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CLASSES OF EXACT SOLUTIONS TO THE TEUKOLSKY MASTER EQUATION

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Классы точных решений уравнения Тюкольского

Уравнение Тюкольского описывает в линейном приближении возмущения метрики Керра. Оно допускает разделение переменных и таким образом порождает радиальное уравнение Тюкольского и угловое уравнение Тюкольского. Мы приводим единое описание всех классов точных решений указанных волновых уравнений в терминах конфлюентных функций Гойна и конфлюентных полиномов Гойна. Найдены и описаны новые классы точных решений и их определяющие свойства. Специальное внимание уделено полиномиальным решениям, которые являются сингулярными и описывают коллимированные волны, бегущие в одном направлении. Показано, что подходящие линейные комбинации таких сингулярных решений могут описывать ограниченные по амплитуде коллимированные волны, бегущие в одном направлении.

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Classes of Exact Solutions to the Teukolsky Master Equation

The Teukolsky Master Equation (TME) describes perturbations of the Kerr metric in linear approximation. It admits separation of variables, thus yielding the Teukolsky Radial Equation (TRE) and the Teukolsky Angular Equation (TAE). We present here a unified description of all classes of exact solutions to these equations in terms of the confluent Heun functions and the confluent Heun polynomials. Large classes of new exact solutions are found and described together with their characteristic properties. Special attention is paid to the polynomial solutions which are singular ones and describe collimated one-way-running waves. It is shown that a proper linear combination of such singular solutions can describe bounded one-way-running waves.

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1. INTRODUCTION

At present the study of different types of perturbations of the gravitational field of black holes, neutron stars and other compact astrophysical objects is a very active field for analytical, numerical, experimental and astrophysical research. Ongoing and nearest future experiments based on perturbative and/or numerical analysis of relativistic gravitational dynamics are expected to provide critical tests of the existing theories of gravity [1].

The study of perturbations of rotating relativistic objects in Einstein GR was pioneered by Teukolsky [2] making use of the famous Teukolsky Master Equation (TME). It describes the perturbations ${}_s\Psi(t,\, r,\, \theta,\, \varphi)$ of all physically interesting spin-weights $s = 0,\, \pm 1/2,\, \pm 1,\, \pm 3/2,\, \pm 2$ to the Kerr background metric in terms of the corresponding Newman–Penrose scalars. The pairs of spin-weights s with opposite signs $\sigma = \mathrm{sign}(s) = \pm 1$ correspond to two different perturbations with opposite helicity and spin $|s| = 0,\, 1/2,\, 1,\, 3/2$, or 2. Under proper boundary conditions for TME one obtains quasinormal modes (QNM) of the Kerr black holes. Various significant results and additional references can be found in [3–5].

The key feature of the TME is that in the Boyer–Lindquist coordinates one can separate the variables using the ansatz $\Psi(t,\, r,\, \theta,\, \varphi) = e^{-i\omega t} e^{im\varphi} S(\theta) R(r)$, i.e., looking for solutions in a specific factorized form. Thus, one obtains a pair of two connected ordinary differential equations for the nontrivial factors ${}_sS_{\omega,E,m}(\theta)$ and ${}_sR_{\omega,E,m}(r)$ — the Teukolsky angular equation (TAE) [2, 3, 6]

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\,{}_sS_{\omega,E,m}(\theta)\right) + {}_sW_{\omega,E,m}(\theta)\,{}_sS_{\omega,E,m}(\theta) = 0, \tag{1.1a}$$

$${}_sW_{\omega,E,m}(\theta) = E + a^2\omega^2\cos^2\theta - 2sa\omega\cos\theta - (m^2 + s^2 + 2ms\cos\theta)/\sin^2\theta \tag{1.1b}$$

and the Teukolsky radial equation (TRE) [2, 3]

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\,{}_sR_{\omega,E,m}(r)\right)+{}_sV_{\omega,E,m}(r)\,{}_sR_{\omega,E,m}(r)=0, \tag{1.2a}$$

$${}_sV_{\omega,E,m}(r)=\frac{1}{\Delta}K^2-is\frac{1}{\Delta}\frac{d\Delta}{dr}K-L. \tag{1.2b}$$

The azimuthal number m has arbitrary integer values $m = 0, \pm1, \pm2, \ldots$, and $\Delta = r^2-2Mr+a^2$, $K = \omega(r^2+a^2)-ma$, $L = E-s(s+1)+a^2\omega^2-2ma\omega-4is\omega r$. The real parameter $a = J/M$ is related with the angular momentum J of the Kerr metric, M being its Keplerian mass. The two *complex* parameters ω and E — the constants of separation, are to be determined using the boundary conditions of the problem.

The negativity of the imaginary part $\omega_I = \Im(\omega) < 0$ of the complex frequency $\omega = \omega_R + i\omega_I$ ensures linear stability of the solutions in the exterior domain of the Kerr metric with respect to the future time direction $t \to +\infty$ [2, 7]. In the interior domain the solutions to the TME are not stable [8].

From a mathematical point of view, the function ${}_s\mathcal{K}_{\omega,E,m}(t,\,r,\,\theta,\,\varphi) \sim e^{-i\omega t}e^{im\varphi}\,{}_sS_{\omega,E,m}(\theta)\,{}_sR_{\omega,E,m}(r)$ actually defines a factorized *kernel* of the general integral representation for the solutions to the TME:

$$\begin{aligned}&{}_s\Psi(t,\,r,\,\theta,\varphi)=\\&=\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}\int d\omega\int dE\,{}_sA_{\omega,E,m}\,e^{-i\omega t}\,e^{im\varphi}\,{}_sS_{\omega,E,m}(\theta)\,{}_sR_{\omega,E,m}(r).\end{aligned} \tag{1.3}$$

The formal mathematical representation (1.3) is written ad hoc as the most general superposition of all particular solutions. In it a summation on all admissible values of the two separation constants ω and E is assumed. Its usefulness will be illustrated by different examples in what follows.

It is well known [9] that the Carter separation constant (which is equivalent to the constant E, used here) may be related with the total angular momentum of the solution ${}_s\Psi(t,\,r,\,\theta,\,\varphi)$. Under proper boundary conditions for the TAE this momentum has discrete values defined by an integer l [2]. When there are physically interesting superpositions ${}_s\Psi(t,\,r,\,\theta,\,\varphi)$ of solutions with a definite total angular momentum, the integration with respect to the constant E must be replaced with summation over the integer l. Thus, instead of the most general linear mixture (1.3) we have to use the representation of the solutions

$${}_s\Psi(t,\,r,\,\theta,\,\varphi)=\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}\int d\omega\sum_{l}\,{}_sA_{\omega,l,m}\,e^{-i\omega t}\,e^{im\varphi}\,{}_sS_{\omega,l,m}(\theta)\,{}_sR_{\omega,l,m}(r), \tag{1.4}$$

introduced in the problem at hand for the first time in [2]. The transition from the representation (1.3) to representation (1.4) is formally equivalent to the use of a *singular* kernel proportional to the sum of Dirac δ-functions: $\sum_l \delta(E - {}_sE(\omega,\, l,\, m))$ in (1.3). Here ${}_sE(\omega,\, l,\, m)$ belongs to some spectrum which is specific for the given problem and defined by the corresponding boundary conditions, see Sec. 7.

