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3D FINITE ELEMENTS WITH HARMONIC BASIS FUNCTIONS FOR APPROXIMATIONS OF HIGH ORDER

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Трехмерные конечные элементы с гармоническими базисными функциями для аппроксимаций высокого порядка

Как известно, наиболее распространенные конечные элементы задаются ячейкой и базисными функциями. В настоящей работе для таких ячеек, как тетраэдр, куб, прямоугольная призма и т.п., разработана методика получения гармонических базисных функций с высокой степенью аппроксимации. В частности, приводятся рекуррентные формулы вычисления и алгоритм генерации базисных функций. Рассматриваются также условия на преобразования координат, сохраняющие свойство гармоничности приближения. Отличительной особенностью полученных конечных элементов является отсутствие внутренних узлов и возможность адаптивного сгущения узлов на границе ячейки. Для тетраэдров и прямых призм построенные базисные функции точно приближают гармонические многочлены второй, третьей, четвертой и пятой степеней.

Предложенные конечные элементы могут использоваться для задач интерполяции и интегрирования гармонических функций, а также для решения проекционными численными методами краевых задач с уравнениями Лапласа и Пуассона в скалярном и векторном случаях. Приводятся примеры hp-интерполяции с высокой точностью дипольных магнитных полей. Приближения, полученные в результате интерполяции, с компьютерной точностью удовлетворяют векторному оператору Лапласа. Для их построения требуется меньшее число узлов по сравнению с обычными лагранжевыми элементами.

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3D Finite Elements with Harmonic Basis Functions for Approximations of High Order

As is known, most wide-spread finite elements are defined as a cell with basis functions. In the present paper for such cells as tetrahedron, cube, rectangular prism, etc., the methods of production of harmonic basis functions with approximation of high order have been developed. In particular, the recursion relations for calculation of the basis functions and the algorithm of their production are presented. The conditions on transformations of coordinates keeping the harmonicity property of the basis functions are also considered. The distinctive peculiarities of obtained finite elements are absence of inner nodes and possibility of adaptive condensation of nodes on a cell boundary. For tetrahedrons and rectangular prisms the constructed basis functions exactly approximate the harmonic polynomials of the second, third, fourth and fifth orders.

The proposed finite elements may be used for problems of interpolation and integration of harmonic functions and also for solving the boundary value problems with Laplace and Poisson equations in scalar and vector cases by means of projective numerical methods. The examples of hp-interpolation of dipole magnetic fields with high accuracy are given. The approximations obtained as a result of the interpolation satisfy the vector Laplacian with computer accuracy. In order to construct the approximations a smaller number of nodes is required in comparison with the usual Lagrange finite elements.

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INTRODUCTION

It is known that finite elements may be used for interpolation and integration of functions [1–5] and also for solving the boundary value problems by projective grid methods [4]. In particular, for construction of approximate solutions of the two-dimensional boundary value problems, in [6–8] it was suggested to apply harmonic functions. Mention the paper [9], in which approximate solutions of the two-dimensional boundary value problems were also constructed in the form of special harmonic functions. At present, spherical and radial harmonic basis functions are used in applied geophysics, medicine, computer graphics, physics of electronic signals, etc., but the accuracy of these functions is not high.

In the present paper the finite-element harmonic basis functions are proposed for construction of approximations of high accuracy in the three-dimensional case. Their peculiarities are absence of inner nodes in a cell and possibility of adaptive condensation of nodes on the cell boundary. The harmonic basis functions have been constructed for tetrahedrons and rectangular prisms with 5 and 6 faces and they exactly approximate the harmonic polynomials of 2, 3, 4 and 5 orders. The algorithm of production of the basis functions, the recursion relations for their calculations are presented and on numerical examples the comparison of convergence rate for the constructed interpolations and the standard Lagrange ones is also given. The example of using the hp-interpolation of high accuracy for a 3D dipole magnetic field with developed finite elements illustrates their efficiency in comparison with the standard Lagrange elements.

A finite element in $\mathbf{R}^n$ $(n = 2, 3)$ can be defined [5, 10] as a triple (ω, P, Φ), where $\omega \in \mathbf{R}^n$ is a closed subset with the Lipschitzian boundary $\partial\omega$ and with a nonempty set $\tilde{\omega}$ of inner points (i.e., a cell, often named as finite element); P is the m-dimensional space of functions defined on ω (usually this is the space of polynomials); Φ is the set ω of linearly independent linear functionals $F_i : P \to \mathbf{R}, i = 1, \dots, m$. In the most wide-spread finite elements $F_i(v)$ is the value of a function v in the node $x_i \in \omega$. If for the set of functions $\{N_j\}_{j=1,\dots,m} \in P$ for every j the system

$$F_i(N_j) = \delta_{ij}, \quad i = 1, \dots, m \tag{1}$$

is solvable, then any function $v \in P$ can be represented in the form

$$v(x) = \sum_{i=1}^{m} F_i(v) N_i(x). \tag{2}$$

In this case N_i are called basis functions. The construction of the functions is connected with construction of interpolational polynomials [3, 10–13] or, in more general case, with using the method of undefined coefficients [1]. In particular, for consctuction of the interpolational polynomial the set of nodes $\{x_i\}_{i=1,\dots,m} \in \omega$ and the set of linearly independent functions $\{f_i\}_{i=1,\dots,m}$ must be given, by which the basis functions are defined

$$N_j = \sum_{i=1}^{m} b_i^{(j)} f_i, \quad j = 1, \dots, m. \tag{3}$$

Here the coefficients $b_i^{(j)}$ are found from the system (1) for every j. Further, the approximate value is defined by formula (2). Note that usually the polynomials, which are the cofactors for partial derivatives in the Taylor series expansion of a function under approximation, are used as the functions f_i and have the form

$$f_i(x) = x_1^{k_{1,i}} x_2^{k_{2,i}} x_3^{k_{3,i}}, \quad k_{1,i} + k_{2,i} + k_{3,i} \geqslant 0.$$

In view of the fact that the Lagrange interpolational polynomials consist of functions in the same form, the basis functions defined in (3) by f_i are said to be the Lagrange functions. An examples of the Lagrange basis functions are given in [5, 10, 12, 13]. The general theorem about solvability of the system (1) for a set of functions $\{f_i\}_{i=1,\dots,m}$ and for a set of nodes $\{x_i\}_{i=1,\dots,m} \in \omega$ is presented in [3].

Because the Taylor series for the functions, satisfying a homogeneous linear differential equation, can have a special form, then it is natural to take into account this peculiarity when forming the approximation to a such function. In particular, for the construction of finite-element nodal basis functions of high order approximation, satisfying the equation, the following approach has been developed. Suppose that the analytical solution of the first boundary value problem is known

$$\begin{cases} \Lambda(u) = 0, & x \in \tilde{\omega}; \\ u = g, & x \in \partial\omega \end{cases}$$

for a linear differential operator Λ in a special region $\tilde{\omega}$. At the nontrivial boundary condition, as a rule, it has the form

$$u(x) = \sum_{i=1}^{\infty} A_i(g) f_i(x),$$

where $A_i(g)$ denotes coefficients and f_i are functions, for which

$$\Lambda(f_i) = 0, \quad x \in \tilde{\omega}.$$

The functions $f_i (i \geqslant 1)$ are linearly independent, therefore, if the stable to round-off errors effective formulas for their calculations are known, then one can use the functions for construction of finite-element basis functions.

Usually basis functions are constructed for «standard» cells, such as n-dimensional cube $[-1,1]^n$, simplex with unit edges in $[0,1]^n$, etc. Further, in order to get basis functions for a real cell, the transformation of the «standard» coordinate system to the real one is used. It is natural to apply the transformations, which do not make worse the accuracy of approximation. Some theorems about this subject are regarded in [5, 10].

Thus, using the different known sets of linearly independent functions for unknown approximation and the methods for construction of interpolational polynomials, in specific cases one can form the different finite elements ensuring more fast convergence to the exact value than usual Lagrange interpolation.

