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NUMERICAL INVESTIGATION OF RENORMALIZATION GROUP EQUATIONS IN A MODEL OF VECTOR FIELD ADVECTED BY ANISOTROPIC STOCHASTIC ENVIRONMENT

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Численное исследование уравнений ренормгруппы в модели векторного поля, адвектированного анизотропной стохастической средой

Рассмотрено влияние сильной одноосевой маломасштабной анизотропии на стабильность скейлинговых режимов в инерционном интервале в модели пассивного поперечного векторного поля, адвектированного несжимаемым турбулентным потоком, с использованием теоретико-полевой ренормгруппы. Предполагается, что поле скоростей имеет гауссовскую статистику с нулевым средним и с определенным шумом с конечными временными корреляциями. Показано, что скейлинговые режимы в инерционном интервале связаны с существованием стабильных инфракрасных неподвижных точек соответствующих уравнений ренормгруппы с определенными угловыми интегралами. Приведен анализ интегралов. Задача решена численно, и граничные значения пространственной размерности $d_c \in (1,3]$, ниже которых скейлинговый режим нестабилен, найдены как функции параметров анизотропии.

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Numerical Investigation of Renormalization Group Equations in a Model of Vector Field Advected by Anisotropic Stochastic Environment

Using the field-theoretic renormalization group, the influence of strong uniaxial small-scale anisotropy on the stability of inertial-range scaling regimes in a model of passive transverse vector field advected by an incompressible turbulent flow is investigated. The velocity field is taken to have a Gaussian statistics with zero mean and defined noise with finite time correlations. It is shown that the inertial-range scaling regimes are given by the existence of infrared stable fixed points of the corresponding renormalization group equations with some angle integrals. The analysis of integrals is given. The problem is solved numerically and the borderline spatial dimension $d_c \in (1,3]$ below which the stability of the scaling regime is not present is found as a function of anisotropy parameters.

The investigation has been performed at the Laboratory of Information Technologies, JINR.

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1. INTRODUCTION

During the last two decades the so-called toy models of advection of a passive scalar field (concentration of an impurity, temperature, etc.) or a vector field (weak magnetic field in a conductive environment) by a given Gaussian statistics of the velocity field have played the main role in the theoretical investigations of intermittency and anomalous scaling in fully developed turbulence [1,2]. The reason for this is twofold. On the one hand, the breakdown of the classical Kolmogorov–Obukhov phenomenological theory of fully developed turbulence [3] is more noticeable for simpler models of a passively advected scalar or vector quantity than for the velocity field itself and, on the other hand, the problem of a passive advection is easier from theoretical point of view (see, e.g., [1] and references therein).

An effective approach for studying self-similar scaling behavior is the method of the field-theoretic renormalization group (RG) [4] which can also be used in the theory of fully developed turbulence and related problems [5, 6]. During the last decade the so-called rapid-change models of a passive scalar or vector quantity advected by a self-similar white-in-time velocity field (also known as Kraichnan model for scalar case and Kazantsev–Kraichnan model for vector field) and their various generalized descendants were analyzed. It was shown that within the field-theoretic RG approach the anomalous scaling is related to the existence of «dangerous» composite operators with negative critical dimensions in the framework of the operator product expansion (OPE) [5,6].

Nevertheless, one particular model of a passive vector advection is much more complicated for theoretical investigations than the others even in the case that the vector field is advected by the velocity field with a Gaussian statistics. It is the model where the so-called stretching term is absent (the so-called $A = 0$ model, see, e.g, [7, 8, 10]). The investigation of the anomalous scaling of correlation functions in this model is essentially more complicated even in the simplest isotropic case and the assumption of the presence of the small-scale anisotropy in the model leads to difficulties even in analysis of the stability of the corresponding asymptotic scaling regimes [9]. The complexity of its analysis is similar to the corresponding problem in the field-theoretic renormalization group approach to the stochastic Navier–Stokes equation [9].

