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FINITE-SIZE SCALING AND STRONG SPACE ANISOTROPY: $O(\infty)$ SPIN MODELS

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Тончев Н. С. E17-2008-159
Конечно-размерное подобие и сильная пространственная
анизотропия: $O(\infty)$ спиновые модели

Получены аналитические результаты для конечно-размерного подобия в d-мерных $O(\infty)$ спиновых моделях с сильной пространственной анизотропией, описываемых модельным гамильтонианом, в котором взаимодействие между спинами в $\mathbf{k}$-пространстве имеет асимптотику вида $\sum_{i=0}^{l} a_i|\mathbf{k}_i|^{2\sigma_i}$. Здесь $\mathbf{k}_i$ — векторы с размерностями p_i такие, что $\sum_{i=0}^{l} p_i = d,\ l > 0$, где $a_i \leqslant 0$ — некоторые константы и σ_i — параметры, характеризующие радиус взаимодействия. Рассмотрены геометрии систем типа бесконечных пластин: $L^{p_0}(\sigma_0) \times \infty^{p_1}(\sigma_1) ... \times \infty^{p_l}(\sigma_l)$ с периодическими граничными условиями. Показано, что существуют размерности d между верхней и нижней критическими размерностями, для которых имеет место конечно-размерное подобие, поведение которого однозначно определяется параметрами $\sigma_0, P_0,\ \gamma_l := 1 - \sum_{i=1}^{l}(p_i/2\sigma_i)$. Типичными примерами являются системы с критическими точками Лифшица и/или с квантовыми критическими точками.

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Finite-Size Scaling and Strong Space Anisotropy: $O(\infty)$ Spin Models

We present analytical results for the finite-size scaling behavior of d-dimensional $O(\infty)$ spin systems with strong space anisotropy, described by a model Hamiltonian with interaction between spins which in the $\mathbf{k}$ space has leading terms of the form $\sum_{i=0}^{l} a_i|\mathbf{k}_i|^{2\sigma_i}$. Here $\mathbf{k}_i$ are p_i-dimensional vectors, such that $\sum_{i=0}^{l} p_i = d,\quad l > 0,\ a_i \leqslant 0$ are some constants and $\sigma_i > 0$ are parameters controlling the range of interaction. We consider systems confined to a d-dimensional layer with the geometry $L^{p_0}(\sigma_0) \times \infty^{p_1}(\sigma_1) ... \times \infty^{p_l}(\sigma_l)$ and periodic boundary conditions across the finite p_0 dimensions with the characteristic length «L». It is shown that there are dimensions d between the lower and upper critical dimensions for which the finite-size scaling behavior of the model is uniquely determined by the parameters $\sigma_0,\ p_0,\ \gamma_l := 1 - \sum_{i=1}^{l}(p_i/2\sigma_i)$. Prominent examples are systems with the Lifshitz point critical behavior and/or quantum critical behavior.

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1. INTRODUCTION

In the anisotropic systems the critical behavior is described by correlation functions that depend on the space direction. Although these systems are known for a long time, their investigation generates one of the most challenging problems of the physics of critical phenomena. One should distinguish between the weakly anisotropic and strongly anisotropic systems. In the former case only metric factors but not critical exponents depend on the direction. Strongly anisotropic systems are omnipresent in soft matter and solid state physics. Prominent examples are ferroelectric liquid crystals, block copolymers, uniaxial ferroelectrics, high-temperature superconductors with columnar defects, dipolar-coupled uniaxial magnetic systems, and even some systems near a quantum critical point (see, e.g., [1–5] and references therein).

One of the key concepts of the modern theory of critical phenomena is the concept of universal finite-size scaling [6, 7]. For the systematic study of finite-size critical behavior of the weakly anisotropic systems, one can see, e.g., [8, 9] and references therein. Much less is known for the description of finite-size effects in strongly anisotropic systems. It has been an exciting area of recent theoretical research [10–19]. Unfortunately, interesting finite-size results for strongly anisotropic systems are rarely accessible to purely analytical analysis. In Refs. [20,21] we suggested a recipe for studying systems with mixed geometry; i.e., some sizes of the sample were finite and some were extended to infinity.