1. GENERAL PROPERTIES OF FINITE ELEMENTS WITH HARMONIC BASIS FUNCTIONS

For more detailed description we shall characterize the harmonic finite elements by a cell ω having the same properties as in the case of Lagrange elements, by a set of nodes $\{x_i\}_{i=1,2,\dots,m} \in \omega$, by a set $P_h = \{f_i\}_{i=1,\dots,m}$ for construction of basis functions, by an algorithm of obtaining the coefficients $b_i^{(j)}$ in equality (3) and by maximal degree of the harmonic polynomial, on which the basis functions are exact. As a cell for proposing finite elements it is natural to use the same cells as for usual Lagrange elements. In view of the fact that harmonic functions have maximal and minimal values on the boundary of domain, for harmonic finite elements it should be used only nodes of the boundary. The maximal degree of harmonic polynomial, on which the basis functions are exact, is defined by the algorithm of obtaining the coefficients. Therefore, the main questions of new elements construction are to choose the set P_h and the algorithm of finding the coefficients $b_i^{(j)}$.

Before turning to choice of the set P_h for standard cells, it should be mentioned the peculiarity of transformation of «standard» coordinate system to a real one. The transformations must keep their harmonicity. Let us give sufficient conditions. Define

$$x_k = x_k(\xi_1, \xi_2, \xi_3), \quad k = 1,2,3,$$

then for the function $q = q(x_1(\xi), x_2(\xi), x_3(\xi))$ we get

$$\frac{\partial q}{\partial \xi_i} = \sum_{k=1}^{3} \frac{\partial q}{\partial x_k} \frac{\partial x_k}{\partial \xi_i}, \quad \frac{\partial^2 q}{\partial \xi_i^2} = \sum_{k=1}^{3} \left(\frac{\partial q}{\partial x_k} \frac{\partial^2 x_k}{\partial \xi_i^2} + \frac{\partial x_k}{\partial \xi_i} \sum_{j=1}^{3} \frac{\partial^2 q}{\partial x_j \partial x_k} \frac{\partial x_j}{\partial \xi_i} \right),$$
$$i = 1,2,3.$$

Therefore,

$$\Delta_\xi q = \sum_{k=1}^{3}\left(\frac{\partial q}{\partial x_k}\sum_{i=1}^{3}\frac{\partial^2 x_k}{\partial \xi_i^2} + \sum_{j=1}^{3}\frac{\partial^2 q}{\partial x_j \partial x_k}\sum_{i=1}^{3}\frac{\partial x_k}{\partial \xi_i}\frac{\partial x_j}{\partial \xi_i}\right).$$

From this the sufficient conditions for realization of $\Delta_\xi q = 0$ are the following:

1) $\Delta_\xi x_k = 0, \quad k = 1,2,3;$

2) $\sum\limits_{i=1}^{3}(\partial x_k/\partial \xi_i)^2 = \sum\limits_{i=1}^{3}(\partial x_j/\partial \xi_i)^2$, $k \neq j$, $k, j = 1,2,3;$

3) $\sum\limits_{i=1}^{3}(\partial x_k/\partial \xi_i)(\partial x_j/\partial \xi_i) = \sum\limits_{i=1}^{3}(\partial x_i/\partial \xi_i)(\partial x_l/\partial \xi_i)$, $k \neq j$, $i \neq l$, $k, j, i, l = 1,2,3.$

The similarity transformation satisfies all these conditions

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c \cdot \Theta(\alpha,\beta,\gamma)\begin{pmatrix} \xi_1 - y_1 \\ \xi_2 - y_2 \\ \xi_3 - y_3 \end{pmatrix},$$

where $c = \text{const}, c \neq 0$, $y = (y_1, y_2, y_3)$ is the center of cell, $\Theta(\alpha,\beta,\gamma) = \Theta_1(\alpha)$ $\Theta_2(\beta)$ $\Theta_3(\gamma)$ is the matrix describing the rotation. Here

$$\Theta_1(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \Theta_2(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix},$$

$$\Theta_3(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix},$$

α, β, γ are the Euler angles [14]. Precisely this transformation will be used later on.

The set of functions P_h for standard cells on plane or in space can be obtained using the representation of the Dirichlet solution in the same special regions as rectangle, circle, parallelepiped, cylinder or sphere. For simplicity, consider the cases of square and circle. According to [15], the solution of the Dirichlet problem in the square $[-1,1] \times [-1,1]$ has the following form:

$$u(x) = \sum_{i=1}^{4} u_i N_i^{(0)}(x) + \sum_{k=1}^{4}\sum_{n=1}^{\infty}\widetilde{u}_n^{(k)}\frac{\mathrm{sh}(\pi n(1 + x_{m,k}x_m)/2)}{\mathrm{sh}\pi n}\sin(\pi n(1+x_{3-m})/2), \tag{4}$$

for $x \in \omega$, where

$$N_i^{(0)}(x) = (1 + x_{1,i}x_1)(1 + x_{2,i}x_2)/4, \quad \widetilde{u}_n^{(k)} = \int_{-1}^{1}\widetilde{u}^{(k)}\sin(\pi n(1 + x_m)/2)dx_m,$$

here $m = 1$, when $k = 1, 3$ and $m = 2$, when $k = 2, 4$. We denoted the function $u - \sum_{i=1}^{4} u_i N_i^{(0)}$ defined on side of the square with number k by $\widetilde{u}^{(k)}$. The numbering of nodes and sides is shown in Fig. 1. Formula (4) can be written in the form:

$$u(x) = \sum_{i=1}^{4} A_i f_i(x) + \sum_{n=1}^{\infty} (A_{4n+1} f_{4n+1}(x) + A_{4n+2} f_{4n+2}(x) + A_{4n+3} f_{4n+3}(x) + \\ + A_{4n+4} f_{4n+4}(x)), \quad (5)$$

where A_i are coefficients and the harmonic functions f_i are elements of the following sequence:

$$1; x_1; x_2; x_1x_2; \sin(k\pi(1+x_1)/2)\mathrm{sh}(k\pi(1+x_2)/2);$$

$$\sin(k\pi(1+x_1)/2)\mathrm{sh}(k\pi(1-x_2)/2); \mathrm{sh}(k\pi(1+x_1)/2)\sin(k\pi(1+x_2)/2);$$
$$\mathrm{sh}(k\pi(1-x_1)/2)\sin(k\pi(1+x_2)/2);\ k = 1, 2, \ldots$$

One more set of linearly independent harmonic functions can be constructed from the representation of the Dirichlet problem inside a circle of radius a. In accordance with [15] in polar coordinate system $x = (r, \varphi)$ the solution of the problem has the form

$$u(x) = C_1 f_1 + C_2 f_2 + C_3 f_3 + \sum_{k=2}^{\infty} (C_{2k} f_{2k} + C_{2k+1} f_{2k+1}), \quad (6)$$

where

$$f_1 = 1/2, \quad f_{2k} = (r/a)^k \cos((k+1)\varphi), \quad f_{2k+1} = (r/a)^k \sin((k+1)\varphi),$$

$$C_{2k} = (1/\pi) \int_0^{2\pi} u(\varphi) \cos((k+1)\varphi) d\varphi, \quad C_{2k+1} = (1/\pi) \int_0^{2\pi} u(\varphi) \sin((k+1)\varphi) d\varphi.$$

Taking into account that $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$ and using the formulas of representation for trigonometrical functions for sum of angles, we get the following recursion relations for calculation of functions f_{2k}, f_{2k+1}:

$$f_{2k} = f_{2k-2}(x_1/a) - f_{2k-1}(x_2/a), \\ f_{2k+1} = f_{2k-2}(x_2/a) + f_{2k-1}(x_1/a),\ k = 2, 3, \ldots, \quad (7)$$

where $f_1 = 1$, $f_2 = x_1/a$, $f_3 = x_2/a$. Notice that for calculation of a harmonic function by means of formula (5), the recursion relations can be also applied.

Using the linearly independent harmonic functions from (5) and (6), the finite-element basis functions can be consructed by different methods. For example, in the case of square, presupposing that a function on the cell boundary is approximated by interpolational Lagrange polynomials of defined degree, from formula (4) we get the corresponding set of harmonic basis functions [17]. One can get basis functions by finding the coefficients in (3) as a solution of the system (1) or by constructing the mean square approximations to approximations of a function by interpolational Lagrange polynomials on the cell boundary. In Figs. 2, 3 the sets of nodes for usual Lagrange elements and for harmonic elements of different approximation orders are shown.