In what follows, we shall concentrate on analysis of the stability of scaling regimes of the model and it will be shown that the inertial-range scaling regimes are given by the infrared (IR) stable fixed points of the system of five differential Gell-Mann–Low equations (also known as flow equations) which contain a special type of integrals. Therefore, their calculation in process of integration of the system of differential equations is needed. In this respect, one effective approach to the integration of the integrals will be discussed in detail.

2. THE MODEL AND THE FIELD THEORY

We consider the so-called $A = 0$ model of the advection of transverse (solenoidal) passive vector field $\mathbf{b} \equiv \mathbf{b}(\mathbf{x}, t)$ given by the stochastic equation

$$\partial_t \mathbf{b} = \nu_0 \Delta \mathbf{b} - (\mathbf{v} \cdot \nabla)\mathbf{b} + \mathbf{f}, \tag{1}$$

where $\partial_t \equiv \partial/\partial t$, $\Delta \equiv \nabla^2$ is the Laplace operator, ν_0 is the diffusivity (a subscript 0 denotes bare parameters of unrenormalized theory), and $\mathbf{v} \equiv \mathbf{v}(\mathbf{x}, t)$ is the incompressible advecting velocity field. The vector field $\mathbf{f} \equiv \mathbf{f}(\mathbf{x}, t)$ is a transverse Gaussian random (stirring) force with zero mean and covariance

$$D^f_{ij} \equiv \langle f_i(\mathbf{x}, t) f_j(\mathbf{x}', t')\rangle = \delta(t - t') C_{ij}(\mathbf{r}/L), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}', \tag{2}$$

where parentheses $\langle \cdots \rangle$ hereafter denote average over corresponding statistical ensemble and L denotes an integral scale related to the stirring. In what follows, the concrete form of the correlator defined in (2) is not essential.

We suppose that the statistics of the velocity field is also given in the form of a Gaussian distribution with zero mean and pair correlation function [8]

$$\langle v_i(x) v_j(x')\rangle \equiv D^v_{ij}(x; x') = \int \frac{d^d\mathbf{k}\, d\omega}{(2\pi)^{d+1}} R_{ij}(\mathbf{k}) D^v(\omega, \mathbf{k}) e^{-i\omega(t-t') + i\mathbf{k}(\mathbf{x}-\mathbf{x}')}, \tag{3}$$

where d is the dimension of the space, $\mathbf{k}$ is the wave vector, and $R_{ij}(\mathbf{k})$ is the uniaxial anisotropic transverse projector taken in the following form [9]:

$$R_{ij}(\mathbf{k}) = \left(1 + \alpha_1 (\mathbf{n} \cdot \mathbf{k})^2/k^2\right) P_{ij}(\mathbf{k}) + \alpha_2 n_s n_l P_{is}(\mathbf{k}) P_{jl}(\mathbf{k}), \tag{4}$$

where $P_{ij}(\mathbf{k}) \equiv \delta_{ij} - k_i k_j / k^2$ is common isotropic transverse projector, the unit vector $\mathbf{n}$ determines the distinguished direction of uniaxial anisotropy, and α_1, α_2 are the parameters characterizing anisotropy. The necessity of positive definiteness of the correlation tensor D^v_{ij} leads to the restrictions on the values of the anisotropy parameters, namely $\alpha_{1,2} > -1$. The function $D^v(\omega, \mathbf{k})$ in (3) is taken in the following form [8]:

$$D^v(\omega, k) = \frac{g_0 u_0 \nu_0^3 k^{4-d-2\varepsilon-\eta}}{(i\omega + u_0\nu_0 k^{2-\eta})(-i\omega + u_0\nu_0 k^{2-\eta})}, \tag{5}$$

where g_0 plays the role of the coupling constant of the model, the parameter u_0 is the ratio of turnover time of scalar field and velocity correlation time, and the positive exponents ε and η are small RG expansion parameters (for details see [8, 9]). The value $\varepsilon = 4/3$ corresponds to the Kolmogorov «two-thirds law» for the spatial statistics of velocity field, and $\eta = 4/3$ corresponds to the Kolmogorov frequency. Simple dimensional analysis shows that g_0 and u_0, which we commonly term as charges, are related to the characteristic ultraviolet (UV) momentum scale Λ (or inner legth $l \sim \Lambda^{-1}$) by relations $g_0 \simeq \Lambda^{2\varepsilon}$ and $u_0 \simeq \Lambda^{\eta}$.