The goal of the present study is two-fold. First, we show that the method of calculations [20, 21] may be extended to systems with another type of lattice anisotropy. Second, we analyze the role of the lattice anisotropy which is a microscopic characteristic in conjunction with the shape anisotropy, which is a geometric characteristic of the sample. Since our consideration involves both anisotropy in space and confined geometry, complex finite-size effects may be expected to occur if a phase transition takes place in the bulk. In view of this complexity, theoretical work on a simplified model will be a rationalizing and guiding step towards an ultimate and realistic description of the critical phenomena in such samples. As we show, the simplest possible consideration is in the framework of the $O(N)$-vector model in the $N \to \infty$ limit. Although this model is of no direct experimental relevance it is preferred as it appears to be the main testground for finite-size scaling theories, see [6, 7, 22, 23].

2. THE MODEL SYSTEMS

The microscopic interactions between spins in a sample usually enter the expressions of the theory through their Fourier transforms. Here we will consider a class of d-dimensional systems whose anisotropic interactions are either of short-ranged or of long-ranged kind in different lattice directions. To describe this, let us split the Euclidean space R^d into $R^{p_0} \times R^{p_1} \times ... \times R^{p_l}$, where the momenta $\mathbf{k}_i$ in R^{p_i} are p_i-dimensional vectors. The nonnegative integers $\{p_i;\ i = 0, ..., l;\ l > 0\}$ that characterize the anisotropy of the interactions are such that $\sum_{i=0}^{l} p_i = d$. We consider the following small-$\mathbf{k}$ expansion of the Fourier transform of the spin–spin coupling:

$$J(\mathbf{k}) \simeq J(0) + \sum_{i=0}^{l} a_i |\mathbf{k}_i|^{2\sigma_i}, \quad \sigma_i > 0. \tag{2.1}$$

Here, a_i are metric factors and $\sigma_i > 0$ are parameters controlling the range of interaction. For simplicity, in our analysis we will assume $a_i = -(1/2)a^{2\sigma_i}$ (a is the lattice spacing), which is not a principal restriction. We assume a d-dimensional system with *mixed* geometry (i.e., p_0 finite and $n = p_1 + ... + p_l$ infinite dimensions) under periodic boundary conditions in the finite dimensions. Further, for the geometry of the system under consideration, the symbol $L^{p_0}(\sigma_0) \times \infty^{p_1}(\sigma_1) ... \times \infty^{p_l}(\sigma_l)$ will be used.

The question is whether any realistic interaction can lead to expression (2.1) in the $\mathbf{k}$-space. Obviously it is the case where the rotational symmetry of the n-dimensional Euclidean space is broken. Prominent examples are related with the Lifshitz point critical behavior [24]. For example, in [25] a model with possible anisotropy of the derivative terms of the corresponding Ginzburg–Landau Hamiltonian has been considered. It is pointed out that the isotropy can be restored (in the long-wavelength limit) by appropriate rescaling of the axes at the level of second derivatives; however, there is no such a possibility at higher orders. For details, see [25] from where one can obtain a model with a concrete realization of the quite general expression (2.1). Such are models with the so-called «n-axial Lifshitz point of character K» [26] (or «n-axial of order K-1 Lifshitz point» [27]), see also [25, 28]. In this case a possible form of the sample in conjunction with the anisotropy of the interaction is $\{p_0 = d - l,\ p_i = 1,\ \sigma_0 = Q,\ \sigma_i = K;\ K, Q > 0,\ i = 1, ...,\ l = n\}$.