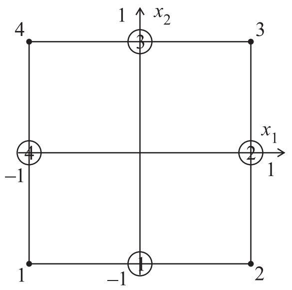

Fig. 1. Numbering of nodes and sides of the standard square

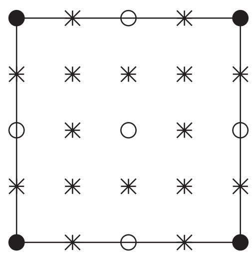

Fig. 2. The nodes of the Lagrange elements: • — linear element, •, ∘ — 9-noded element, •, ∘, ∗ — 25-noded element

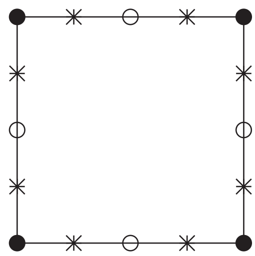

Fig. 3. The nodes of the harmonic elements: • — linear element, •, ∘ — 8-noded element, •, ∘, ∗ — 16-noded element

Let us regard an example of solving the problem of interpolation with high accuracy for a harmonic function using different sets of basis functions. Notice that the problems of interpolation are important for estimating the solutions accuracy of the boundary value problems by means of projective grid methods too, because according to the lemma Cea [10] and to the main theorems about convergence of projective grid methods [4], the estimate of convergence of approximate solutions of the boundary value problems is proportional, as a rule, to the convergence rate of interpolations of the exact solutions by means of basis functions.

For example, we shall interpolate the following function:

$$u^*(x) = B_1^S(x) + B_2^S(x), \quad \vec{B}^S(x) = \frac{1}{4\pi} \int\limits_{\Omega_S} \vec{J} \times grad \frac{1}{|x-y|} d\Omega_y.$$

Here $|x - y|$ is the distance between points x and y. When calculating we presupposed that the region Ω_S has infinity extension along $0x_3$ axis and in the plane $x_3 = 0$ it has the section D in the form

$$D = \{53. \leqslant \mathrm{sign}(x_1)x_1 \leqslant 100., 60. \leqslant \mathrm{sign}(x_2)x_2 \leqslant 85.\}.$$

The vector $\vec{J} = \mathrm{sign}(x_1)\pi(116.348)\vec{i}_3$, $x \in D$.

Note that usually for accelerators and spectrometers symmetric fields of dipole magnets without iron yoke are calculated by the above-mentioned formula for $\vec{B}^S$. For the considering case analytical formulas for calculations of $\vec{B}^S$ is contained, for example, in [16]. In Fig. 4 the partition of the interpolation region by 4 elements $\omega_1, \dots, \omega_4$ is shown. The behavior of u^* in the region is given in Fig. 5.

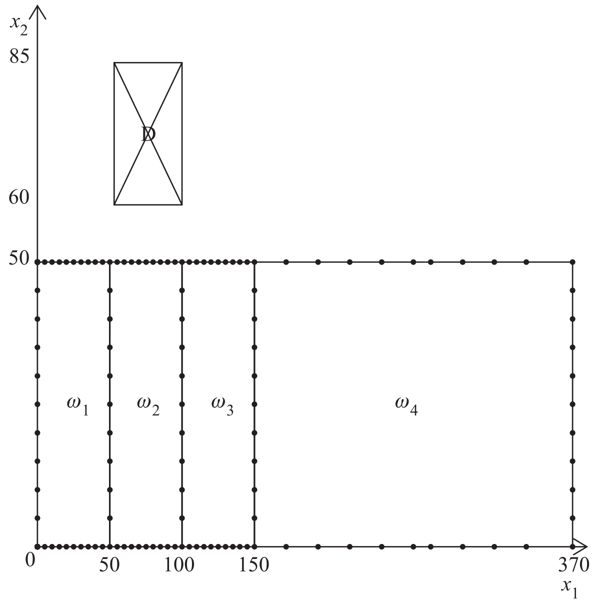

Fig. 4. The partition of interpolation region by elements and the symmetrical part of the region D

In Table 1 the results are presented characterizing the convergence of approximations obtained by different methods for the cell ω_2 with strong variation of the function and for the prolonged cell ω_4 with smooth behavior of the function. Here the following notations are used:

$$\sigma_i^{(k)} = \max_{x \in \hat{\omega}_i} |u^{(k)} - u^*|/u_0, \quad k = 1, 2, 3, \quad i = 2, 4,$$

where $u_0 = 0.2163557E + 04$, $\hat{\omega}_i$ is the grid of nodes obtained by dividing the cell ω_i along every direction of the Cartesian coordinate system into 10 parts.

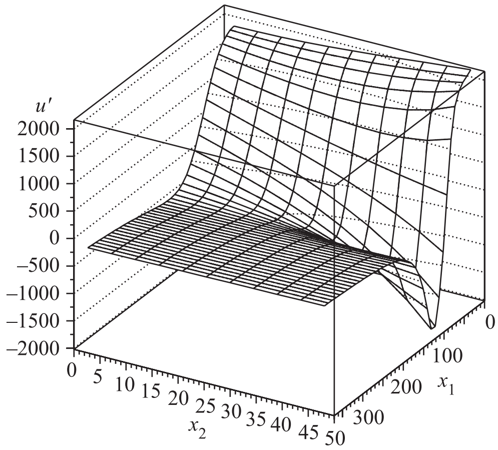

Fig. 5. The behavior of the function under interpolation

Table 1

Element $\omega_2 = \{50. \leqslant x_1 \leqslant 100.; 0. \leqslant x_2 \leqslant 50.\}$					
Number of interpolation nodes	8	16	24	32	40
$\sigma_2^{(1)}$	0.5766E-01	0.1338E-01	0.9945E-03	0.1056E-03	0.3832E-05
$\sigma_2^{(3)}$	0.8761E-01	0.2176E-01	0.6797E-02	0.1641E-02	0.1841E-04
Number of interpolation nodes	9	16	25	36	49
$\sigma_2^{(2)}$	0.7912E-01	0.2082E-01	0.1372E-01	0.4066E-02	0.8893E-03
Number of interpolation nodes	9	25	49		
$\sigma_2^{(4)}$	0.7912E-01	0.2432E-01	0.9918E-02		
Element $\omega_4 = \{150. \leqslant x_1 \leqslant 370.; 0. \leqslant x_2 \leqslant 50.\}$					
Number of interpolation nodes	8	16	24	32	40
$\sigma_4^{(1)}$	0.3147E-01	0.6029E-02	0.3925E-03	0.4002E-04	0.3406E-06
$\sigma_4^{(3)}$	0.4029E-01	0.3168E-01	0.3116E-01	0.3108E-01	0.3105E-01
Number of interpolation nodes	9	16	25	36	49
$\sigma_4^{(2)}$	0.3605E-01	0.1743E-01	0.7247E-02	0.2549E-02	0.7626E-03
Number of interpolation nodes	9	25	49		
$\sigma_4^{(4)}$	0.3605E-01	0.1521E-01	0.7475E-02		

The value $u^{(1)}$ is calculated for every cell with the basis functions constructed by formulas (3) and (7). When calculating $u^{(2)}$, the Lagrange interpolation of approximation orders 2, 3, 4, 5, 6 was used. For $u^{(3)}$ the basis functions were calculated by formula (3) and f_i was the same as in representation (5). At last, the result of 9-point Lagrange interpolation (the order of approximation equals 2) is denoted as $u^{(4)}$.

In Fig. 6 the dependence of convergence of approximations on general number of nodes N is shown in logarithmic scale. The approximations have been constructed by different methods. Here $\sigma^{(k)} = \max\limits_{i} \sigma_i^k$, $k = 1, 2, 3$, $i = 1, 2, \ldots, 4$. When the constructing 9-point nodal Lagrange interpolation, for dividing every element $\omega_1, \ldots, \omega_4$ the sequence 1×1, 2×2 and 3×3 was used.

As it follows from Table 1 and Fig. 6, most fast convergence is observed when using the basis functions formed by means of the set of functions (7). The set of functions (5) also gives the better result of interpolation than the Lagrange

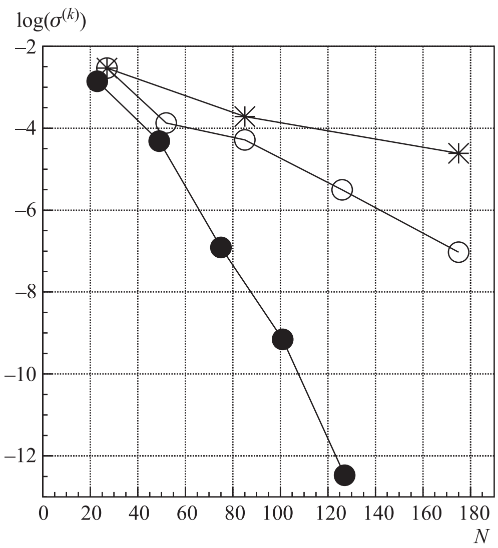

Fig. 6. The convergence of interpolations (from the left to the right and from top to bottom): $*$ — 9-noded Lagrange; $\circ$ — 9, 17, 25, 36, 49-noded Lagrange; $\bullet$ — 8, 16, 24, 32, 40-noded harmonic

one, approximately at the same number of nodes, with exception of the cell in the form of prolonged rectangle. Probably in this case a special condensation of nodes is required.