Further on, the boundary conditions may fix some discrete spectrum for the frequencies ω in (1.4). Then the integral on ω will be replaced by discrete summation over some ω_n. This is equivalent to the use once more of a singular kernel, now proportional to $\sum_n \delta(\omega - \omega_n)$.

The study of the QNM [5] not only illustrates the above situation but also shows that the kernel ${}_s\mathcal{K}_{\omega,E,m}(t,\, r,\, \theta,\, \varphi)$ can be singular with respect to the variable r at infinity and at the horizons.

In the existing literature only regular with respect to the variable θ kernels ${}_s\mathcal{K}_{\omega,E,m}(t,\, r,\, \theta,\, \varphi)$ are in use. In the present work, we start the consideration of both regular and singular with respect to the angle θ kernels in the integral representation (1.3) of the solutions to the TME. Different types of kernels are to be used for solution of different boundary problems. Note, that from a physical point of view, the regularity of the very solution ${}_s\Psi(t,\, r,\, \theta,\, \varphi)$ in equations (1.3) and (1.4) is important. The kernels like ${}_s\mathcal{K}_{\omega,E,m}(t,\, r,\, \theta,\, \varphi)$ are auxiliary mathematical objects. One is often forced to use singular kernels in the natural integral representations of the solution to physical problems. The regularity of the very physical solution ${}_s\Psi(t,\, r,\, \theta,\, \varphi)$ with respect to the variable θ depends on the choice of the amplitudes ${}_sA_{\omega,E,m}$. It can be guaranteed by a suitable choice of these amplitudes, as shown in Sec. 10.

Despite the essential progress both in the numerical study [10] of the solutions to equations (1.1a) and (1.2a) and in the investigation of their analytical properties [11], at present there exists a number of basic questions remaining unanswered. For example, it has been well known for a long time [12] that the TAE (1.1a) and TRE (1.2a) can be reduced to the confluent Heun ordinary differential equation [13] written here in the following simplest uniform shape [16, 17]:

$$H'' + \left(\alpha + \frac{\beta+1}{z} + \frac{\gamma+1}{z-1}\right) H' + \left(\frac{\mu}{z} + \frac{\nu}{z-1}\right) H = 0. \tag{1.5}$$

The constants μ and ν in Eq. (1.5) are related with the constants $\alpha,\, \beta,\, \gamma,\, \delta,\, \eta$, accepted in the notation $\mathrm{HeunC}(\alpha,\, \beta,\, \gamma,\, \delta,\, \eta,\, z)$ as follows:

$$\delta = \mu + \nu - \alpha\frac{\beta+\gamma+2}{2}, \tag{1.6a}$$

$$\eta = \frac{\alpha(\beta+1)}{2} - \mu - \frac{\beta+\gamma+\beta\gamma}{2}. \tag{1.6b}$$

To the best of our knowledge we still do not have a detailed description of the exact analytical solutions to the TAE (1.1a) and TRE (1.2a) in terms of the *confluent Heun function* $\mathrm{HeunC}(\alpha,\,\beta,\,\gamma,\,\delta,\,\eta,\,z)$ — the unique particular local solution of Eq. (1.5) which is regular in the vicinity of the regular singular point $z=0$ and obeys the normalization condition $\mathrm{HeunC}(\alpha,\,\beta,\,\gamma,\,\delta,\,\eta,\,0)=\;1$ [13]*. Note that other particular solutions to equation (1.5), as well as its general solution, are *not* termed «confluent Heun's functions», according to the accepted modern terminology [13]. The reason is that, in general, other solutions can be represented in a nontrivial way in terms of solutions $\mathrm{HeunC}(\alpha,\,\beta,\,\gamma,\,\delta,\,\eta,\,z)$ of the corresponding arguments. Hence, from a computational point of view, it is sufficient to study *only* the Taylor series of this standard local solution and its analytical continuation in the complex plane $\mathbb{C}_z$. Thus, the instrumental use of the confluent Heun function $\mathrm{HeunC}(\alpha,\,\beta,\,\gamma,\,\delta,\,\eta,\,z)$ — the basic purpose of the present work, is much more advantageous than the simple fact, recognized already in [12], that the TRE and TAE can be reduced to the confluent Heun equation (1.5).

In the late 2006 a program for filling the above gaps in the study of the TME was started as a natural extension of the articles [14], where a similar approach was developed for the Regge–Wheeler equation (RWE). The first results were quite stimulating [15], but serious difficulties came across in both analytical and numerical studies. This is because the theory of Heun's functions, as well as numerical tools for calculations with them still are not developed enough.

Here we pay special attention to the polynomial solutions of Eq. (1.5). According to [13], the confluent Heun function $\mathrm{HeunC}(\alpha,\,\beta,\,\gamma,\,\delta,\,\eta,\,z)$ reduces to a polynomial of degree $N\geqslant 0$ of the variable z, if and only if the following two conditions are satisfied:

$$\frac{\delta}{\alpha}+\frac{\beta+\gamma}{2}+N+1=0, \tag{1.7a}$$

$$\Delta_{N+1}(\mu)=0. \tag{1.7b}$$

We call the first condition (1.7a) a «δ_N-condition», and the second one (1.7b) — a «Δ_{N+1}-condition». An explicit form of the «Δ_{N+1}-condition» in form of a determinant useful for practical calculations, as well as a novel derivation of confluent Heun's polynomials can be found in [16]. A recurrent procedure for calculation of Δ_{N+1} (1.7b) and its relation with Starobinsky's constants are presented in [17].

On the other hand, the so-called algebraically special solutions to the RWE and TRE were discovered long time ago [18]. These are of a generalized poly-

*In the present work, we use the Maple-computer-package notation for the Heun functions. Basically, this notation is borrowed from the two mile-stone articles on modern theory of Heun's functions by Decarreau et al. in [13]; at present, it seems to be most popular, since the Maple package is the only one for analytical and numerical work with Heun's functions.

nomial type. According to the existing literature, these solutions describe pure incoming or pure outgoing waves. The algebraically special solutions still are not discussed in terms of Heun's polynomials. To our knowledge, attempts for application of this class of solutions to real physical problems cannot be found in the existing literature on gravitational physics. The only exception are the recent articles [15, 19], where one can find some very preliminary results.

