-qubit Hadamard operator, and I — the identity operator.

Figure 2 shows the results of calculations of the phase shift of different 3-qubit states for the following values of parameters: $\Delta 2 = \Delta 3 = -0.5$, $\Delta 4 = 0.3$, $\Delta 5 = 0$, $\Omega_{12} = 0.54$, $\Omega_{23} = 1.5$, $\Omega_{34} = 0.33$, $\Omega_{45} = 0.36$. Here $10^8 c^{-1}$ is the unit of frequency. One can see that there exist values for which $|111\rangle$-state (governed by the Rabi frequency) differs significantly from the others.

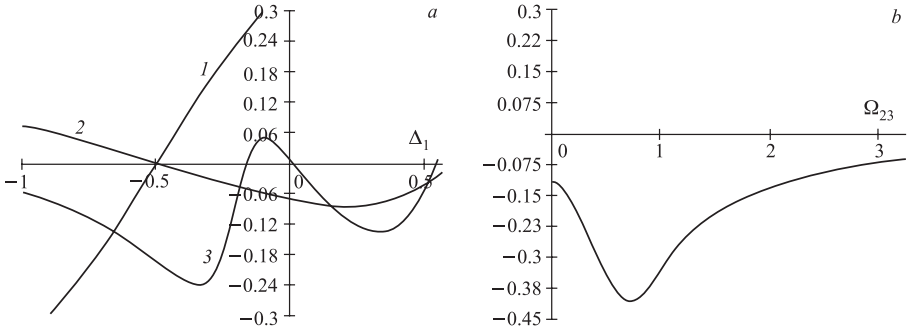

Fig. 2. *a*) Phase shift via detuning Δ_1: *1* — for $|100\rangle$-state; *2* — for $|110\rangle$-state; *3* — for $|111\rangle$-state. *b*) Phase shift for $|111\rangle$-state via Rabi frequency Ω_{23}

2. CNOT GATE

In this part of the paper we suggest a way of implementation of an important quantum two-qubit gate CNOT (XOR). It is based on using of two coupled double quantum dots (quantum wells). We deal with nonsymmetric double well. The parabolic approximation is used for the parts of the double wells. It is not essential, but it allows us to simplify calculations. Consider two-dimensional nonsymmetric quantum well formed by two parabolic potentials

$$V_{1,2} = 2^{-1}\omega_{1,2}^2((x - x_{1,2})^2 + (y - y_{1,2})^2)$$

with different parameters $\omega_{1,2}^2$. Here (x, y) are the Cartesian coordinates on the plane. More precisely, we have the parabolic potential curves outside some neighbourhood of the parabola's intersection and some smoothing in this neighbourhood. In this article we use simple solvable model of the double well which allows us to take into account tunnelling between the parts of the double well. It is sufficient for the purposes of the present paper. More detailed (but essentially more complicated) description based on other methods mentioned above gives us some corrections but does not change the main result — a possibility to construct CNOT gate using this system. That is why here we prefer to use more simple (but mathematically correct) solvable model. The model is based on the theory of self-adjoint extensions of symmetric operators [4,5]. The structure of the model is as follows. The starting point is the orthogonal sum of the Hamiltonians of two independent parabolic quantum wells: $H_0 = H_1 + H_2$ in $L_2(R^2) \oplus L_2(R^2)$, $H_{1,2} = -\dfrac{\hbar}{2m}\Delta + V_{1,2}$. Let us restrict H_0 on the set of smooth functions vanishing at $(x_1, y_1), (x_2, y_2)$. The closure of this operator is a symmetric one with the deficiency indices $(2, 2)$. It has self-adjoint extension which gives us the model in question. The family of the extensions is parameterized by Hermitian matrices of the second order. Of course, there is the operator H_0 among other extensions, but we are interested in those model operators that correspond to some interaction between the wells. One can describe electron tunnelling by choosing appropriate values on the model parameters which play the roles of coupling constants. The Green function $G_{1,2}(r, r', z)$ for the Hamiltonian with parabolic potential is well known. It allows us to get the result for the model operator in an explicit form. Particularly, the resolvent $R(z)$ $(R(z) = (H - z)^{-1})$ is obtained using the Krein formula and has the following form:

$$R(z)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}R_{11}(z) & R_{12}(z)\\ R_{21}(z) & R_{22}(z)\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

where

$$\begin{pmatrix}R_{11}(z) & R_{12}(z)\\ R_{21}(z) & R_{22}(z)\end{pmatrix} = \begin{pmatrix}G_1(r, r', z) & 0\\ 0 & G_2(r, r', z)\end{pmatrix} -$$
$$\begin{pmatrix}G_1(r, 0, z) & 0\\ 0 & G_2(r, 0, z)\end{pmatrix}\begin{pmatrix}a(z) & 0\\ 0 & a(z) + \mu^{-1}\end{pmatrix}^{-1}\begin{pmatrix}G_1(r, 0, \overline{z}) & 0\\ 0 & G_2(r, 0, \overline{z})\end{pmatrix}, \quad (1)$$

$$a(z) = \lim_{r\to 0}\left(G_{1,2}(r, 0, z) - \frac{m}{\pi\hbar^2}\ln\frac{1}{r}\right) = -\frac{m}{2\pi\hbar^2}\left(\frac{\Gamma'\left(\dfrac{1}{2} - \dfrac{z}{2\hbar\omega_{1,2}}\right)}{\Gamma\left(\dfrac{1}{2} - \dfrac{z}{2\hbar\omega_{1,2}}\right)} + \ln\frac{m\omega_{1,2}}{\pi\hbar} + 2C_E\right).$$