Other examples of using harmonic basis functions for interpolation in $\mathbf{R}^2$, in particular, for triangular elements are presented in [17].

Let us notice one more distinctive peculiarity of harmonic basis functions construction in the proposed approach. The property can essencially simplify the formation of adaptive interpolations and the solving the boundary value problems with defined accuracy by means of them, because it does not require creation of a special structure of data [18–20]. The question is possibility of adaptive condensation of nodes on boundaries of cells. Present the example of adaptive interpolation on an «irregular» element ω_0 shown in Fig. 7. As a function under interpolation the function u^* was taken. For elements with chosen 8, 16, 24 and

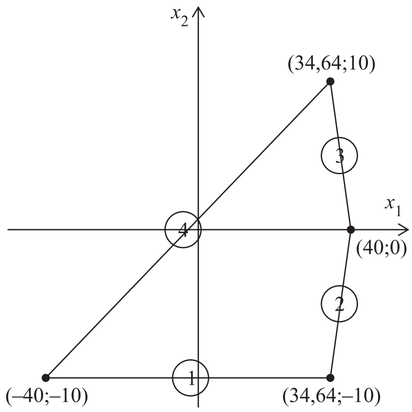

Fig. 7. «Irregular» quadrangular element, coordinates of angular points and numbering of sides

32 nodes the values of relative errors $\max\limits_{x \in \hat{\omega}_0} |u^{(1)} - u^*|/u_0$, where $\hat{\omega_0}$ is the grid of nodes obtained by dividing the cell into 1600 similar cells, equal 0.1792E-02, 0.1265E-04, 0.1453E-06, 0.9400E-08, respectively. In addition, according to the numbering of sides shown in Fig. 7, on every side the following number of nodes is uniformly distributed: 2,0,0,2; 5,1,1,5; 9,1,2,8; 11,2,3,12 — for every element, respectively.

Let us enumerate general properties of the proposed finite elements with harmonic basis functions as a result of the section:

1) they approximate harmonic polynomials of corresponding order exactly;

2) there is a general form of coordinate transformation, keeping the harmonicity property;

3) basis functions are formed by linearly independent sets of harmonic functions, by means of which the solutions of the Dirichlet problems are represented in special regions;

4) there are recursion relations for construction of basis functions;

5) they ensure the absence of inner nodes and the possibility of adaptive choice of nodes on the boundary of a cell.

2. FINITE ELEMENTS IN $\mathbf{R}^3$

Let us regard two algorithms of obtaining the harmonic basis functions for cells in $\mathbf{R}^3$. Both of them are based on using the representation of solution of the Dirichlet problem in a sphere of radius a. According to [15], in the spherical coordinate system $x = (r, \theta, \varphi)$ the solution of the problem has the form

$$u(x) = \sum_{n=0}^{\infty} (\frac{r}{a})^n \sum_{k=0}^{n} (A_{nk}\cos(k\varphi) + B_{nk}\sin(k\varphi))P_n^k(\cos\theta),$$

where $P_n^k(\cos\theta)$ are the Legendre joined functions of argument $\cos\theta$ and

$$A_{00} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(\theta,\varphi)\sin\theta d\theta d\varphi,$$

also for $n > 0$

$$A_{nk} = \frac{(2n+1)(n-k)!}{2\pi(n+k)!}\int_0^{2\pi}\int_0^{\pi} u(\theta,\varphi)P_n^k(cos\theta)\cos(k\varphi)\sin\theta d\theta d\varphi,$$

$$B_{nk} = \frac{(2n+1)(n-k)!}{2\pi(n+k)!}\int_0^{2\pi}\int_0^{\pi} u(\theta,\varphi)P_n^k(cos\theta)\sin(k\varphi)\sin\theta d\theta d\varphi.$$

As f_j we choose the following functions for construction of harmonic basis functions:

$$a_{n+1,k+1} = \frac{(2n+1)(n-k)!}{(n+k)!}(\frac{r}{a})^n\cos(k\varphi)P_n^k(cos\theta), \tag{8}$$

$$b_{n+1,k+1} = \frac{(2n+1)(n-k)!}{(n+k)!}(\frac{r}{a})^n\sin(k\varphi)P_n^k(cos\theta).$$

First, we shall prove the recursion relations for calculation of $a_{n,k}$ and $b_{n,k}$ and further give the algorithm of production of basis functions and the results obtained by the algorithm for cells in the form of tetrahedrons, rectangular prisms of 5 and 6 faces.

Lemma

For calculation of function $a_{n,k}$ and $b_{n,k}$ the following recursion formulas are valid:

$$a_{n+1,1} = \frac{2n+1}{n}\left(\widetilde{x}_3 a_{n,1} - \widetilde{r}^2\frac{n-1}{2n-3}a_{n-1,1}\right),\ n = 1, 2, \ldots,\ a_{1,1} = 1, a_{2,1} = 3\widetilde{x}_3; \tag{9}$$

$$a_{n+1,n+1} = \frac{2n+1}{(2n-1)2n}(\widetilde{x}_1 a_{n,n} - \widetilde{x}_2 b_{n,n}), \quad n = 1, 2, \ldots, \quad a_{1,1} = 1, b_{1,1} = 0.; \tag{10}$$

$$b_{n+1,n+1} = \frac{2n+1}{(2n-1)2n}(\widetilde{x}_2 a_{n,n} + \widetilde{x}_1 b_{n,n}), \quad n = 1, 2, \ldots, \quad a_{1,1} = 1, b_{1,1} = 0.; \tag{11}$$

$$a_{n+1,k+1} = \frac{2n+1}{n+k}\left(\frac{1}{2n-1}(\widetilde{x}_1 a_{n,k} - \widetilde{x}_2 b_{n,k}) + \frac{n-k}{2n-1}\widetilde{x}_3 a_{n,k+1}\right), n = 1, 2, \ldots, \tag{12}$$

$$b_{n+1,k+1} = \frac{2n+1}{n+k}\left(\frac{1}{2n-1}(\widetilde{x}_2 a_{n,k} + \widetilde{x}_1 b_{n,k}) + \frac{n-k}{2n-1}\widetilde{x}_3 b_{n,k+1}\right), n = 1, 2, \ldots, \tag{13}$$

where $\widetilde{x}_i = x_i/a$, $i = 1, 2, 3$ *and* $\widetilde{r} = r/a$.

Proof. Let us prove formula (9) using the recursion formulas [14] for the Legendre polynomials $P_n = P_n^0$. From definition (8) we have

$$a_{n+1,1} = (2n+1)\widetilde{r}^n P_n(\cos\theta) = (2n+1)\widetilde{r}^n\left(\frac{2n-1}{n}\cos\theta P_{n-1} - \frac{n-1}{n}P_{n-2}\right) =$$

$$= \frac{2n+1}{n}(\widetilde{r}\cos\theta(\widetilde{r}^{n-1}(2n-1)P_{n-1}) - (\widetilde{r}^2\frac{n-1}{2n-3}\widetilde{r}^{n-2}(2n-3)P_{n-2})).$$

Therefore, (9) holds. For the proof of formula (10) we shall use the explicit form of P_n^n [14] and usual trigonometrical formulas. We have

$$a_{n+1,n+1} = \frac{2n+1}{(2n)!}\widetilde{r}^n \cos(n\varphi)P_n^n = \frac{2n+1}{(2n)!}\widetilde{r}^n(\cos((n-1)\varphi)\cos\varphi -$$

$$-\sin((n-1)\varphi)\sin\varphi)\prod_{l=1}^{n}(2l-1)\sin^n\theta.$$

From this formula (10) follows. Formula (11) can be proved analogously. For the proof of (12) and (13) we reduce the following recursion formula for the Legendre joined functions:

$$P_n^k(\cos\theta) = (n+k-1)\sin\theta P_{n-1}^{k-1} + \cos\theta P_{n-1}^k. \tag{14}$$

Let us use the two known recursion formulas from [14]

$$P_j^m(t) - P_{j-1}^m(t) - (2j+1)(1-t^2)^{1/2}P_j^{m-1}(t) = 0,$$

$$(2j+1)tP_j^m(t) - (j-m+1)P_{j+1}^m(t) - (j+m)P_{j-1}^m(t) = 0,$$

Here $t = \cos\theta$, $|t| < 1$ and $0 \leqslant m \leqslant j-1$.