It can be shown that the stochastic problem (1)–(3) can be treated as a field theory with the following action functional [4, 5]:

$$\begin{aligned} S(\Phi) &= b'_j[(-\partial_t - v_i\partial_i + \nu_0\Delta + \\ &+ \nu_0\chi_{10}(\mathbf{n}\cdot\partial)^2)\delta_{jk} + n_j\,\nu_0\left(\chi_{20}\Delta + \chi_{30}(\mathbf{n}\cdot\partial)^2\right) n_k]b_k - \\ &- \frac{1}{2}\left(v_i[D^v_{ij}]^{-1}v_j - b'_i D^f_{ij} b'_j\right), \end{aligned} \tag{6}$$

where D^v_{ij} and D^f_{ij} are given in (3) and (2) respectively, $\mathbf{b}'$ is an auxiliary vector field (see, e.g., [5]), and the required integrations over $x = (\mathbf{x}, t)$ and summations over the vector indices are implied. In action (6) the terms with new parameters χ_{10}, χ_{20}, and χ_{30} are related to the presence of small-scale anisotropy and they are necessary to make the model multiplicatively renormalizable. The model (6) corresponds to a standard Feynman diagrammatic technique (see, e.g., [8] for details), and the standard analysis of canonical dimensions then shows which one-irreducible Green functions can possess UV superficial divergences.

The functional formulation (6) gives possibility to use the field-theoretic methods, including the RG technique to solve the problem. By means of the RG approach it is possible to extract large-scale asymptotic behavior of the correlation functions after an appropriate renormalization procedure which is needed to remove UV divergences.

Using the standard RG analysis (see, e.g., [5,8]), one concludes that possible scaling regimes of the model are given by the IR stable fixed points of the system of five nonlinear RG differential equations (flow equations) for five scale-dependent effective variables (charges) $C = \{g, u, \chi_1, \chi_2, \chi_3\}$ of the model which are functions of the dimensionless scale parameter $t = k/\Lambda$ [5]. In our model the system of the flow equations has the following form:

$$t\frac{dg}{dt} = g(-2\varepsilon+2\gamma_1), \quad t\frac{du}{dt} = u(-\eta+\gamma_1), \quad t\frac{d\chi_i}{dt} = \chi_i(\gamma_1-\gamma_{i+1}), \quad i = 1, 2, 3, \tag{7}$$

where the functions $\gamma_i, i = 1, 2, 3, 4$ are given by the following expressions (one-loop approximation):

$$\gamma_1 = -g\frac{S_{d-1}}{(2\pi)^d}\frac{1}{(d-1)(d+1)}\int_0^1 dx \frac{(1-x^2)^{(d-3)/2}}{w_1 w_2}K_1, \tag{8}$$

$$\gamma_{i+1} = -\frac{g}{\chi_i}\frac{S_{d-1}}{(2\pi)^d}\frac{1}{(d-1)(d+1)}\int_0^1 dx \frac{(1-x^2)^{(d-3)/2}}{w_1 w_2}K_{i+1}, \; i = 1, 2, 3, \tag{9}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface of the d-dimensional sphere, $w_1 = (1+u+\chi_1 x^2)$, $w_2 = (1+u+\chi_1 x^2+(\chi_2+\chi_3 x^2)(1-x^2))$, and the coefficients $K_i, i = 1, 2, 3, 4$ are given as follows:

$$\begin{aligned}
K_1 \;\; = \;\; & 2(1+\chi_2+u)+2(\chi_1-\chi_2+\chi_3+\alpha_1(1+\chi_2+u))x^2-(1+2\chi_3- \\
& -2\alpha_1(\chi_1-\chi_2+\chi_3)+u+\alpha_2(1+u))x^4-(\chi_1+\alpha_2(-1+\chi_1-u)+ \\
& +\alpha_1(1+2\chi_3+u))x^6-(\alpha_1-\alpha_2)\chi_1 x^8+d(-1+x)(1+x)(-2(1+ \\
& +\chi_2+u)-(2\chi_1-\chi_2+2\chi_3+2\alpha_1(1+\chi_2+u)-\alpha_2(1+\chi_2+u))\times \\
& \times x^2+(\alpha_1(-2\chi_1+\chi_2-2\chi_3)+\chi_3+\alpha_2(\chi_1-\chi_2+\chi_3))x^4+(\alpha_1- \\
& -\alpha_2)-\chi_3 x^6)+d^2(1+\alpha_1 x^2)(-1-u-(\chi_1+\chi_3)x^2+\chi_3 x^4+ \\
& +\chi_2(-1+x^2)),
\end{aligned}$$

$$\begin{aligned}
K_2 \;\; = \;\; & \alpha_2(-1+x^2)((-2+d)(1+d)(1+\chi_2+u)+(3-2\chi_1+4\chi_2- \\
& -2\chi_3+3u+d(1-\chi_1+\chi_2-\chi_3+d(-1+\chi_1-2\chi_2+\chi_3-u)+ \\
& +u))x^2-(-3\chi_1+2(1+\chi_2-2\chi_3+u)+d(1+(-1+d)\chi_1- \\
& -d\chi_2-\chi_3+2d\chi_3+u))x^4-((2+d)\chi_1-(-2+d^2)\chi_3)x^6)- \\
& -(1+\alpha_1 x^2)(d(1+\chi_2+u)-(-2\chi_2-3(1+u)+d(-\chi_1+ \\
& +\chi_2-\chi_3+d(1+\chi_2+u)))x^2-(-3\chi_1+2(1+\chi_2-\chi_3+u)+ \\
& +d(1+\chi_3+d(\chi_1-\chi_2+\chi_3)+u))x^4-((2+d)\chi_1- \\
& -(-2+d^2)\chi_3)x^6),
\end{aligned}$$

$$\begin{aligned}
K_3 \;\; = \;\; & -d(1+u)+(d^2-2d-2)\chi_2+(-3+2\chi_2-2\chi_3+\alpha_2(-1+d\chi_2- \\
& -u)-3u+\alpha_1(-2\chi_2+d^2\chi_2-d(1+2\chi_2+u))+d(-\chi_1+3\chi_2- \\
& -2\chi_3+d(1-\chi_2+\chi_3+u)))x^2+(-3\chi_1+2(1+\chi_3+u)+d(1+ \\
& +d\chi_1-\chi_2+3\chi_3-d\chi_3+u)+\alpha_2(3-\chi_1+3u+d(1-2\chi_2+\chi_3+
\end{aligned}$$

$$+u)) + \alpha_1(-3 + 2\chi_2 - 2\chi_3 - 3u + d(-\chi_1 + 3\chi_2 - 2\chi_3 + d(1 -$$
$$+\chi_2 - 2\chi_3 - u) - 2(1 + u)) + \alpha_1(-3\chi_1 + 2(1 + \chi_3 + u) + d(1 +$$
$$+d\chi_1 - \chi_2 + 3\chi_3 - d\chi_3 + u)))x^6 + (\alpha_1 - \alpha_2)((2 + d)\chi_1 - d\chi_3)x^8,$$