Very recently the algebraically special solutions of the RWE and TME were proved to be relevant for the study of instabilities of different kind of some more or less «exotic» solutions to the Einstein equations [20]. Physical manifestation of the instabilities of the mathematical solutions are the explosions of the corresponding objects. Therefore, it seems natural to look for a perturbative description of explosions in terms of solutions of the TME, which are stable in the future and instable in the past. The confluent Heun functions give a rigorous mathematical basis for analysis of these problems.

On the other hand, the recently found properties of confluent Heun's function [16] show that one can introduce a new subclass of «$\delta_N$confluent Heun's functions», which obey *only* the δ_N-condition — Eq. (1.7a). In [17] is shown that such «δ_N-solutions» of the TRE and TAE define the most general class of solutions, for which properly generalized Teukolsky–Starobinsky's identities exist. Here we study in more detail the δ_N-solutions to the TRE and TAE. In particular, we show that the regular solutions to the TAE, which are the only class of solutions to the TAE, used up to now [2, 3, 6], are precisely nonpolynomial δ_N-solutions. In contrast, the polynomial solutions to the TAE of all spins are shown to be singular ones, at least around one of the poles ($\theta = 0, \pi$) of the unit sphere $\mathbb{S}^{(2)}_{\theta,\varphi}$. This new situation reflects the properties of the confluent Heun function. It is not consistent with our experience, based on the work with hypergeometric functions, solving the angular part of the Laplace equation in celestial and quantum mechanics, or in electrostatics. It is well known that in the last case solutions regular on both the poles are polynomial.

In the limit $a \to 0$, when the Kerr metric approaches the nonrotating Schwarzschild one, there exists a smooth transition from perturbations of the Kerr metric to perturbations of the Schwarzschild metric in terms of the Weyl scalars, but a simple transition from the solutions of the TME to the solutions of the RWE (see [14, 17]) is not possible [3]. Nevertheless, the mathematical analogy between the corresponding solutions becomes quite transparent when the solutions are represented in terms of the confluent Heun functions [14, 17]. The limit $a \to 0$ is traced in more detail in Subsec. 4.1.2 — for the TRE and in Sec. 7 — for the TAE.

This way, using confluent Heun's function, we hope to obtain a more clear picture of the quite complicated present-day state of the art in the perturbation theory under consideration and its possible further developments.

The main purposes of the present work is to report some of the basic results, obtained for a detailed description of the exact solutions of the TME in terms of the confluent Heun function $\mathrm{HeunC}(\alpha,\,\beta,\,\gamma,\,\delta,\,\eta,\,z)$, to introduce a large number of new classes of such solutions, and to formulate some interesting boundary problems for the TRE, TAE, and TME in terms of confluent Heun's functions.

Besides the already stressed new developments, in the present work for the first time we introduce and study the differential invariants of the Weyl tensor, which indicate in an invariant way both the event and Cauchy horizons of the Kerr metric as singular points of the TRE (Subsec. 2.1), the explicit form of 16 classes of exact solutions to the TRE in terms of confluent Heun's functions (Subsec. 2.2), a new classification of the solution to the TRE, based on specific properties of confluent Heun's functions (Sec. 3), especially, the class of δ_N-radial solutions and, in particular, two unknown infinite classes of exact solutions with equidistant complex spectra of frequencies, two novel classes of polynomial solutions to the TRE (Sec. 4), the explicit form of 16 classes of exact solutions to the TAE in terms of confluent Heun's functions (Sec. 5), a new concomitant confluent Heun's function and its application to the TAE (Sec. 5), a new classification of the solution to the TAE, based on specific properties of confluent Heun's functions (Sec. 6), especially, the class of δ_N-angular solutions, a novel description of the regular solutions to the TAE in terms of confluent Heun's functions (Sec. 7), two classes of singular polynomial solutions to the TAE (Sec. 8), 256 classes of exact solutions to the TME (Sec. 9), an explicit construction of exact bounded solutions to the TME with spin $1/2$, using the singular kernel, built from the polynomial solutions to the TAE (Sec. 10), and novel general exact solutions of the TME in the form of one-way running waves (Sec. 10). Some general conclusions and perspectives for further developments are outlined in the concluding Sec. 11.

2. EXACT SOLUTIONS TO THE TEUKOLSKY RADIAL EQUATION IN TERMS OF THE CONFLUENT HEUN FUNCTIONS

2.1. Explicit Form of the TRE and Geometrical Character of Its Singularities. Much like in the case of the Schwarzschild solution, for the Kerr one we have a complicated space-time structure and a different physical meaning of the space-time coordinates in the different domains. For example, consider, as usual, only the real values of r. In the interior of the Kerr metric: $0 \leqslant r_- < r < r_+$ – between the zeros $r_\pm = M \pm \sqrt{M^2 - a^2}$, $a \leqslant M$ of the function Δ (i.e., between the Cauchy horizon r_- and the event horizon r_+), two of the eigenvalues: λ_t and λ_r of the metric in the Boyer–Lindquist coordinates simultaneously change their signs. Indeed, one pair of eigenvalues is $\lambda_\theta = g_{\theta\theta} = r^2 + a^2\cos^2\theta$ and $\lambda_r = g_{rr} = (r^2 + a^2\cos^2\theta)/\Delta$. The second pair of eigenvalues is the roots λ_t, λ_ϕ of the equation $\lambda^2 - (g_{tt} + g_{\phi\phi})\lambda + g_{tt}g_{\phi\phi} - g_{t\phi}^2$. Their product equals

$\lambda_t\lambda_\phi = -\Delta\sin^2\theta$. The last expression, together with the form of g_{rr} proves the simultaneous change of the signs of the two eigenvalues λ_t, λ_r, when the variable r crosses the horizons $r_\pm$, since the determinant $g = -(r^2+a^2\cos^2\theta)^2\sin^2\theta$ of the metric does not vanish there. As a result, between the two horizons $r_\pm$ the variable $t_{\rm in} = x \in (-\infty, \infty)$ plays the role of the interior time and the variable $r_{\rm in} = t$ is the interior radial variable. We use the following Kerr-metric-tortoise coordinate: $x = r + a_+ \ln|(r-r_+)/(r_+-r_-)| - a_- \ln|(r-r_-)/(r_+-r_-)| \in (-\infty,\infty)$, where $a_\pm = \dfrac{r_+ + r_-}{r_+ - r_-} r_\pm$. It is a straightforward generalization of the tortoise variable for the exterior domain $r \in (r_+, \infty)$ proposed in [2]. Since our expression is valid in the interior domains, too, the inverse function defines $r = r(t_{\rm in})$ when $r \in (r_-, r_+)$. In the second interior domain $r < r_-$ the variables r and t restore their original meaning. For a detailed analysis of the light cones in the Kerr geometry see [21]. This consideration is necessary for understanding of the physical meaning of the solutions to the TME in the different Kerr-space-time domains.