Multiplying the first formula on $j+m$ and subtracting the obtained result from the second one, after reduction of similar terms we get

$$P_{j+1}^m(t) = (j+m)(1-t^2)^{1/2} P_j^{m-1}(t) + t P_j^m(t).$$

From this the representation (14) follows at $j+1=n$ and $m=k$.

Let us prove formula (12). We have

$$a_{n+1,k+1} = \frac{(2n+1)(n-k)!}{(n+k)!} \widetilde{r}^n \cos(k\varphi) P_n^k = \frac{(2n+1)(n-k)!}{(n+k)!} \widetilde{r}^n (\cos((k-1)\varphi)$$

$$\cos\varphi - \sin((k-1)\varphi)\sin\varphi)((n+k-1)\sin\theta P_{n-1}^{k-1} + \cos\theta P_{n-1}^k) =$$

$$= \frac{2n+1}{(2n-1)(n+k)} \widetilde{r} \cos\varphi \sin\theta (\frac{(2n-1)(n-k)!}{(n+k-2)!} \widetilde{r}^{n-1} \cos((k-1)\varphi) P_{n-1}^{k-1}) -$$

$$- \frac{2n+1}{(2n-1)(n+k)} \widetilde{r} \sin\varphi \sin\theta (\frac{(2n-1)(n-k)!}{(n+k-2)!} \widetilde{r}^{n-1} \sin((k-1)\varphi) P_{n-1}^{k-1}) +$$

$$+ \frac{(2n+1)(n-k)}{(2n-1)(n+k)} \widetilde{r} \cos\theta (\frac{(2n-1)(n-k-1)!}{(n+k-1)!} \widetilde{r}^{n-1} \cos(k\varphi) P_{n-1}^k).$$

Therefore, (12) holds. Formula (13) is proved analogously.

Go to the algorithm of production of basis functions. For its formulation we introduce the functions g_j, $j \geqslant 1$, which are elements of the following sequence:

$$1; a_{2,1}; a_{2,2}; b_{2,2}; a_{3,1}; a_{3,2}; b_{3,2}; a_{3,3}; b_{3,3}; a_{4,1}; a_{4,2}; b_{4,2}; a_{4,3}; b_{4,3}; a_{4,4}; b_{4,4}; \ldots$$

As initial values of nodes and functions f_j in the algorithm the known linear finite elements are used [10, 12, 21]. The standard cells and nodes of the elements are shown in Figs. 8–10. For construction of basis functions the following sets are used: for the standard simplex — $\{g_1, g_2, g_3, g_4\}$; for the standard right five-faced prism - $\{g_1, g_2, g_3, g_4, g_6, g_7\}$; for the standard cube — $\{g_1, g_2, g_3, g_4, g_6, g_7, g_9, g_{14}\}$.

Let us formulate the algorithm of production of a finite element with n nodes and n basis functions.

1) Let a linear finite element is defined by a standard cell with $n1$ nodes and functions $f_1, f_2, \ldots, f_{n1}$, between which functions $g_1, g_2, \ldots, g_{n2}$, $n2 \leqslant n1$ have sequential numbering. Set $k = n1 + 1$ and $j = n2 + 1$.

2) Add the node with number k to the existing set of nodes.

3) Add the function g_j to the existing set of functions. Obviously that it must not be contained in the existing set.

4) Form system (1) and check its solvability for every $j = 1, 2, \ldots, k$.

5) If the system is solvable and $k = n$, then the set of nodes and the set of functions are formed, $f_k = g_j$ and the problem of production has been solved. If $k < n$, then we set $f_k = g_j$, $k = k+1$,$j = j+1$ and go to step 2.

6) If the system is not solvable, then we set $j = j + 1$ and go to step 3 at $j + 1 \leqslant (M)^2$, where M is enough great defined number. In the case when the last inequality is not valid, we conclude that it is impossible to construct an interpolational polynomial at the given sets of nodes and functions $g_1, g_2, \ldots, g_{(M)^2}$.

The algorithm was used for production of harmonic basis functions for the cells shown in Figs. 8–10. In Table 2 the description of obtained finite elements is given. For check of the basis function accuracy on harmonic polynimials of different degrees, the algorithm of their construction from [22] was used. The total number of linearly independent harmonic polynomials of degree, not higher than n_h, equals $(n_h + 1)^2$. The Table contains only such basis functions that have enough nice approximating properties even at relatively large dimensions of cells. Most full series of finite elements with harmonic basis functions was obtained for cells in the form of simplex. In order to construct elements of higher approximation order, recursion formulas, more stable to round-off error than (9)–(13), are required.

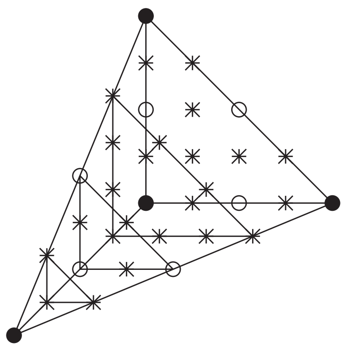

Fig. 8. The standard simplex: $\bullet$ — nodes of linear element, $\bullet, \circ$ — nodes of 10-noded element, $\bullet, \circ, *$ — nodes of 34-noded element

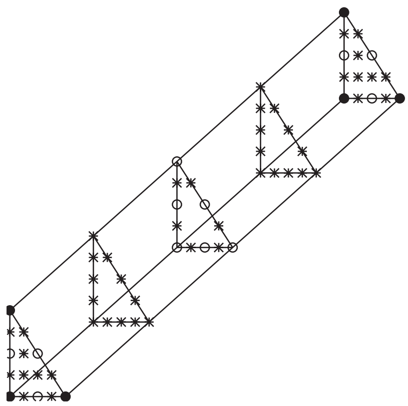

Fig. 9. Five-faced right prism: $\bullet$ — nodes of linear element, $\bullet, \circ$ — nodes of 18-noded element, $\bullet, \circ, *$ — nodes of 66-noded element

Let us regard one more way of obtaining the harmonic functions based on construction of mean square approximations for the Lagrange basis functions defined on the boundary of cell. We interpolate a function f, that is approximated with high accuracy on the cell boundary by means of the Lagrange basis functions L_j, $j = 1, 2, \ldots, m$:

$$f(x) = \sum_{j=1}^{m} F_j(f) L_j(x), \quad x \in \partial\omega.$$

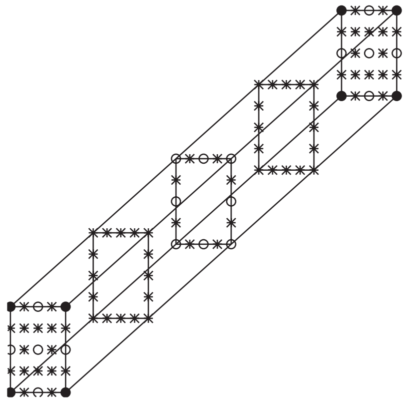

Fig. 10. Parallelepiped: $\bullet$ — nodes of linear element, $\bullet, \circ$ — nodes of 26-noded element, $\bullet, \circ, *$ — nodes of 98-noded element