Expression (2.1) can be adapted to describe some quantum models. In quantum models, the bosonic Matsubara frequencies in the free propagator may be regarded as an additional component (dimension) of the wavevector space (see, e.g., [29–31] and references therein). Furthermore, in quantum systems one can identify the finite $p_0 = 1$ direction with «imaginary time» and so $|\mathbf{k}_0| = \omega_k = 2\pi Tk$ $(k = 1, 2, ...)$. In particular we have models with $\sigma_0 = 1$ in the case of structural phase transitions or $\sigma_0 = 1/2$ in the case of superconducting phase transitions [29] and $\{l = 1, p_1 = n, \sigma_1 < 2(= 2)\}$ for the long-range (short-range) inter-spin interaction. Thus, the long-range interaction in the real space provides strongly anisotropic behavior [14] and, as a consequence, expression (2.1) takes place in the theory like an effective highly anisotropic interaction. Another example is the anisotropic quantum Lifshitz point model considered in [32] with $\{\sigma_0 = p_0 = 1,\ p_1 = m,\ \sigma_1 = 2,\ \sigma_i = p_i = 1,\ i = m+1, ...,\ d-1\}$.

Our aim is to present an approach which allows one to consider analytically isotropic and a wide class of strongly anisotropic systems with different geometries on an equal footing. That is why we will try to keep the values σ_i, p_i as general as possible, for the moment, imposing no other restrictions than the obvious ones: $\sigma_i > 0$ and $p_i > 0$.

3. THE GAP EQUATION

Let (2.1) describe the interaction between classical N-vector spins couple in an $O(N)$ symmetric fashion. In the large N-limit, the theory is solved in terms of the gap equation (obtained by a steepest descent calculation, see, e.g., [22]) for the parameter λ_V related with the finite-volume correlation length of the system. Such a type of systems (with $l = 1,\ p_1 = n$), focusing on the shape dependence of the finite-size scaling limit, is studied in [18]. In [21] the case $l = 1, p_1 = n$ has been considered where the main interest is focused on the *explicit form* of the scaling equation analytically tractable in different regimes. For more details in this particular case, see [18, 21].

For a system with spin–spin interaction (2.1) and mixed geometry $L^{p_0}(\sigma_0) \times \infty^{p_1}(\sigma_1) ... \times \infty^{p_l}(\sigma_l)$, the gap equation has the form

$$\beta = \frac{1}{(2\pi)^n L^{p_0}} \int\limits_{B(n)} \sum_{\mathbf{k}_0 \in \Lambda^{p_0}} \frac{d^{p_1}\mathbf{k}_1 d^{p_2}\mathbf{k}_2 ... d^{p_l}\mathbf{k}_l}{|\mathbf{k}_0|^{2\sigma_0} + \sum\limits_{i=1}^{l} |\mathbf{k}_i|^{2\sigma_i} + \lambda_V}, \tag{3.1}$$

where the infinite dimensions $n = p_1 + ... + p_l$ and β is the appropriately redefined inverse temperature. In the finite directions the corresponding summations are over the vector $\mathbf{k}_0 = \{k_{0_1}, ..., k_{0p_0}\}$ that takes values in Λ^{p_0} defined by $k_{0\nu} = 2\pi n_\nu/(aN_0)$ and $-(N_0 - 1)/2 \leqslant n_\nu \leqslant (N_0 - 1)/2$, where N_0 is an odd integer and $\nu = 1, ..., p_0$. In the infinite directions the sums are replaced by normalized integrals (in accordance with the rule $\frac{\pi}{a} \to \frac{\pi}{a^{2\sigma_0}}$) over the corresponding part of the first Brillouin zone $B(d)$.

The critical point of the infinite d-dimensional system is characterized by a vanishing λ_∞, so that the appropriately scaled inverse critical temperature

$$\beta_c = \frac{1}{(2\pi)^d} \int\limits_{B(d)} \frac{d^d\mathbf{k}}{|\mathbf{k}_0|^{2\sigma_0} + \sum\limits_{i=1}^{l} |\mathbf{k}_i|^{2\sigma_i}} \tag{3.2}$$

is finite whenever the effective dimensionality $D = \sum\limits_{i=0}^{l} p_i/\sigma_i$ is greater than 2.