The explicit form of the TRE

$$\frac{d^2R_{\omega,E,m}}{dr^2} + (1+s)\left(\frac{1}{r-r_+} + \frac{1}{r-r_-}\right)\frac{dR_{\omega,E,m}}{dr} + \\ + \left(\frac{\left(\omega\left(a^2+r^2\right)-am\right)^2}{(r-r_-)(r-r_+)} - is\left(\frac{1}{r-r_+}+\frac{1}{r-r_-}\right)\left(\omega\left(a^2+r^2\right)-am\right) - \\ - E + s(s+1) - a^2\omega^2 + 2ma\omega + 4is\omega r\right)\frac{R_{\omega,E,m}}{(r-r_+)(r-r_-)} \tag{2.1}$$

shows that it has three singular points: $r = r_\pm$ and $r = \infty$. In the present work, we consider only the *non-extremal* Kerr metric with *real* $r_+ > r_- \geqslant 0$. Then the first two are regular singular points, and the third one (the physical infinity $r = \infty$) is an irregular singular point. The symmetry of Eq. (2.1) under the interchange $r_+ \leftrightarrows r_-$ is obvious. Thus, we see that the two horizons of the Kerr metric are singularities for the TRE which are to be treated on equal footing. Do these singularities have an invariant meaning independent of the coordinate choice?

The algebraic invariants of the Riemann curvature tensor $\mathcal{R}_{ijkl}$ are not able to indicate the horizons of the Kerr black hole and one usually considers them as pure coordinate singularities of the metric in the Boyer–Lindquist coordinates. In contrast, the circle $r = 0$, $\theta = \pi/2$ is a singularity of the algebraic invariants of the Riemann tensor [3]. Since the pure *algebraic* invariants of the tensor $\mathcal{R}_{ijkl}$ do not fix completely the geometry, their consideration is not sufficient to recover all gauge-invariant space-time properties. For this purpose one must consider a large enough number of high-order differential invariants of the Riemann tensor [22].

It is not difficult to find differential invariants of the Riemann tensor of the Kerr metric which are able to distinguish both the horizons $r_\pm$ and the ergosphere $g_{tt} = 0$. Indeed, let us consider the following algebraic invariants of the Weyl tensor $\mathcal{W}_{ijkl}$: $I_1 = \frac{1}{48}\mathcal{W}_{ijkl}\mathcal{W}^{ijkl}$ — the density of the Euler characteristic class, and $I_2 = \frac{1}{48}\mathcal{W}_{ijkl}\,{}^*\mathcal{W}^{ijkl}$ — the density of the Chern–Pontryagin characteristic class [23]. Let us put $(I_1 - iI_2)^{1/2} = \lambda = |\lambda|\exp(i\psi)$. Then $r = \left(\frac{M}{|\lambda|}\right)^{1/6}\cos(\psi/6)$ and $\rho = \left(\frac{M}{|\lambda|}\right)^{1/6}\cos(\psi/6)^{-1}$ are obviously invariants of the Weyl tensor — nonalgebraic and nondifferential ones. In the Boyer–Lindquist coordinates one obtains $\rho = r + \frac{a^2}{r}\cos\theta$ and $g_{tt} = 1 - 2M/\rho$. The differential invariants of the first order

$$DI_1 = -\left(\nabla \ln r\right)^2 = \frac{1}{r\rho}\left(1 - \frac{2M}{r} + \frac{a^2}{r^2}\right), \tag{2.2a}$$

$$DI_2 = \left(\nabla \ln \rho\right)^2 - \left(\nabla \ln r\right)^2 = \frac{4}{\rho^2}\left(\frac{\rho}{r} - 1\right)\left(1 - \frac{2M}{\rho}\right) \tag{2.2b}$$

indicate the two Kerr black hole horizons, the ergosphere and some other geometrical objects in the Kerr space-time. Thus, the two horizons $r_\pm$ of the Kerr metric are shown to be invariant objects, being singularities of the same kind in equation (2.1).

In the limit $a \to 0$ we have $\rho \to r$ and the differential invariant in equation (2.2b) becomes trivial: $DI_2 \to 0$. In the same limit, the differential invariant (2.2a) produces a nontrivial result $\left(1 - 2M/r\right)/r^2$ for the Schwarzschild metric which is similar to the one derived already in the third of the articles [22], as well as in the very recent sixth one.

2.2. Explicit Form of the Local Solutions to the TRE. The analytical study of the solutions to the TRE and TAE was started in [4] and continued by different approximate methods [5,11] without utilizing of Heun's functions. Using the confluent Heun function one can write down 16 exact local Frobenius-type solutions to the TRE (2.1) in the form:

$$\begin{aligned} &{}_sR^{\pm}_{\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r;\, r_+,\, r_-)\Delta^{s/2} = \\ &\quad = e^{\sigma_\alpha\frac{\alpha_\pm z_\pm}{2}} z_\pm^{\sigma_\beta\frac{\beta_\pm}{2}} z_\mp^{\sigma_\gamma\frac{\gamma_\pm}{2}} \mathrm{HeunC}(\sigma_\alpha\alpha_\pm,\, \sigma_\beta\beta_\pm,\, \sigma_\gamma\gamma_\pm,\, \delta_\pm,\, \eta_\pm,\, z_\pm), \end{aligned} \tag{2.3}$$

which is very similar to the form of the solutions to the RWE [14, 17]. Here*

$$\alpha_+ = {}_s\alpha_{\omega,E,m}(r_+,\, r_-) = 2i\omega(r_+ - r_-) = ip\,\omega/\Omega_a, \tag{2.4a}$$

$$\beta_+ = {}_s\beta_{\omega,E,m}(r_+,\, r_-) = s + 2i\left(m - \omega/\Omega_-\right)/p, \tag{2.4b}$$

$$\gamma_+ = {}_s\gamma_{\omega,E,m}(r_+,\, r_-) = s - 2i\left(m - \omega/\Omega_+\right)/p, \tag{2.4c}$$

$$\delta_+ = {}_s\delta_{\omega,E,m}(r_+,\, r_-) = \alpha_+\left(s - i\omega(r_+ + r_-)\right) = \alpha_+\left(s - i\omega/\Omega_g\right), \tag{2.4d}$$