Table 2

Standard simplex $= \{x_i \geqslant 0, i = 1, 2, 3; x_1 + x_2 + x_3 \leqslant 1\}$		
Number of nodes	Exact for harmonic polynomial of order	Set P_h
10	2	$g_1, \ldots, g_{10}$
20	3	$g_1, \ldots, g_{19}, g_{21}$
34	4	$g_1, \ldots, g_{33}, g_{37}$
52	5	$g_1, \ldots, g_{47}, g_{49}, \ldots, g_{52}, g_{54}$
Five-faced right prism $= \{x_1, x_2 \geqslant 0; x_1 + x_2 \leqslant 1; -1 \leqslant x_3 \leqslant 1\}$		
Number of nodes	Exact for harmonic polynomial of order	Set P_h
18	2	$g_1, \ldots, g_{15}, g_{17}, g_{20}, g_{21}$
38	4	$g_1, \ldots, g_{34}, g_{38}, g_{39}, g_{42}, g_{43}$
Standard cube $= [-1, 1]^3$		
Number of nodes	Exact for harmonic polynomial of order	Set P_h
26	3	$g_1, \ldots, g_{21}, g_{24}, g_{26}, \ldots, g_{28}, g_{37}$
56	5	$g_1, \ldots, g_{39}, g_{41}, \ldots, g_{43}, g_{45}, \ldots, g_{47}, g_{49}, \ldots, g_{52}, g_{54}, g_{58}, g_{62}, g_{66}, g_{67}, g_{69}, g_{86}$
98	7	$g_1, \ldots, g_{69}, g_{71}, \ldots, g_{74}, g_{76}, \ldots g_{80}, g_{82}, \ldots, g_{84}, g_{86}, \ldots g_{88}, g_{93}, g_{95}, \ldots, g_{97}, g_{101}, \ldots, g_{105}, g_{116}, g_{122}, \ldots, g_{124}, g_{145}$

Construct the approximation for every L_j in the form

$$L_j \approx \sum_{i=1}^{m} b_i^{(j)} g_i.$$

We find the coefficients $b_i^{(j)}$, $1 \leqslant i \leqslant m$ from algebraic system of equations arising at differentiation of the functional

$$\Phi_j = (1/2) \int\limits_{\partial\omega} (L_j - \sum_{i=1}^{m} b_i^{(j)} g_i)^2 dS.$$

We have

$$\frac{\partial \Phi_j}{\partial b_k^{(j)}} = \sum_{i=1}^{m} b_i^{(j)} \int\limits_{\partial\omega} g_k g_i dS - \int\limits_{\partial\omega} g_k L_j dS = 0, \quad k = 1, 2, \dots m. \tag{15}$$

Table 3

Standard simplex $= \{x_i \geqslant 0, i = 1,2,3; x_1 + x_2 + x_3 \leqslant 1\}$		
Number of nodes	Exact for harmonic polynomial of order	Set P_h
10	2	$g_1, \dots, g_{10}$
20	3	$g_1, \dots, g_{20}$
34	4	$g_1, \dots, g_{34}$
52	5	$g_1, \dots, g_{52}$

Five-faced right prism = $\{x_1, x_2 \geqslant 0; x_1 + x_2 \leqslant 1; -1 \leqslant x_3 \leqslant 1\}$		
Number of nodes	Exact for harmonic polynomial of order	Set P_h
18	2	$g_1, \dots, g_{18}$
38	4	$g_1, \dots, g_{38}$

Standard cube = $[-1, 1]^3$		
Number of nodes	Exact for harmonic polynomial of order	Set P_h
26	2	$g_1, \dots, g_{26}$
56	3	$g_1, \dots, g_{56}$
98	4	$g_1, \dots, g_{98}$

For calculation of integrals of the system the iterated quadrature formulas with preassigned nodes from [11, 13] were used. Notice the peculiarity of using such formulas at integration over standard simplex $S_1 \in \mathbf{R^2}$. Let T be a triangle in $\mathbf{R^3}$ and v be an enough smooth function, then we have

$$\int\limits_T v(x)dS = \int\limits_{S_1} v(\xi)|\vec{r}_1 \times \vec{r}_2|d\xi = (1/2) \int\limits_{[0,1]^2} \hat{v}(\xi)d\xi =$$

$$= \sum_{0 \leqslant \xi_{1,i}+\xi_{2,i}<1} p_i\hat{v}(\xi_i) + (1/2) \sum_{\xi_{1,i}+\xi_{2,i}=1} p_i\hat{v}(\xi_i).$$

Here $\vec{r}_j = \sum\limits_{k=1}^{3} \vec{i}_k(\partial x_k/\partial \xi_j)$, $j = 1, 2$ [14, 23] and $\hat{v}$ is the symmetric extension on the square $[0, 1]^2$ of the function $v(\xi)|\vec{r}_1 \times \vec{r}_2|$ relatively to the diagonal $\xi_1+\xi_2 = 1$, p_i are weights and ξ_i are nodes of the quadrature formula for $[0, 1]^2$.

In the algorithm of production of basis functions the step 3 must be changed. Now instead of forming and testing the system (1), it is necessary to form and solve the systems (15) for $j = 1, 2, \ldots, m$. Further, the approximate value to the function is used in the form (2). It is obvious that an accuracy of such approximations depends on the accuracy of approximation of the function on the cell boundary by means of the Lagrange basis functions $\{L_j\}_{j=1,2,\ldots,m}$. In Table 3 the results obtained by the algorithm are presented.

3. AN EXAMPLE OF THE PROBLEM OF 3D DIPOLE MAGNETIC FIELD INTERPOLATION

It is known that for magnetic field $\vec{B}$ in the working region of a dipole magnet the equations $\mathrm{div}\vec{B} = 0$ and $\mathrm{rot}\vec{B} = 0$ hold, therefore

$$\nabla^2\vec{B} = \mathrm{grad}(\mathrm{div}\vec{B}) - \mathrm{rot}(\mathrm{rot}\vec{B}) = 0,$$

i.e, the field components are harmonic functions.

In physical experiments with magnetic spectrometers (for example, [24]) for reconstruction of tracks of charged particles it is necessary to restore momenta of the particles using the knowledge of magnetic field in every point of the magnet working region. If it is a parallelepiped, then the measurements on a rectangular grid are carried out and further the interpolational formulas are used. In this case the problem of interpolation can be examined as a particular boundary value problem with special boundary conditions [25] and with conditions inside the region, when the function is defined on a grid of measurements and must satisfy the differential operator.

There are other approaches to solving the interpolation problem for dipole magnetic fields. For example, in the paper [26] the particular attention is given to continuity of field gradient in the points of common boundary between two adjoining finite elements. Notice that in this case the difference of gradients, calculated in adjoining finite elements in the common boundary points, is used as a criterion of estimate of calculations. It was applied in [27] for linear elements. Evidently, that if a region of interpolation Ω is devided by cells and in the every cell a function is the polynomial of order p, then such a function belongs to the space of functions, for which the integral of square of all partial derivatives of order p over the region Ω is limited, i.e., the function belongs to the Sobolev space $W_2^p(\Omega)$. In according to the general embedding theorems for the spaces [28, 10, 7] the partial derivatives of first order of the functions are continuous on boundaries of adjoining elements at $p \geqslant 3$. In the example, given below, the partial derivatives calculated in adjoining elements in the common points of the boundary, coincide not less than in two signes already at $p = 2$ and not less than in four signes at $p = 4$. Some characteristics, making possible to estimate results of the Lagrange interpolation and to conclude about the accuracy of measurments of magnetic field for cells in the form of parallelepiped, are presented in [29]. More general characteristics were used in [30] for calculated field of a dipole magnet including the iron yoke.

Let us give an example of using the new harmonic finite elements for hp-interpolation of two coaxial coils magnetic field (the paper [31] and the book [13] can serve as a good introduction in the main ideas of the hp-finite element method). In Fig. 11 the coils, forming the magnetic field, are shown. And the interpolation region, devided by cells, is given in Fig. 12.

The partition of interpolation region by cells and coaxial coils (the view along the direction of axis Ox_3).

We interpolate a function in the form

$$u^*(x) = B_1^S(x) + B_2^S(x) + B_3^S(x), \quad \vec{B}^S(x) = \frac{1}{4\pi} \int\limits_{\Omega_S} \vec{J} \times grad \frac{1}{|x-y|} d\Omega_y.$$

As a rule, $|x - y|$ is the distance between points x and y. The region Ω_S is defined by the following set:

$$\Omega_S = \{x = (r, \varphi, z) : 53. \leqslant r \leqslant 100.; 0 \leqslant \varphi \leqslant 2\pi; 60. \leqslant sign(x_3)z \leqslant 85.\},$$

and the vector $\vec{J}(x) = \pi(116.348)\vec{i}_\varphi$, $x \in \Omega_S$. The formulas of double analytical integration for calculation of $\vec{B}^S$ in the considered case are contained in [16]. The behavior of the function at $z = 50$ is shown in Fig. 13.