The infinite n-dimensional system, with a finite size L in the remaining p_0 dimensions, can be found in three qualitatively different states depending on the value of its effective dimension D:

(i) If $2 < \sum\limits_{i=1}^{l} \frac{p_i}{\sigma_i}$, then the system is above its lower critical dimension and, therefore, it exhibits true critical behavior. A crossover from n-dimensional to d-dimensional critical behavior takes place when $L \to \infty$.

(ii) In the borderline case of $2 = \sum\limits_{i=1}^{l} \frac{p_i}{\sigma_i}$, the system is at its lower critical dimension and may have only a zero-temperature critical point.

(iii) When $\sum\limits_{i=1}^{l} \frac{p_i}{\sigma_i} < 2$, the system is below its lower critical dimension and (d-dimensional) critical behavior appears only in the thermodynamic limit $L \to \infty$.

We assume that there is no phase transition for finite L, and henceforth in the present study the following additional principal restriction on the parameters σ_i, p_i is imposed:

$$\sum_{i=1}^{l} \frac{p_i}{2\sigma_i} < 1. \tag{3.3}$$

In the continuous limit, in which $a \to 0$ keeping all other lengths fixed, the gap equation (3.1) may be presented in the form (see appendix A)

$$K = \frac{1}{L^{p_0}} \sum_{\mathbf{k}_0 \in \Lambda^{p_0}} \frac{1}{(\lambda_V + |\mathbf{k}_0|^{2\sigma_0})^{\gamma_l}}, \quad 0 < \gamma_l < 1, \quad l \geqslant 1, \tag{3.4}$$

where K is an effective inverse critical temperature (see Eq. (A.8)), and

$$\gamma_l := 1 - \sum_{i=1}^{l} \frac{p_i}{2\sigma_i}. \tag{3.5}$$

Several comments are in order. The above result is an $l > 1$ generalization of the gap equation recently studied in [21]. One can relate Eq. (3.4) with a fictitious p_0-dimensional fully finite-size isotropic reference system in which the memory of the anisotropic characteristics of the initial system is retained only in the parameter γ_l. As a result, only three model parameters p_0, σ_0 and γ_l are relevant for describing the model.

Let us note that Eq. (3.4) embodies quite different physics. If $\{\sigma_i = \sigma,\ p_i = 1,\ i = 1, ..., l\}$, then $l = n$ and the result γ_n coincides with the result γ_1 obtained in Ref. [21] for $l = 1$ and $\{\sigma_1 = \sigma,\ p_1 = n\}$ which correspond to a different symmetry of the lattice. For example, in bulk systems with cubic spatial anisotropy, one distinguishes [33] between the critical behaviors of the case $\{l = n,\ \sigma_i = 2,\ p_i = 1,\ i = 1, ..., l\}$

$$\tilde{k}^4 \equiv \sum_{i=1}^{n} k_i^4 \tag{3.6}$$

and the usual isotropic case with $\{l = 1,\ \sigma_1 = 2,\ p_1 = n\}$

$$k^4 \equiv (\mathbf{k}.\mathbf{k})^2, \qquad \mathbf{k} \in R^n. \tag{3.7}$$

However, this is not the case in the geometry under consideration, having $\gamma_1 = \gamma_n$. In other words, the anisotropy of the interaction in the subspace $R^{p_1} \times ... \times R^{p_l}$ does not affect the finite-size behavior if the system has a geometry elongated (up to infinity) in the same directions. The shape anisotropy diminishes the effect of spatial anisotropy of the interaction at least in the spherical limit $N \to \infty$.

4. FINITE-SIZE SCALING

Although Eq. (3.4) seems to be very simple in form, in the presence of $\gamma_l \neq 1$ the actual analysis of it is complicated. This complication may be overcome by the approach presented in Refs. [20, 21].