Here C_E is the Euler constant; Γ is the Euler Γ function; α, μ are the model parameters. The positive parameter α is related to the tunnelling probability p_t between the wells, $p_t = 0$ ($p_t = 1$) corresponds to zero (infinite) value of the parameter α. Parameter μ is related to the height of the barrier between two parts of double well.

If the system is in homogeneous electric field F, then energy levels in independent parabolic wells shift by

$$\Delta E = \frac{F}{\sqrt{(2m\omega_{1,2}^2)}}. \tag{2}$$

Relation (1) shows that the corresponding level shift is approximately the same for small value of the parameters α, μ^{-1}, i.e., for first levels (near the bottom of the wells). Note that the shift ΔE depends on the well parameter $\omega_{1,2}$ (2). It allows us to change the relative positions of the levels corresponding to different parts of the double well. There are not more than two electrons at each level. In the case of odd number of electrons the localization of the last non-paired electron in the first or the second part of the double well depends on the relative position of the levels which varies if the electric field exists. For some level shifts the position of non-paired electron changes (Fig. 3), and one can use this effect for quantum computation. The double well can be considered as a qubit. Namely, let the state $|1\rangle$ correspond to the localization of the non-paired electron in the left part of the double well and the state $|0\rangle$ — in the right part. Hence, the switching on of the electric field leads to the change of the qubit state, i.e., we do the operation NOT. Figure 3, *b, c* shows the electron transitions corresponding to the change of the qubit state.

Let us consider the analogous second qubit which is situated near the first one and is oriented in such a way that the part of the double well that is chosen as the state $|1\rangle$, is close to the first qubit (see Fig. 3, *a*). The second qubit will be the control qubit and the first one will be the target qubit. If the second qubit is in the state $|1\rangle$ then the corresponding electron

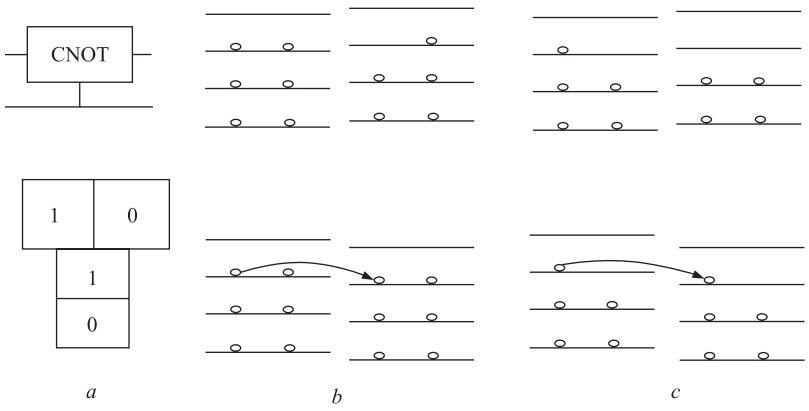

Fig. 3. *a*) Geometrical configuration of the system of two double quantum dots. Indices 0, 1 show the place of non-paired electron localization in the corresponding state. In the picture the upper qubit is the target one and the lower qubit is the control one. *b*) The scheme of electron transition during the transformation of the target qubit from the state $|0\rangle$ to the state $|1\rangle$. *c*) The scheme of electron transition during the transformation of the target qubit from the state $|1\rangle$ to the state $|0\rangle$

is situated near the first qubit, hence it creates in the first double well an electric field which causes the shift of the levels in the first (target) qubit and the change of the state of this qubit. If the control qubit is in the state $|0\rangle$ then the corresponding electron is far from the first double well and its electric field is very weak and for the corresponding values of the wells parameters is nonsufficient for the proper levels shifts causing change of the state of the target qubit. Hence, we do the operation CNOT.

For calculations we use some approximation. Tunnelling between parts of the double well is described in the framework of the solvable model based on the operator extension theory. It is appropriate for the levels far from the top of the barrier between the parts of the well. We consider one-particle Schrödinger equation and mean field approximation for the interaction between the particles. More detailed investigation gives us corrections but does not change the effect. To ensure the coherence it is necessary to keep the system under low temperature — kT should be less than the energy of the level excitations (i.e., approximately 1 mK). The decoherence can be caused by inclusions, i.e., the sample should have high purity. Note that in quantum dot implementations of qubits suggested earlier the qubit control was related to the time during which the external field exists. It is very difficult to determine this time precisely, and the variation of this parameter leads to error. In our case this source of errors is absent. Due to another implementation of qubit (the position of non-paired electron) this parameter does not play any role (we need only that this time be sufficient to ensure electron transitions). It is an essential advantage of our way.

Acknowledgements. The work is supported by RFBR grant.

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