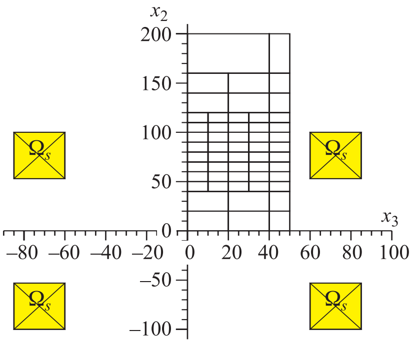

Fig. 11. The partition of interpolation region by cells and coaxial coils forming the magnetic field (the section by plane $x_1 = 0$)

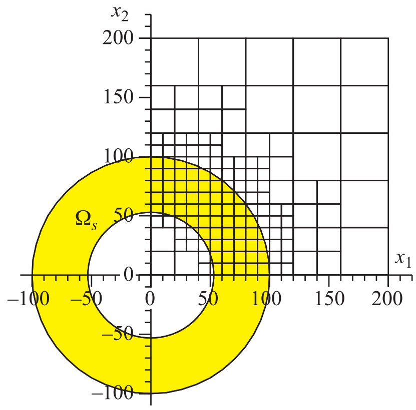

Fig. 12. The partition of interpolation region by cells and coaxial coils (the view along the direction of axis Ox_3)

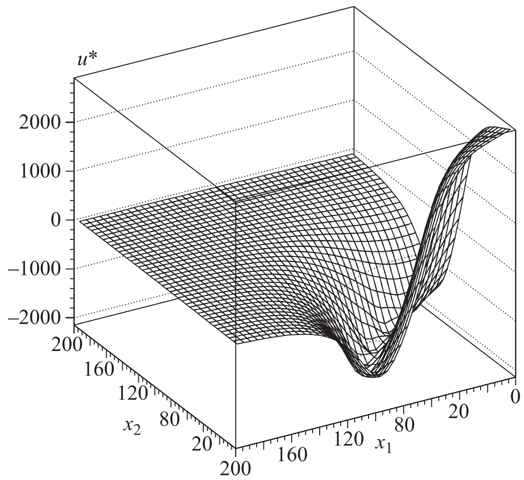

Fig. 13. The behavior of the function under interpolation at $z = 50$

In Tables 4 and 5 the results are presented characterizing the convergence of approximations obtained by different methods for cells with different diameters and strong variation of the function. Here the following notations are used:

$$\delta^{(k)} = \max_{x \in \hat{\omega}} |u^{(k)} - u^*|/u_0, \quad k = 1, 2, 3,$$

Table 4

Tetrahedron $= \{x_i \geqslant 0, i = 1,2,3; x_1 + x_2 + x_3 \leqslant 25\}$				
Number of nodes	10	20	35	56
$\delta^{(1)}$	0.8704E-03	0.1788E-03	0.3059E-04	0.2926E-05
Number of nodes	10	20	34	52
$\delta^{(2)}$	0.1026E-02	0.7903E-04	0.2683E-04	0.2100E-06
$\delta^{(3)}$	0.1026E-02	0.1193E-03	0.2870E-04	0.2692E-05
Tetrahedron $= \{x_i \geqslant 0, i = 1,2,3; x_1 + x_2 + x_3 \leqslant 50\}$				
Number of nodes	10	20	35	56
$\delta^{(1)}$	0.2374E-01	0.4959E-02	0.2132E-02	0.3672E-03
Number of nodes	10	20	34	52
$\delta^{(2)}$	0.2629E-01	0.4855E-02	0.9584E-03	0.2320E-03
$\delta^{(3)}$	0.2305E-01	0.5537E-02	0.2004E-02	0.4154E-03
Tetrahedron $= \{x_1, x_2 \geqslant 50, x_3 \geqslant 0, (x_1 - 50) + (x_2 - 50) + x_3 \leqslant 50\}$				
Number of nodes	10	20	35	56
$\delta^{(1)}$	0.3847E-01	0.1036E-01	0.2378E-02	0.7144E-03
Number of nodes	10	20	34	52
$\delta^{(2)}$	0.3616E-01	0.9045E-02	0.2056E-02	0.7314E-03
$\delta^{(3)}$	0.3909E-01	0.1051E-01	0.2095E-02	0.7892E-03

Table 5

Cube $= [0,25]^3$			
Number of nodes	27	64	125
$\delta^{(1)}$	0.1113E-02	0.1221E-03	0.1830E-04
Number of nodes	26	56	98
$\delta^{(2)}$	0.1534E-03	0.2671E-05	0.3064E-06
$\delta^{(3)}$	0.9778E-03	0.1572E-03	0.1762E-04
Cube $= [0,50]^3$			
Number of nodes	27	64	125
$\delta^{(1)}$	0.6275E-01	0.2996E-01	0.6936E-02
Number of nodes	26	56	98
$\delta^{(2)}$	0.5156E-01	0.4600E-01	0.2268E-01
$\delta^{(3)}$	0.6158E-01	0.2717E-01	0.7454E-02
Cube $= \{50 \leqslant x_1 \leqslant 100, 50x_2 \leqslant 100, 0x_3 \leqslant 50\}$			
Number of nodes	27	64	125
$\delta^{(1)}$	0.6336E-01	0.3938E-01	0.1921E-01
Number of nodes	26	56	98
$\delta^{(2)}$	0.1510E-00	0.6194E-01	0.4724E-01
$\delta^{(3)}$	0.9928E-01	0.3440E-01	0.1712E-01

where $u_0 = 0.2909375E + 04$, $\hat{\omega}$ is the grid of nodes obtained by dividing the cell ω into 10 parts along every direction in the Cartesian coordinate system. The value $u^{(1)}$ is calculated by means of the usual Lagrange interpolation with orders of approximation 2, 3, 4 and, in addition, with order 5 for tetrahedrons. For calculation of $u^{(2)}$ and $u^{(3)}$ the elements, similar to elements from Table 2 and Table 3, are used, respectively. As is obvious from Tables 4 and 5, the convergence of approximations, which is observed in Table 1 and in Fig. 6, is characterized for the regions with smooth behavior of the function.

For construction of interpolational function of high accuracy, the following algorithm is used:

1) Calculation of detailed volume map of gradient of the function.

2) Dividing the interpolation region by cells, depending on the gradient distribution (nodes of the partitions can be not conforming)

3) Using the p-interpolation on the chosen partition, i.e., the order of approximation increases until the required accurary is reached.

For illustration of the algorithm the distribution of $|\mathrm{grad}(u^*)|$ at $z = 50$ is presented in Fig. 14. As is obvious from Fig. 15, the module of gradient is maximal precisely on the plane. In Figs. 11, 12 the partitions by cells in the plane $x_1 = 0$ and in the plane $z = 50$ are given, respectively.

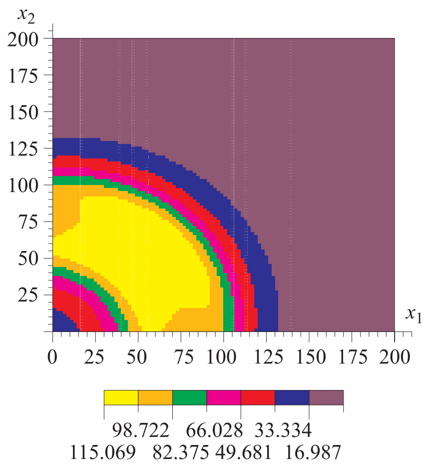

Fig. 14. The distribution of $|\mathrm{grad}(u^*)|$ at $z = 50$

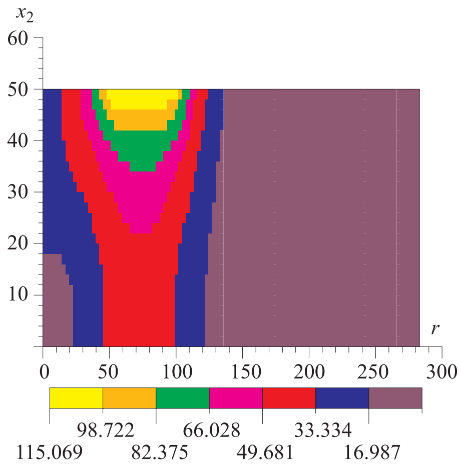

Fig. 15. The distribution of $|\mathrm{grad}(u^*)|$ at $x_1 = x_2$ ($\varphi = 45^o$)

In view of symmetry of the problem the 1/2 part of the grid is used for interpolation. Figures 16, 17 show the result of convergence of different interpolations, depending on total nodes N, for cubes and tetrahedrons, respectively (every cube is devided by 5 tetrahedrons).