$$\eta_+ = {}_s\eta_{\omega,E,m}(r_+,\, r_-) = -E + s^2 + m^2 + \\ + \frac{2m^2\Omega_a^2 - \omega^2}{p^2\Omega_a^2} - \frac{(2m\Omega_a - \omega)^2}{p^2\Omega_g^2} - \frac{1}{2}\left(s - i\,\frac{\omega\,\Omega_+}{\Omega_a\Omega_g}\right)^2, \tag{2.4e}$$

$$z_+ = z_+(r;\, r_+,\, r_-) = \frac{r - r_-}{r_+ - r_-}, \quad z_- = z_-(r;\, r_+,\, r_-) = \frac{r_+ - r}{r_+ - r_-}, \tag{2.4f}$$

$$z_+ + z_- = 1,\; z_+ z_- = \frac{-\Delta}{(r_+ - r_-)^2}.$$

The discrete parameters $\sigma_\alpha, \sigma_\beta, \sigma_\gamma$ have values ± 1**. In equations (2.4a)–(2.4e) we use the following quantities: the angular velocity of the event horizon $\Omega_+ = \sqrt{r_-/r_+}\big/\,(r_+ + r_-)$, the angular velocity of the Cauchy horizon $\Omega_- = \sqrt{r_+/r_-}\big/\,(r_+ + r_-)$, the arithmetically-averaged angular velocity $\Omega_a = (\Omega_+ + \Omega_-)/2 = 1/(2a)$, the geometrically-averaged angular velocity $\Omega_g = \sqrt{\Omega_+\Omega_-} = 1/(2M)$, and the new dimensionless parameter $p = \sqrt{r_+/r_-} - \sqrt{r_-/r_+} = \sqrt{\Omega_-/\Omega_+} - \sqrt{\Omega_+/\Omega_-} = (r_+ - r_-)/a = 2\sqrt{M^2/a^2 - 1}$. Note that the inverse relation $r_\pm = \sqrt{\Omega_\mp/\Omega_\pm}\big/\,(\Omega_+ + \Omega_-)$ permits us to replace $r_\pm$ with $\Omega_\pm$ wherever it is necessary, thus making transparent the duality of the parameters $r_\pm$ and $\Omega_\pm$, as well as the behavior of the above quantities under interchange of the two horizons: $r_+ \leftrightarrows r_- \Rightarrow \Omega_+ \leftrightarrows \Omega_-$, $p \mapsto -p$, $\Omega_{a,g} \mapsto \Omega_{a,g}$ — invariant.

The parameters $\alpha_-, \beta_-, \gamma_-, \delta_-, \eta_-$ can be obtained by interchanging the places of the two horizons: $r_+ \leftrightarrows r_-$ in (2.4a)–(2.4e). This procedure may be substantiated using the known properties of the confluent Heun function under changes of parameters [13]. One can check directly that this way we obtain indeed solutions of equation (2.1).

According to equations (2.3) and (2.4f), the behavior of the solutions ${}_sR^{\pm}_{\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r;\, r_+,\, r_-)$ around the corresponding singular points $z =$

---

*Note that the notation $z_\pm$ in Eq. (2.4f) is consistent with the limits $z_\pm \to \pm\infty$ for $r \to \infty$. Their relation with the notation of the parameters of the Kerr metric $r_\pm$ is illustrated by the equations $z_\pm(r_\mp; r_+, r_-) = 0$. The labels $\pm$ in the notation $R^\pm$ in Eq. (2.3) are related with the labels of their arguments $z_\pm$, not with the labels of the parameters $r_\pm$.

**Further on, $\sigma_x = \mathrm{sign}(x)$ denotes the sign of the *real* quantity x. The only exception is $\sigma \equiv \sigma_s$, where we skip the index s.

$z_\pm(r_\mp;\, r_+,\, r_-) = 0$ is defined by the dominant factor $\left(z_\pm\right)^{\sigma_\beta\beta_\pm/2}$. All other factors in equation (2.3) are regular around these points. The same solutions are in general singular around the corresponding singular points $z = z_\pm(r_\pm;\, r_+,\, r_-) = 1$.

Only two of the sixteen solutions (2.3) are linearly independent. Nevertheless, it is necessary to know all of them since for different purposes one has to use different pairs of independent local solutions.

Using the known asymptotic expansion of the confluent Heun function [13] we obtain two asymptotic solutions of Tomè type. These are local solutions of the TRE around its irregular singular point $|r| = \infty$ in the complex plane $\mathbb{C}_r$:

$$
\begin{aligned}
{}_sR^{\pm\infty}_{\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r;\, r_+,\, r_-) &\sim e^{i\sigma_\alpha\omega(r+(r_++r_-)\ln r)}\times \\
&\times \sum_{j\geqslant 0} a_j \left(\pm\frac{r_+ - r_-}{r}\right)^{j+1+(1+\sigma_\alpha)s}, \quad a_0 = 1. \qquad (2.5)
\end{aligned}
$$

The notation $\pm\infty$ in (2.5) denotes the two directions: $r \to +\infty$ and $r \to -\infty$ on the real r axis for approaching the irregular singular point $|r| = \infty$ in the complex plane $\mathbb{C}_r$. For the coefficients $a_j = a_{j,\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}$ one has a recurrence relation [13] which shows that they increase together with the integer j. Hence, the asymptotic series (2.5) is a divergent one.

As seen from (2.4)

$$
{}_sR^{-}_{\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r;\, r_+,\, r_-) = {}_sR^{+}_{\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r;\, r_-,\, r_+).
$$

Hence, one can introduce a new parity property of the solutions and construct symmetric and antisymmetric (with respect to the interchange $r_+ \rightleftarrows r_-$) solutions of the TRE:

$$
\begin{aligned}
&{}_sR^{\mathrm{SYM}}_{\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r;\, r_+,\, r_-) = \\
&= \frac{1}{2}\left({}_sR^{+}_{\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r;\, r_+,\, r_-) + {}_sR^{-}_{\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r;\, r_+,\, r_-)\right), \qquad (2.6)
\end{aligned}
$$

$$
\begin{aligned}
&{}_sR^{\mathrm{ASYM}}_{\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r;\, r_+,\, r_-) = \\
&\quad = \frac{1}{2}\left({}_sR^{+}_{\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r;\, r_+,\, r_-) - {}_sR^{-}_{\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r;\, r_+,\, r_-)\right).
\end{aligned}
$$

Clearly, these solutions are singular at both horizons in the general case. When one considers the two-singular-point boundary problem [13] on the interval $[r_-, r_+]$ in the Kerr black hole interior, the solutions (2.6) may be regular at one, or at both the ends for some values of the separation constants ω and E. Since this boundary problem is still not studied, at present we are not able to make more definite statements about this case.