$$\begin{aligned}
K_4 \quad = \quad & \alpha_2(-1 + x^2)(1 + 2\chi_2 + u + (\chi_1 - 2(4 + 3\chi_2 - \chi_3 + 4u))x^2 + \\
& +2(4 - 4\chi_1 + 2\chi_2 - 3\chi_3 + 4u)x^4 + 4(2\chi_1 + \chi_3)x^6 + d(1 + \chi_2 + \\
& +u + (-6 + \chi_1 - \chi_2 + \chi_3 - 6u)x^2 - (-6 + 6\chi_1 + \chi_3 - 6u)x^4 + \\
& +6\chi_1 x^6) - d^2(x^2 - 1)(-(1 + \chi_3 + u)x^2 + (-\chi_1 + \chi_3)x^4 + \\
& +\chi_2(x^2 - 1))) - (1 + \alpha_1 x^2)(3 - (12 - 3\chi_1 - 2\chi_3 + d(6 + \chi_3))x^2 + \\
& +((2 + d)(4 + d - 6\chi_1) + (-6 + d + d^2)\chi_3)x^4 + (2 + d)((4 + \\
& +d)\chi_1 - (-2 + d)\chi_3)x^6 - (d - 2)\chi_2(x^2 - 1)((2 + d)x^2 - 1) + \\
& +u(3 + (2 + d)x^2((4 + d)x^2 - 6))).
\end{aligned}$$

In (7), the scale parameter t belongs to the interval $0 \leqslant t \leqslant 1$ with the initial conditions given at $t = 1$ and the IR stable fixed point corresponds to the limit $t \to 0$, i.e., $C|_{t=0} = C^*$.

Before we perform the analysis and solution of the system of differential equations (7), it is necessary to guarantee the convergence of the integrals which are present in (8) and (9) within the interval $x \in [0, 1]$. Another question is to find an effective method to solve the integrals. Both questions are briefly discussed in the next section.

3. NUMERICAL AND ANALYTICAL ANALYSIS OF INTEGRALS

The integrals in (8) and (9) are linear combinations of the following integrals:

$$I = \int_0^1 dx \frac{(1 - x^2)^{\frac{d-3}{2}} x^{2n}}{w_1 w_2}, \tag{10}$$

where the explicit form of functions w_1 and w_2 is given in the text below (8) and (9), and n is a natural number, i.e., $n = 0, 1, 2, \ldots$ Therefore, the γ functions in (8) and (9) will be convergent if and only if integrals (10) are convergent. The necessary and sufficient conditions for the convergence of the integrals (10) are subject of the following theorem:

Theorem 1: *The integrals (10) are convergent within integration interval* $x \in [0, 1]$ *if and only if the following conditions are satisfied*:

i) $\chi_1 \in (-1 - u, \infty)$;

ii) $\chi_2 \in (-1 - u, \infty)$;

iii) $\chi_3 \in \left(-\left(\sqrt{1 + u + \chi_1} + \sqrt{1 + u + \chi_2}\right)^2, \infty\right)$.

Proof: The proof of the theorem is similar to the proof of an analogous theorem which was proven in [9], therefore we shall not present it here. ■

In principle, there are a few ways of solveing the integrals (10). In what follows, we shall try to transform them to the form which is more appropriate for their numerical calculations, i.e, the procedure improves their convergent properties. The approach is based on the following theorem:

Theorem 2: *Let α be a real number and let $P_0(x)$ and $Q(x)$ be polynomials of real variable x such that $d(P_0(x)) \leqslant d(Q(x))$, where $d(R(x))$ denotes the degree of a polynomial $R(x)$ and $Q(x)$ is nonzero for $x \in [0,1]$. Then for arbitrary $m \in \mathbb{Z}_0^+$ the following formula holds:*

$$I = \int_0^1 \frac{P_0(x)\left(1-x^2\right)^{\alpha}}{Q(x)}\,dx = \sum_{i=1}^{m}\left[\frac{1}{4\,(\alpha+i)}\left(\frac{P_{i-1}(1)}{Q(1)} - \frac{P_{i-1}(-1)}{Q(-1)}\right) + \right.$$
$$\left. + \frac{\sqrt{\pi}}{4}\frac{\Gamma(\alpha+i)}{\Gamma(\alpha+i+1/2)}\left(\frac{P_{i-1}(1)}{Q(1)} + \frac{P_{i-1}(-1)}{Q(-1)}\right)\right] + \int_0^1 \frac{P_m(x)}{Q(x)}\left(1-x^2\right)^{\alpha+m} dx, \tag{11}$$