In order to obtain the scaling form of Eq. (3.4), let us first introduce the convenient scaling variables:

$$x = L^{2\sigma_0(p_0/2\sigma_0 - \gamma_l)}(K - K_\infty^c), \qquad y = \lambda_V L^{2\sigma_0}, \tag{4.1}$$

where K_∞^c is the inverse critical temperature of the bulk system. Now the calculation procedure proceeds as in the $l = 1$ case with the substitution $\gamma \to \gamma_l$ in the corresponding equations of [21]. The final result, valid in the large N_0 asymptotic regime, is

$$x \approx -a(p_0; \sigma_0, \gamma_l) y^{(p_0/2\sigma_0 - \gamma_l)} + F^{\gamma_l}_{p_0, 2\sigma_0}(y) + \frac{1}{y^{\gamma_l}}. \tag{4.2}$$

In Eq. (4.2),

$$a(p_0;\sigma_0,\gamma_l) = -\frac{1}{(4\pi)^{p_0/2}\sigma_0}\frac{\Gamma\left(\frac{p_0}{2\sigma_0}\right)\Gamma\left(\gamma_l-\frac{p_0}{2\sigma_0}\right)}{\Gamma\left(\frac{p_0}{2}\right)\Gamma(\gamma_l)} \tag{4.3}$$

and

$$F^{\gamma}_{\alpha,\beta}(y) = \frac{1}{(2\pi)^{\beta\gamma}}\int\limits_0^{\infty} dz z^{\frac{1}{2}\beta\gamma-1}E^{\gamma}_{\frac{\beta}{2},\frac{\beta\gamma}{2}}\times$$
$$\times\left(-\frac{z^{\frac{\beta}{2}}y}{(2\pi)^{\beta}}\right)\left[A^{\alpha}(z)-1-\left(\frac{\pi}{z}\right)^{\frac{\alpha}{2}}\right] \tag{4.4}$$

is a generalization of the so-called universal scaling function, introduced earlier in the finite-size scaling theory by a number of authors (see, e.g., [7] and references therein). Here the difference is the appearance of the generalized Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$ in the integrand (for a definition and some properties of $E^{\gamma}_{\alpha,\beta}(z)$, see, e.g., [21, 34] and references therein). As usual

$$A(z) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-zn^2}. \tag{4.5}$$

Knowledge of the properties of the generalized Mittag-Leffler function allows one to carry out further calculations analytically.

Our consideration is limited to systems with an effective dimension $D = \sum\limits_{i=0}^{l} p_i/\sigma_i$ below the upper critical dimension $D_u = 4$ and above the lower one, $D_l = 2$, i.e., for real dimensions $d = p_0 + n$:

$$2\sigma_0 + n\left[1-\frac{\sigma_0}{n}\sum_{i=1}^{l}\frac{p_i}{\sigma_i}\right] < d < 4\sigma_0 + n\left[1-\frac{\sigma_0}{n}\sum_{i=1}^{l}\frac{p_i}{\sigma_i}\right]. \tag{4.6}$$

The conditions (4.6), together with the conditions for γ_l, take the form

$$\max\left[0,\frac{p_0}{2\sigma_0}-1\right] < \gamma_l < \min\left[1,\frac{p_0}{2\sigma_0}\right]. \tag{4.7}$$

In this way we demonstrate that finite-size scaling in its standard form with scaling variable $y^{1/2\sigma_0} = L\lambda_V^{1/2\sigma_0} \equiv L/\xi_{p_0,L}$ takes place for the considered class of systems regardless of the nature of their anisotropic properties. It is important to note that this statement is established under the conditions (4.7) for the model parameters $\sigma_0,\ p_0,\ \gamma_l$.

We emphasize that the invariance of the model under a lower symmetry group leads to minimal mathematical complications in comparison with the case $l = 1$. This allows the study of the finite-size scaling and the derivation of the finite-size corrections to the bulk results of the considered models.