3. A NEW CLASSIFICATION OF THE SOLUTIONS TO THE TRE, BASED ON THE δ_N-CONDITION. NOVEL RADIAL δ_N-SOLUTIONS

For the TRE the δ_N-condition reads:

$$
{}_s\omega^{\pm}_{m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}\,\mathcal{L}^{\pm}_{\sigma_\alpha,\sigma_\beta,\sigma_\gamma} = \Omega_g\left(\mathcal{M}^{\pm}_{m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma} + i\,{}_s\mathcal{N}^{\pm}_{\sigma_\alpha,\sigma_\beta,\sigma_\gamma}\right), \tag{3.1}
$$

where

$$
\begin{aligned}
\mathcal{L}^{\pm}_{\sigma_\alpha,\sigma_\beta,\sigma_\gamma} &= \frac{\sigma_\beta\Omega_\pm - \sigma_\gamma\Omega_\mp}{\Omega_\pm - \Omega_\mp} - \sigma_\alpha,\\
\mathcal{M}^{\pm}_{m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma} &= m(\sigma_\beta - \sigma_\gamma)\frac{\Omega_g}{\Omega_\pm - \Omega_\mp},\\
{}_s\mathcal{N}^{\pm}_{\sigma_\alpha,\sigma_\beta,\sigma_\gamma} &= N + 1 + \left(\sigma_\alpha + \frac{\sigma_\beta + \sigma_\gamma}{2}\right)s.
\end{aligned}
$$

We call *radial δ_N-solutions* the solutions defined via the δ_N-condition (3.1).

The calculation of the values of the coefficients in equation (3.1) yields two very different cases:

1. In the first case $\mathcal{L}^{+}_{\pm,\pm,\pm} = \mathcal{L}^{-}_{\pm,\pm,\pm} = 0$ and we see that one is not able to fix the frequencies ${}_s\omega^{+}_{m,\pm,\pm,\pm}$ and ${}_s\omega^{-}_{m,\pm,\pm,\pm}$. Instead, choosing $\sigma_\alpha = \sigma_\beta = \sigma_\gamma = -\sigma$ and using (3.1) one fixes the non-negative integer N in the form

$$
{}_sN + 1 = 2|s| \geqslant 1 \quad \text{for} \quad |s| \geqslant 1/2. \tag{3.2}
$$

Thus, the degree of the polynomial Δ_{N+1}-condition is fixed, too.

2. In the second case the coefficients $\mathcal{L}^{\pm}_{\sigma_\alpha,\sigma_\beta,\sigma_\gamma}$ are nonzero and one can fix the values of the frequencies ${}_s\omega^{\pm}_{m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}$ from equation (3.1). Thus, one obtains two different types of exact *equidistant* spectra:

a) For $\mathcal{L}^{+}_{\mp,\pm,\pm} = \mathcal{L}^{-}_{\mp,\pm,\pm} = \pm 2$, $\mathcal{M}^{+}_{\mp,\pm,\pm} = \mathcal{M}^{-}_{\mp,\pm,\pm} = 0$ and $\mathcal{N}^{+}_{\mp,\pm,\pm} = \mathcal{N}^{-}_{\mp,\pm,\pm} = (N+1)$ the δ_N-condition (3.1) produces the pure imaginary equidistant frequencies

$$
{}_s\omega^{+}_{N,m,\mp,\pm,\pm} = {}_s\omega^{-}_{N,m,\mp,\pm,\pm} = \pm i\,\frac{N+1}{4M}, \quad N \geqslant 0 \quad \text{— integer}. \tag{3.3}
$$

Note that these frequencies depend neither on the spin-weight s and azimuthal number m, nor on the rotation parameter a. The spectrum is not influenced by the rotation of the waves and of the very Kerr metric. The frequencies (3.3) are defined only by the monopole term in multipole expansion of the metric.

b) For all other cases the coefficients $\mathcal{L}^{\pm}_{\sigma_\alpha,\sigma_\beta,\sigma_\gamma}$, and $\mathcal{M}^{\pm}_{\sigma_\alpha,\sigma_\beta,\sigma_\gamma}$ are not fixed integers and one obtains the following two similar double-equidistant spectra of frequencies:

$$ {}_s\omega^{+}_{N,m,\mp,\mp,\pm} = {}_s\omega^{-}_{N,m,\mp,\pm,\mp} = m\Omega_+ \pm \frac{i}{4M}\left(1 - \frac{r_-}{r_+}\right)(N+1\mp s); \quad (3.4a) $$

$$ N \geqslant 0, \quad m \text{ — integers}, $$

$$ {}_s\omega^{+}_{N,m,\pm,\mp,\pm} = {}_s\omega^{-}_{N,m,\pm,\pm,\mp} = m\Omega_- \pm \frac{i}{4M}\left(\frac{r_+}{r_-} - 1\right)(N+1\pm s); \quad (3.4b) $$

$$ N \geqslant 0, \quad m \text{ — integers}. $$

A set of important new mathematical properties of the radial δ_N-solutions can be found in [16,17]. In [17] it is shown that these solutions define the most general class of solutions to the TRE for which the properly generalized Teukolsky–Starobinsky identities exist. The solutions which satisfy the relation (3.2) were studied in [2, 3] without utilizing the Heun functions and the δ_N-condition. The last condition turns to be valid automatically for the solutions to the TRE studied in [2,3]. The infinite series of the solutions with equidistant spectra (3.3) and (3.4) are introduced and considered for the first time in the present work.

4. POLYNOMIAL SOLUTIONS TO THE TRE

The δ_N-condition yields the basic classification of the solutions described in the previous Sec. 3. As a result, one obtains two classes of polynomial solutions to the TRE, imposing in addition the Δ_{N+1}-condition (1.7b). In what follows we will use the determinant form of the Δ_{N+1}-condition given in [16].

4.1. The First Class of Polynomial Solutions to the TRE. The solutions of this class correspond to the first case in Sec. 3 and obey equation (3.2). The inequality ${}_sN = 2|s| - 1 \geqslant 0$ excludes the existence of scalar perturbations ($|s| = 0$) of the first polynomial class.