where

$$P_i(x) = \frac{P_{i-1}(x) - (A_i x + B_i)\,Q(x)}{1-x^2},$$
$$A_i = \frac{1}{2}\left(\frac{P_{i-1}(1)}{Q(1)} - \frac{P_{i-1}(-1)}{Q(-1)}\right), \quad B_i = \frac{1}{2}\left(\frac{P_{i-1}(1)}{Q(1)} + \frac{P_{i-1}(-1)}{Q(-1)}\right)$$

for $i = 1, 2, \ldots, m$.

Proof: The proof of the theorem is done by the mathematical induction with respect to m. The case $m = 0$ is evident. Further, let us denote as $T(n)$ the proposition of the theorem for $m = n$ and suppose that the theorem holds for $n \geqslant 0$. Thus, it is necessary to prove the validity of the theorem for $m = n + 1$.

According to the assumption of validity of $T(n)$, it follows that

$$I = \int_0^1 \frac{P_0(x)\left(1-x^2\right)^{\alpha}}{Q(x)}\,dx = \sum_{i=1}^{n}\left[\frac{1}{4(\alpha+i)}\left(\frac{P_{i-1}(1)}{Q(1)} - \frac{P_{i-1}(-1)}{Q(-1)}\right) + \right.$$
$$\left. + \frac{\sqrt{\pi}}{4}\frac{\Gamma(\alpha+i)}{\Gamma(\alpha+i+1/2)}\left(\frac{P_{i-1}(1)}{Q(1)} + \frac{P_{i-1}(-1)}{Q(-1)}\right)\right] +$$
$$+ \int_0^1 \frac{P_n(x)}{Q(x)}\left(1-x^2\right)^{\alpha+n} dx = I_{\sum,n} + I_n, \tag{12}$$

where

$$P_i(x) = \frac{P_{i-1}(x) - (A_{i-1}x + B_{i-1})\, Q(x)}{1 - x^2},$$

$$A_{i-1} = \frac{1}{2}\left(\frac{P_{i-1}(1)}{Q(1)} - \frac{P_{i-1}(-1)}{Q(-1)}\right), \quad B_{i-1} = \frac{1}{2}\left(\frac{P_{i-1}(1)}{Q(1)} + \frac{P_{i-1}(-1)}{Q(-1)}\right),$$

for all $i \in [1, n]$ and as I_n we have denoted the integral part of (12). Further, integral I_n in (12) can be written as follows:

$$I_n = \int_0^1 \frac{P_n(x)}{Q(x)} \left(1 - x^2\right)^{\alpha+n} \, dx =$$
$$= \int_0^1 \frac{A_n x + B_n}{1 - x^2} \left(1 - x^2\right)^{\alpha+n+1} \, dx + \int_0^1 \frac{P_{n+1}(x)}{Q(x)} \left(1 - x^2\right)^{\alpha+n+1} \, dx, \quad (13)$$

where P_{n+1} is defined by the relation

$$\frac{P_n(x)}{(1 - x^2)\, Q(x)} = \frac{A_n x + B_n}{1 - x^2} + \frac{P_{n+1}(x)}{Q(x)}, \quad (14)$$

therefore,

$$P_{n+1}(x) = \frac{P_n(x) - (A_n x + B_n)\, Q(x)}{1 - x^2} \quad (15)$$

with identities

$$\frac{P_n(1)}{Q(1)} = A_n + B_n, \quad \frac{P_n(-1)}{Q(-1)} = -A_n + B_n. \quad (16)$$