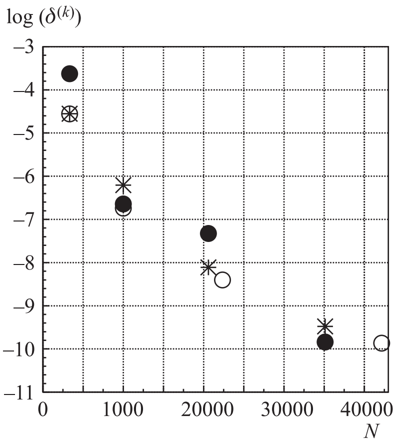

Fig. 16. The convergence of interpolations for tetrahedrons (from the left to the right, from top to bottom): ○ — 10, 20, 35, 56-noded Lagrange; ● — 10, 20, 34, 52-noded harmonic (the first method), ∗ — 10, 20, 34, 52-noded harmonic (the second method)

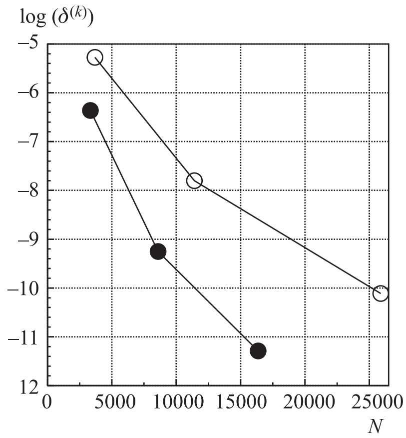

Fig. 17. The convergence of interpolations for hexahedrons (from the left to the right, from top to bottom): ○ — 27, 64, 125-noded Lagrange; ● — 26, 56, 98-noded harmonic (the first method)

For tetrahedrons the best accuracy is reached when using as 42176 points and 56-noded Lagrange elements as 35116 points 52-noded harmonic elements. For hexahedrons the convergence to the exact solution is better than for tetrahedrons, and the best accuracy is reached on smaller number of points at using the harmonic elements. The comparison of interpolations with the Lagrange and harmonic hexahedrons shows that the best accuracy $\approx 0.1 \cdot 10^{-4}$ is reached on 98-noded harmonic elements at the total number of points 16358, that is lesser on 9531 points than for interpolation by 125-noded Lagrange elements.

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REFERENCES

1. *Bahvalov N. S., Zhidkov N. P., Kobelkov G. M.* Numerical Methods. M.: Nauka, 1987 (in Russian).

2. *Fedorenko R. P.* Introduction in Computational Physics. M.: MPTI, 1994 (in Russian).
3. *Mysovskih I. P.* Interpolation Cubature Formulas. M.: Nauka, 1981 (in Russian).
4. *Marchuk G. I., Agoshkov V. I.* An Introduction into Projection-Mesh Methods. M.: Nauka, 1981 (in Russian).
5. *Shaidurov V. V.* Multigrid Methods for Finite Elements. The Netherland, Kluwer Academic, 1995.
6. *Mihlin S. G., Smolicki H. L.* Approximate Methods for Solving Differential and Integral Equations. M.: Nauka, 1965 (in Russian).
7. *Rektorys K.* Variational Methods in Mathematics, Science and Engineering. Dordrecht-Holland, etc., Dr. Reidel, 1980.
8. *Melenk J. M., Babuška I.* Approximation with Harmonic and Generalized Harmonic Polinomials in the Partition of Unity Method // Comput. Assist. Mech. Eng. Sci. (CAMES). 1997. V. 4. P. 607–632.
9. *Volkov E. A.* Approximate Solving Laplace Equation by Block Method on Polygons with Analytic Mixed Boundary Conditions. // Proc. V. A. Steklov Inst. Math. V. 201. M.: Nauka, 1992. P. 165–185 (in Russian).
10. *Ciarlet P.* The Finite Element Method for Elliptic Problems. Amsterdam: North-Holland, 1978.
11. *Krylov V. I.* Approximate Calculation of Integrals. 2nd ed. M.: Nauka, 1962 (in Russian).
12. *Zienkiewicz O. C., Morgan K.* Finite Elements and Approximation. N.Y.: A. Wiley-Interscience, 1983.
13. *Szabó B., Babuška I.* Finite Element Analysis. N.Y.: A. Wiley-Interscience, 1991.
14. *Korn G. A., Korn T. M.* Mathematical Handbook for Scientists and Engineers. N.Y.: McGraw-Hill Book, 1968.
15. *Budak B. M., Samarskii A. A., Tikhonov A. N.* A Collection of Problems on Mathematical Physics. Oxford: Pergamon Press, 1964.
16. *Gyimesi M. et al.* Biot-Savart Integration for Bars and Arcs // IEEE Trans. Mag. 1993. V. 29, No. 6. P. 2389–2391.
17. *Yuldashev O. I., Yuldasheva M. B.* Harmonic Basis Functions for Finite Elements of High Order Approximations. JINR LIT Scientific Report 2006–2007. Dubna: JINR, 2007. P. 317–320 (in Russian).
18. *Demkowicz L., Oden J. T., Rachowicz W., Hardy O.* Toward a Universal h–p Adaptive Finite Element Strategy. Part 1. Constrained Approximation and Data Structure // Comp. Meth. Appl. Mech. Engin. 1989. V. 77. P. 79–122.
19. *Šolin P., Červený J., Doležel I.* Arbitrary-Level Hanging Nodes and Automatic Adaptivity in the hp-FEM // Math. Comp. Simul. 2008. V. 77, No. 1. P. 117–132.
20. *Chetverushkin B. N., Churbanova N. G., Soukhinov A. A., Trapeznikova M. A.* Simulation of Contaminant Transport in a Heterogeneous Porous Medium. M.: Moscow State Univ., 2006. P. 48.

21. *Sabonnadiere J. C., Coulomb J. L.* The Method of Finite Elements and CADs. M.: Mir, 1989 (in Russian).

22. *Tikhonov A. N., Samarskii A. A.* Equations of Mathematical Physics. Oxford: Pergamon Press, 1963.

23. *Brebbia C. A., Telles J. C. F., Wroubel L. C.* Boundary Element Techniques. Berlin: Springer, 1984.

24. *Aleev A. N., Arefiev V. A., Balandin V. P. et al.* EXCHARM Spectrometer // Instrum. Exp. Tech. 1999. V. 42, No. 4. P. 481–492.

25. *Baranov D. A., Belov A. V., Beljakova T. F. et al.* Effective Program FLRECON for Reconstructing 3D Field with Data on Boundary of Calculation Region // Proc. of XV Workshop on Charged Particle Accelerators. Protvino, IHEP, 1996 (in Russian).

26. *Akishin P. G., Ivanov V. V., Litvinenko E. I.* 3D B-Spline Approximation of Magnetic Fields in Inclined Dipole Magnets. JINR LIT Scientific Report 2006-2007. Dubna: JINR, 2007. P. 209–212.

27. *Vodopianov A. S., Shishov Yu. A., Yuldasheva M. B., Yuldashev O. I.* Computer Models of Dipole Magnets of a Series VULCAN for the ALICE Experiment. JINR Commun. E11-98-385. Dubna, 1998. 18 p.; http://preprints.cern.ch/cgi-bin/setlink?base=preprint&categ=scan&id=SCAN-9909054

28. *Sobolev S. L.* Some Applications of Functional Analysis in Mathematical Physics. Providence, Rhode Island, Amer. Math. Soc., 1991.

29. *Zhidkov E. P., Yuldashev O. I., Yuldasheva M. B.* A Projection Method for Solving Linear Problems with the Divergence, Curl Operators and Its Application in Magnetostatics // Bulletin of Peoples' Frendship University of Russia. Series Appl. and Comp. Math. 2002. N0. 1(1). P. 79–86.

30. *Ritman J., Yuldashev O. I., Yuldasheva M. B.* An Algorithm for Construction of Dipole Magnets Computer Models with Quality Control and Its Application for the PANDA Forward Spectrometer. JINR Commun. E11-2005-49. Dubna, 2005. 18 p.; http://www1.jinr.ru/Preprints/2005/049(E11-2005-49).pdf

31. *Babuška I.* On the h, p and h–p Version of the Finite Element Method. Czech-Slovak Conference on Differential Equations and Their Applications, Bratislava, 1993 // Tatra Mountains Mathematical Publications. 1994. V. 4. P. 5–18.

32. PAW — Physics Analysis Workstation. CERN Program Library. Geneva: CERN, 1999.

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