4.1.1. The General Case. For brevity, we denote the solutions ${}_sR^{\pm}_{\omega,E,m,-\sigma,-\sigma,-\sigma}(r;\, r_+,\, r_-)$ as ${}_sR^{\pm}_{\omega,E,m}(r;\, r_+,\, r_-)$. For them the parameter μ takes the values $\mu = {}_s\mu^{\pm}_{\omega,k,m}(r_+,\, r_-)$ $k = 1, \ldots, 2|s|$ — the solutions of the algebraic equation (1.7b), which now takes the form: $\Delta^{\pm}_{2|s|}(\mu) = 0$. Its degree is $2|s| = 1$, 2, 3, or 4, depending on the spin of the perturbations $|s| = 1/2$, 1, 3/2, 2. Making use of (1.6b), and (2.4a)–(2.4e), we obtain for the separation constant $E = {}_sE^{\pm}_{\omega,k,m}(r_+,\, r_-)$, $k = 1, \ldots, 2|s|$ the expressions

$$ {}_sE^{\pm}_{\omega,k,m}(r_+, r_-) = {}_s\mu^{\pm}_{\omega,k,m}(r_+, r_-) + |s|(|s|-1) - {} $$
$$ - a\omega(a\omega - 2m) + 2i\sigma(2|s|-1)\omega r_{\mp}. \quad (4.1) $$

Applying the explicit expressions for the roots ${}_s\mu^{\pm}_{\omega,k,m}(r_+,\, r_-)$, we obtain

$$
{}_sE^{\pm}_{\omega,m}(r_+,\, r_-) = -a^2\omega^2 + 2a\omega m - \frac{1}{4}\, : \tag{4.2}
$$
$$
\text{for} \quad |s| = \frac{1}{2};
$$
$$
{}_sE^{\pm}_{\omega,k,m}(r_+,\, r_-) = -a^2\omega^2 + 2a\omega\left(m - (-1)^k\sqrt{1 - m/a\omega}\right) : \tag{4.3}
$$
$$
\text{for} \quad |s| = 1,\; k = 1, 2.
$$

For the gravitational waves ($|s| = 2$) one has to find the quantities ${}_s\mu^{\pm}_{\omega,k,m}(r_+,\, r_-)$ solving algebraic equation of the fourth degree $\Delta^{\pm}_4(\mu) = 0$. The explicit form of its roots is too complicated and not necessary for the purposes of the present work. It is more instructive to demonstrate here the result, obtained using the Taylor series expansion of the solutions ${}_s\mu^{\pm}_{\omega,k,m}(r_+,\, r_-)$ around the zero frequency $\omega = 0$.

Thus, we obtain for $|s| = 2$, $k = 1$, 2 the eight values:

$$
\begin{aligned}
{}_sE^{\pm}_{\omega,k,m} &= 2 - 4\left(m - i(-1)^k\frac{3M}{2a}\right)a\omega + \\
&+ 6\left(m^2 + i(-1)^k 2m\left((m^2-1)\frac{a}{M} + \frac{2M}{a}\right) + \frac{3M^2}{a^2} - \frac{7}{6}\right)(a\omega)^2 + \mathcal{O}_3(a\omega).
\end{aligned} \tag{4.4}
$$

For $|s| = 2$, $m \neq 0$, $k = 3$, 4 we have other eight values:

$$
\begin{aligned}
{}_sE^{\pm}_{\omega,k,m} &= i(-1)^k 4\sqrt{ma\omega}\left(1 + i3\left(1 + \left(\frac{3M^2}{8a^2} - \frac{2}{3}\right)\frac{1}{m^2}\right)ma\omega + \right. \\
&\left. + \mathcal{O}_2(a\omega)\right) + 8ma\omega - 6\left(1 + \left(\frac{3M^2}{a^2} - \frac{5}{6}\right)\frac{1}{m^2}\right)(ma\omega)^2 + \mathcal{O}_3(a\omega).
\end{aligned} \tag{4.5}
$$

Clearly, these series describe two kinds of solutions with a completely different behavior around the origin $\omega = 0$. In particular, the series (4.4) and (4.5) have different limits: 2 and 0, respectively, when $\omega \to 0$. For the solutions (4.5) the origin $\omega = 0$ is a branching point, etc.

The independence of the values of ${}_sE^{\pm}_{\omega,k,m}$ in (4.4) and (4.5) on the upper labels ($\pm$) is a result of the polynomial character of the solutions, i.e., of the regularity of the corresponding HeunC-factor simultaneously on both the horizons $r_{\pm}$.

For a complete solution of the problem one has to determine the frequency ω. Hence, one needs an additional relation between the parameters E and ω. This relation may appear when one solves the TAE (See the next Secs. 5–8.).

The first class of polynomial solutions to the TRE is introduced and studied in detail for the first time in the present work.

4.1.2. The Special Case of the Schwarzschild Metric. For the special value of the parameter $a = 0$ we have $r_- = 0$, $r_+ = 2M$. This is the case of perturbations to the nonrotating Schwarzschild black hole described in terms of the Weyl scalars. For simplicity, here we use units in which $2M = 1$. The parameters in the solution (2.3) acquire the limiting values

$$\alpha_+ = 2i\omega, \quad \beta_+ = s, \quad \gamma_+ = s + 2i\omega, \quad \delta_+ = 2i\omega(s - i\omega), \quad \eta_+ = -E + \frac{s^2}{2};$$
$$\alpha_- = -2i\omega, \quad \beta_- = s + 2i\omega, \quad \gamma_- = s, \quad \delta_- = -2i\omega(s - i\omega),$$
$$\eta_- = -E + \frac{s^2}{2} + 2\omega^2 + 2is\omega. \tag{4.6}$$

These differ from the values of the parameters of confluent Heun's functions in the Regge–Wheeler approach to the perturbations of the Schwarzschild metric [14].

In the limit $a \to 0$ equation (3.1) does not define the frequency ω, if $\sigma_\alpha = \mp\sigma_\beta = \pm\sigma_\gamma = -\sigma$, because then one obtains $\mathcal{L}^{\pm}_{-\sigma,\pm\sigma,\mp\sigma} = 0$. If, in addition, $\sigma = \mathrm{sign}(s)$, then the δ_N-condition is fulfilled for the special polynomial solutions of the first class denoted as ${}_sR^{\pm}_{\omega,E,m}(r) = {}_sR^{\pm}_{\omega,E,m,-\sigma,\pm\sigma,\mp\sigma}(r;\, 1,\, 0)$. Equation (3.1) yields the relation ${}_sN = |s| - 1 \geqslant 0$. Scalar perturbations of this type do not exist.