By solving the previous system of equations, one obtains

$$A_n = \frac{1}{2}\left(\frac{P_n(1)}{Q(1)} - \frac{P_n(-1)}{Q(-1)}\right), \quad B_n = \frac{1}{2}\left(\frac{P_n(1)}{Q(1)} + \frac{P_n(-1)}{Q(-1)}\right) \quad (17)$$

and by insertion of A_n and B_n from (17) into (13), one obtains the following expression for integral I_n:

$$I_n = A_n \int_0^1 x \left(1 - x^2\right)^{\alpha+n} \, dx + B_n \int_0^1 \left(1 - x^2\right)^{\alpha+n} \, dx +$$
$$+ \int_0^1 \frac{P_{n+1}(x)}{Q(x)} \left(1 - x^2\right)^{\alpha+n+1} \, dx = A_n \frac{1}{2\,(\alpha + n + 1)} +$$

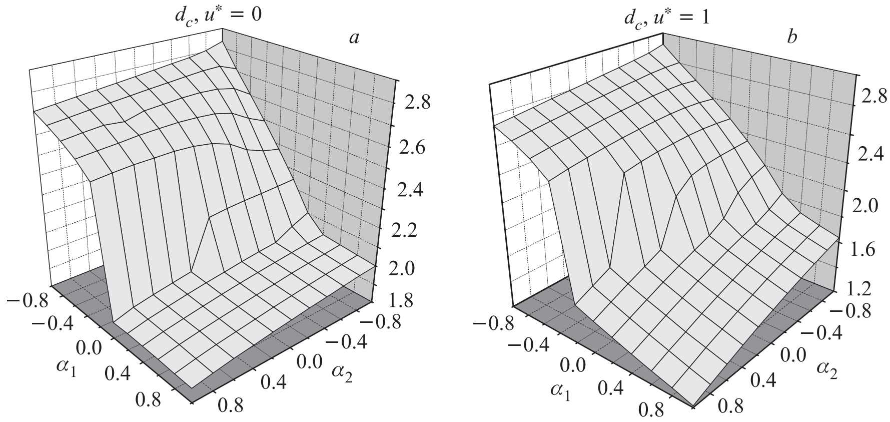
$$
\begin{aligned}
&+ B_n \frac{\sqrt{\pi}}{2} \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+n+3/2)} + \int_0^1 \frac{P_{n+1}(x)}{Q(x)} \left(1-x^2\right)^{\alpha+n+1} dx = \\
&= \frac{1}{4\,(\alpha+n+1)} \left(\frac{P_n(1)}{Q(1)} - \frac{P_n(-1)}{Q(-1)} \right) + \frac{\sqrt{\pi}}{4} \frac{\Gamma(\alpha+(n+1))}{\Gamma(\alpha+(n+1)+1/2)} \times \\
&\qquad \times \left(\frac{P_n(1)}{Q(1)} + \frac{P_n(-1)}{Q(-1)} \right) + \int_0^1 \frac{P_{n+1}(x)}{Q(x)} \left(1-x^2\right)^{\alpha+n+1} dx.
\end{aligned}
$$

Now, one can return to (12) to obtain

$$
\begin{aligned}
I = I_{\sum,n} + I_n = \sum_{i=1}^{n} \Bigg[& \frac{1}{4(\alpha+i)} \left(\frac{P_{i-1}(1)}{Q(1)} - \frac{P_{i-1}(-1)}{Q(-1)} \right) + \\
& + \frac{\sqrt{\pi}}{4} \frac{\Gamma(\alpha+i)}{\Gamma(\alpha+i+1/2)} \left(\frac{P_{i-1}(1)}{Q(1)} + \frac{P_{i-1}(-1)}{Q(-1)} \right) \Bigg] + \\
& + \frac{1}{4\,(\alpha+(n+1))} \left(\frac{P_n(1)}{Q(1)} - \frac{P_n(-1)}{Q(-1)} \right) + \\
& + \frac{\sqrt{\pi}}{4} \frac{\Gamma(\alpha+(n+1))}{\Gamma(\alpha+(n+1)+1/2)} \left(\frac{P_n(1)}{Q(1)} + \frac{P_n(-1)}{Q(-1)} \right) + \\
& + \underbrace{\int_0^1 \frac{P_{n+1}(x)}{Q(x)} \left(1-x^2\right)^{\alpha+n+1} dx}_{I_{n+1}} = \\
= \sum_{i=1}^{n+1} \Bigg[& \frac{1}{4(\alpha+i)} \left(\frac{P_{i-1}(1)}{Q(1)} - \frac{P_{i-1}(-1)}{Q(-1)} \right) + \\
& + \frac{\sqrt{\pi}}{4} \frac{\Gamma(\alpha+i)}{\Gamma(\alpha+i+1/2)} \left(\frac{P_{i-1}(1)}{Q(1)} + \frac{P_{i-1}(-1)}{Q(-1)} \right) \Bigg] + I_{n+1} = \\
& = I_{\sum,n+1} + I_{n+1}. \qquad (18)
\end{aligned}
$$