5. CROSSOVER RULES

Since the numbers of the dimensions p_i and the parameters σ_i enter only in combination $\sum\limits_{i=1}^{l} \dfrac{p_i}{\sigma_i}$ in terms of the reference system (up to a renormalization of the temperature), some specific *crossover rules* take place. Symbolically, we can write

$$L^{p_0}(\sigma_0) \times \infty^{p_1}(\sigma_1) ... \times \infty^{p_l}(\sigma_l) \Longleftrightarrow L^{p_0}(\sigma_0) \times \infty^{\tilde{n}}(\sigma_0), \qquad (5.1)$$

where $\tilde{n} := \sigma_0 \sum\limits_{i=1}^{l} \dfrac{p_i}{\sigma_i}$; i.e., the finite-size behavior of the strongly anisotropic system ($\sigma_i \neq \sigma_0$) with p_0 finite and n infinite dimensions is equivalent to an «effective isotropic» system ($\sigma_i = \sigma_0$) with p_0 finite and $\tilde{n}$ infinite dimensions, and vice versa. Likewise, we can write

$$L^{p_0}(\sigma_0) \times \infty^{p_1}(\sigma_1) ... \times \infty^{p_l}(\sigma_l) \Longleftrightarrow L^{p_0}(\sigma_0) \times \infty^{p_0}(\tilde{\sigma}), \qquad (5.2)$$

where $\tilde{\sigma} := p_0 \sum\limits_{i=1}^{l} \dfrac{\sigma_i}{p_i}$. Indeed, (5.1) and (5.2) are true under the conditions (4.7) for the model parameters $\sigma_0,\ p_0,\ \gamma_l$.

And in the end, let us demonstrate the usefulness of relations (5.1) and (5.2) in calculations of some critical point amplitudes $\bar{y} \equiv y(p_0,\ \sigma_0,\ \gamma_l) = L/\xi_{p_0,\ L}$.

i) If $p_0 = \sigma_0 = 1$, we have $0 < \gamma_l < 1/2$. From the crossover rule (5.1) it follows that the gap equation for λ_V, apart from a trivial rescaling of the temperature and the number of infinite dimensions, i.e., $1 < \tilde{n} < 2$, is independent of the anisotropy. As a result, it is possible to apply the usual theory for isotropic systems. For example, the value of the universal scaling amplitude $\bar{y} \equiv y(1, 1, \gamma_l)$ may be taken directly from the study of the quantum crossover counterpart of the model (2.1), i.e., the quantum spherical spin model [7, 35, 36] or the $O(N)$ quantum φ^4 model in the spherical limit [30]. In the case under consideration, the universal scaling amplitude $\bar{y}$ is the solution of the equation (for details, see [30, 35])

$$\left|\Gamma\left(\frac{1-\tilde{n}}{2}\right)\right| + 4 \sum_{m=1}^{\infty} \frac{K_{(\tilde{n}-1)/2}(my)}{(\frac{1}{2}my)^{(\tilde{n}-1)/2}} = 0. \qquad (5.3)$$

Here $K_\nu(x)$ is the Mac Donald function (second modified Bessel function).

ii) If $p_0 = 2,\ \sigma_0 = 1$, we have $\gamma_l = 1/2$. Due to the crossover rule (5.1) and «the quantum to classical crossover», from the numerical analysis of the two-dimensional quantum spherical model in the case of block geometry at zero temperature ($\tilde{n} = 1$), one gets for the universal amplitude the result $\bar{y} \equiv y(2, 1, 1/2) = 1.5119...$ [30, 35].

iii) With the help of crossover rule (5.2), one can utilize the result obtained in the framework of the one-dimensional $O(\infty)$ quantum φ^4 model [30], i.e., where $p_0 = 1,\ \sigma_0 = 1/2$ and $\tilde{\sigma} = 1$. For the cases when $\{p_i\}$ and $\{\sigma_i\}$ obey the conditions $0 < \sum\limits_{i=1}^{l} p_i/\sigma_i < 2$ and $\sum\limits_{i=1}^{l} \sigma_i/p_i = 1$, one gets the result $\bar{y} \equiv y(1, 1/2, \gamma_l) = 0.6247...$ [30].

6. DISCUSSION

A list of some systems exhibiting finite-size scaling governed by the corresponding parameters $\sigma_0,\ p_0,\ \gamma_l$ is systematized in Table.

Notice that, due to the conditions (4.7), the interval in d when finite-size scaling is established, not always coincides with the lower and upper critical dimensions. Illustrative in this sense is the considered simplest case of isotropic short-range interaction. It is well known that in this case the finite-size scaling takes place between $2 < d < 4$, for both slab and cylinder geometries (see, e.g., [7]). Apparently in the former case our consideration is not complete since it discards the interval $3 \leqslant d < 4$ (see the last column of Table). More generally, «windows» appear in the values of d between the lower and upper critical dimensions where our approach works. Formally, a sufficient condition to avoid this problem is $p_0 = 2\sigma_0$. For such systems standard finite-size scaling between its lower and upper critical dimensions can be established.

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APPENDIX A: DERIVATION OF EQ. (3.4)

Under the conditions (3.3) and $\lambda_V \to 0$, due to the convergence of the integrals in (3.1) over $\{\mathbf{k}_i;\ i = 1, ..., l\}$, one can extend the integration over all R^n in consistence with the underlying continuum field theory. Let us recall that a finite linear dimension $L = N_0 a$ in the continuous limit means that the lattice spacing $a \to 0$ and simultaneously $N_0 \to \infty$.

Following Refs. [20, 21], let us *first* integrate over the infinite dimensions in Eq. (3.1) that reduces the system to a fictitious fully finite-size one. Our assertion is that in the case of anisotropic systems this approach has some advantages allowing analytical treatment of the problem.

Doing the angular integrations in Eq. (3.1), the corresponding n-dimensional integral can be presented as

$$\frac{S_{p_1}...S_{p_l}}{(2\pi)^n L^{p_0}} \int_0^\infty ... \int_0^\infty \sum_{\mathbf{k}_0 \in \Lambda^{p_0}} \frac{x_1^{p_1-1}...x_l^{p_l-1} dx_1...dx_l}{|\mathbf{k}_0|^{2\sigma_0} + \sum_{i=1}^{l} x_i^{2\sigma_i} + \lambda_V}, \tag{A.1}$$

where $S_{p_i} = 2(\pi)^{p_i/2}/\Gamma(p_i/2)$ is the surface of the p_i-dimensional unit sphere. Now, the l-dimensional integral (A.1) is easily performed with the help of the

Some examples of real systems with anisotropic spin–spin interactions and mixed geometry. In the last column the dimensional windows in which the considered standard finite-size scaling takes place are presented