In the end, from (14), (17), and (18) it follow that $T(n+1)$ holds, which proves the theorem. ■

The formula given in (11), which was proven in Theorem 2, allows one to compute the integrals (10) in the form of a sum of the gamma functions, which can be calculated exactly, and one integral which is convenient for integration with respect to needed precision and computing time of calculations. It is clear that in our case, $d \in (1,3]$, it is sufficient to take $m = 1$ and the integral becomes more convenient for integration, namely, the exponent p in $(1-x^2)^p$ part of the integrand becomes a positive real number and the integral can be simply



Dependence of the borderline dimension d_c on the parameters α_1 and α_2 for $u^* = 0$ (a) and $u^* = 1$ (b). The corresponding scaling regime is stable above the given surfaces

calculated with the high precision in a very short time by appropriate numerical method of integration.

4. SCALING REGIMES OF THE MODEL

We have performed a numerical analysis of the system of differential flow equations and we have found all possible fixed points which drive the corresponding scaling regimes of the model. The model exhibits five different scaling regimes. Two of them correspond to the rapid-change model limit: one is trivial with $g^*/u^* = 0$, $1/u^* = 0$ which is stable for $\eta > 0$ and $2\varepsilon < \eta$ and the other is non-trivial with $g^*/u^* > 0$, $1/u^* = 0$ which is stable for $\varepsilon < \eta$ and $2\varepsilon > \eta$. Two of the scaling regimes correspond to the so-called «frozen» limit: one is again trivial with $g^* = 0$, $u^* = 0$ which is stable for $\varepsilon < 0$ and $\eta < 0$ and the other is non-trivial with $g^* > 0$, $u^* = 0$ which is stable for $\varepsilon > 0$ and $\varepsilon > \eta$. The last and the most interesting scaling regime corresponds to the case with finite time correlations of velocity field and it is given by nonzero u^* and $g^* > 0$ (see, e.g., [8] and references therein) which is stable for $\varepsilon = \eta$. Further, we are interested in the dependence of the so-called borderline dimension $d_c \in (1, 3]$ on the anisotropy parameters α_1 and α_2 under which the corresponding scaling regime is unstable. Some results are shown in the figure. One can see that the presence of small-scale anisotropy leads to the violation of the stability of the corresponding scaling regimes below $d = 2$ for appropriate values of anisotropy parameters. But from the viewpoint of further investigation into anomalous scaling of the correlation functions of the advected vector field, the most important conclusion is that all the three-dimensional scaling regimes remain stable under the influence of small-scale uniaxial anisotropy.

5. CONCLUSIONS

Using the field-theoretic RG, we have studied the influence of small-scale uniaxial anisotropy on the stability of the scaling regimes in the model of a passive vector advected by a given stochastic environment with finite time correlations. The existence of five possible scaling regimes as functions of parameters ε and η has been briefly discussed. It has been shown that the stability of the scaling regimes under the influence of small-scale uniaxial anisotropy is driven by the system of five nonlinear differential flow equations which contain angle integrals. The conditions for the convergence of the integrals have been found and one convenient method for their numerical calculation has been worked out. It has been shown that the anisotropy does not disturb the three-dimensional scaling regimes, but the two-dimensional scaling regimes could be destroyed by the small-scale anisotropy. The results will be used in the further investigations of the anomalous scaling of the model.

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