	$J(k) - J(0)$	$L^{p_0}(\sigma_0)\times$ $\times\infty^{p_1}(\sigma_1)... \times \infty^{p_l}$	$p_0,\ \sigma_0,\ \gamma_l$	d
i)	$k_0^2 + k_1^2 ... + k_{d-1}^2$	$k_0 \in \Lambda^1$, $k_i \in R^1$, $l = d-1$	$1, 1, \frac{3-d}{2}$	$2 < d < 3$
	$\|\mathbf{k}_0\|^2 + k_1^2$	$\mathbf{k}_0 \in \Lambda^{d-1}$, $k_1 \in R^1$, $l = 1$	$d-1, 1, \frac{1}{2}$	$2 < d < 4$
ii)	$k_0^{2\sigma} + k_1^{2\sigma} ... + k_{d-1}^{2\sigma}$	$k_0 \in \Lambda^1$, $k_i \in R^1$, $l = d-1$	$1, \sigma, 1 - \frac{d-1}{2\sigma}$	$2\sigma < d < 2\sigma + 1$, $\sigma > \frac{1}{2}$
				$1 < d < 4\sigma, \sigma \leqslant \frac{1}{2}$
	$\|\mathbf{k}_0\|^{2\sigma} + \|\mathbf{k}_1\|^{2\sigma}$	$\mathbf{k}_0 \in \Lambda^{d-n}$, $\mathbf{k}_1 \in R^n$, $l = 1$	$d-n, \sigma, 1 - \frac{n}{2\sigma}$	$2\sigma < d < 4\sigma$, $n < 2\sigma$
iii)	$\|\mathbf{k}_0\|^2 + \|\mathbf{k}_1\|^4$	$\mathbf{k}_0 \in \Lambda^{d-n}$, $\mathbf{k}_1 \in R^n$, $l = 1$	$d-n, 1, 1 - \frac{n}{4}$	$2 + n < d < 4 + \frac{n}{2}$, $n < 4$
	$\|\mathbf{k}_0\|^4 + \|\mathbf{k}_1\|^2$	$\mathbf{k}_0 \in \Lambda^{d-n}$, $\mathbf{k}_1 \in R^n$, $l = 1$	$d-n, 2, 1 - \frac{n}{2}$	$4 - n < d < 8 - n$, $n < 2$
iv)	$k_0^{2Q} + k_1^{2K} ... + k_{d-1}^{2K}$	$k_0 \in \Lambda^1$, $k_i \in R^1$, $l = d-1$	$1, Q, 1 - \frac{d-1}{2K}$	$2K - \frac{K}{Q} < d - 1 < 2K$, $Q > \frac{1}{2}$
				$1 < d < 1 + 4K - \frac{K}{Q}$, $Q \leqslant \frac{1}{2}$
v)	$k_0^2 + \|\mathbf{k_1}\|^4 + \|\mathbf{k_2}\|^2$	$k_0 \in Z$, $\mathbf{k}_1 \in R^m$, $\mathbf{k}_2 \in R^{d-m}$, $l = 2$	$1, 1, 1 + \frac{m}{4} - \frac{d}{2}$	$1 + \frac{m}{2} < d < 2 + \frac{m}{2}$

identity:

$$\int_0^\infty ... \int_0^\infty \frac{x_1^{p_1-1}...x_l^{p_l-1}dx_1...dx_l}{|\mathbf{k}_0|^{2\sigma_0} + \sum_{i=1}^{l} x_i^{2\sigma_i} + \lambda_V} = \frac{a(\{p_i\},\{\sigma_i\};l)}{(|\mathbf{k}_0|^{2\sigma_0} + \lambda_V)^{\gamma_l}}, \tag{A.2}$$

where

$$a(\{p_i\},\{\sigma_i\};l) := \frac{\Gamma(\frac{p_1}{2\sigma_1})...\Gamma(\frac{p_l}{2\sigma_l})}{2^l\sigma_1...\sigma_l}\Gamma(\gamma_l), \tag{A.3}$$

and

$$\gamma_l := 1 - \sum_{i=1}^{l} \frac{p_i}{2\sigma_i}. \tag{A.4}$$

From $\sigma_i > 0$, $p_i > 0$ and inequality (3.3) immediately follow the conditions

$$0 < \gamma_l < 1. \tag{A.5}$$

So for Eq. (A.1) we end up with the result

$$\frac{A(\{p_i\},\{\sigma_i\};l)}{L^{p_0}} \sum_{\mathbf{q}_\perp \in \Lambda^{p_0}} \frac{1}{(\lambda_V + |\mathbf{k}_0|^{2\sigma_0})^{\gamma_l}}, \tag{A.6}$$

where

$$A(\{p_i\},\{\sigma_i\};l) := \frac{S_{p_1}...S_{p_l}}{(2\pi)^n} a(\{p_i\},\{\sigma_i\};l). \tag{A.7}$$

If we absorb $A(\{p_i\},\{\sigma_i\};l)$ in the temperature and introduce the notation

$$K := K(\{\sigma_i\},\{p_i\};l) \equiv \beta/A(\{p_i\},\{\sigma_i\};l), \tag{A.8}$$

from (A.6) and (A.8) one obtains the gap equation (3.4).

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