In the case of integer spins $|s| = 1,\, 2$ the roots $\mu = {}_s\mu^{\pm}_{\omega,k,m}$, $k = 1, \ldots, |s|$ of the equations $\Delta^{\pm}_{|s|}(\mu) = 0$, (1.6b), and (2.4a)–(2.4e) with $r_+ = 1$, $r_- = 0$ and $a = 0$ produce the following simple expressions for $E = {}_sE^{\pm}_{\omega,k,m}$, $k = 1, \ldots, |s|$:

$${}_sE^{\pm}_{\omega,m} = 0: \quad \text{for} \quad |s| = 1, \tag{4.7}$$
$${}_sE^{\pm}_{\omega,k,m} = 1 - (-1)^k\sqrt{1 - i6\sigma\omega}: \quad \text{for} \quad |s| = 2,\, k = 1,\, 2. \tag{4.8}$$

For a complete solution of the problem, one needs an additional relation between the parameters E and ω. This relation may be found by solving the TAE, see Secs. 5–8.

The above considerations of the limit $a \to 0$ and the corresponding results for the Schwarzschild black hole in terms of confluent Heun's functions are new and obtained for the first time in the present work.

4.2. Second Class of Polynomial Solutions to the TRE. According to the results of Sec. 3, the solutions of this class originate from the second case of the δ_N-condition and fall into two subclasses: a) and b). The complete definite frequencies ${}_s\omega^{\pm}_{N,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}$ — formulae (3.3) and (3.4), yield algebraic equations $\Delta^{\pm}_{N+1}(\mu) = 0$ with $(N + 1)$ roots $\mu = {}_s\mu^{\pm}_{N,n,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r_+,\, r_-)$,

$n = 0, 1, \ldots, N$. It seems difficult to derive explicit analytic expressions for these roots, but their numerical values can be easily obtained. Using the values of ${}_s\mu^{\pm}_{N,n,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r_+, r_-)$ and equations (1.6b), (2.4a)–(2.4e) we obtain complete *definite* values for the parameter $E = {}_sE^{\pm}_{N,n,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r_+, r_-)$:

a) In the case of frequencies (3.3) we obtain

$$
\begin{aligned}
{}_sE^{\pm}_{N,n,m,\sigma,-\sigma,-\sigma} = {}_s\mu^{\pm}_{N,n,m,\sigma,-\sigma,-\sigma} + |s|(|s|-1) + a\omega(3a\omega - 2m) + \\
+ 4\omega^2 r_{\mp}^2 + 2i\sigma\omega\left(2M|s| - r_{\pm}\right). \quad (4.9)
\end{aligned}
$$

b) In the case of frequencies (3.4a), (3.4b) we have, respectively:

$$
\begin{aligned}
&{}_sE^{\pm}_{N,n,m,+,-,+} - {}_s\mu^{\pm}_{N,n,m,+,-,+} = {}_sE^{\mp}_{N,n,m,-,+,-} - {}_s\mu^{\mp}_{N,n,m,+,-,+} = \\
&\quad = \pm i2(2a\omega - m)/p - \left(m^2 + 8m\left(1 + M^2/a^2\right)a\omega + \right. \\
&\quad \left. + \left(1 + 10r_{\mp}/r_{\pm} + 9(r_{\mp}/r_{\pm})^2 - 4(r_{\mp}/r_{\pm})^3\right)\omega^2 r_{\pm}^2\right)/p^2, \quad (4.10a)
\end{aligned}
$$

$$
\begin{aligned}
&{}_sE^{\pm}_{N,n,m,+,+,-} - {}_s\mu^{\pm}_{N,n,m,+,+,-} = {}_sE^{\mp}_{N,n,m,-,-,+} - {}_s\mu^{\mp}_{N,n,m,-,-,+} = \\
&\quad = \pm i2\left(m + 2a\omega\left(1 - 2M^2/a^2\right)\right)/p - \\
&- i2psa\omega - 4\left(m^2 + 2m\left(1 - 3M^2/a^2\right)a\omega - \left(1 - 5M^2/a^2\right)(a\omega)^2\right)/p^2. \quad (4.10b)
\end{aligned}
$$

With ω and E given by equations (3.3), (3.4) and (4.9), (4.10) we have no more free parameters in the problem at hand. As a result, the corresponding solutions to the TAE are fixed unambiguously by the designated group of equations obtained for the second class of polynomial solutions to the TRE. This situation is completely new, unexpected and described here for the first time.

5. EXACT SOLUTIONS TO THE TEUKOLSKY ANGULAR EQUATION IN TERMS OF THE CONFLUENT HEUN FUNCTIONS

In terms of the variable $x = \cos\theta$ the TAE has three singular points. Two of them: $x_- = -1$ (i.e., $\theta_S = \pi$ — South (S-)pole) and $x_+ = 1$ (i.e., $\theta_N = 0$ — North (N-)pole) are regular singular points. The third one $x_\infty = \infty$ is an irregular singular point. It is remarkable that introducing the notation

$$
z_+ = z_+(\theta) = \left(\cos(\theta/2)\right)^2, \quad z_- = z_-(\theta) = \left(\sin(\theta/2)\right)^2, \quad z_+ + z_- = 1 \quad (5.1)
$$

and

$$
\begin{gathered}
a_\pm = \pm 4a\omega, \quad b_\pm = s \mp m, \quad c_\pm = s \pm m, \quad d_\pm = \pm 4sa\omega, \quad (5.2) \\
n_\pm = \frac{m^2 + s^2}{2} \mp 2sa\omega - a^2\omega^2 - E,
\end{gathered}
$$

we can write down 16 local solutions of the TAE in the form

$$
\begin{aligned}
&{}_sS^{\pm}_{\omega,E,m,\sigma_a,\sigma_b,\sigma_c} = \\
&\qquad = e^{\sigma_a \frac{a_\pm z_\pm}{2}}\, z_\pm^{\sigma_b \frac{b_\pm}{2}}\, z_\mp^{\sigma_c \frac{c_\pm}{2}}\, \mathrm{HeunC}(\sigma_a a_\pm,\, \sigma_b b_\pm,\, \sigma_c c_\pm,\, d_\pm,\, n_\pm,\, z_\pm), \quad (5.3)
\end{aligned}
$$

which is very similar to the form of Eq. (2.3).

Following the corresponding properties of the TAE (1.1a) [2], the solutions (5.3) have the symmetries

$$
{}_{-s}S^{\pm}_{\omega,E,m,\sigma_a,\sigma_b,\sigma_c}(\pi-\theta) = {}_sS^{\mp}_{\omega,E,m,-\sigma_a,-\sigma_b,-\sigma_c}(\theta), \qquad (5.4a)
$$

$$
{}_sS^{\pm}_{-\omega,E,-m,\sigma_a,\sigma_b,\sigma_c}(\pi-\theta) = {}_sS^{\mp}_{\omega,E,m,\sigma_a,\sigma_b,\sigma_c}(\theta). \qquad (5.4b)
$$

Note that according to Eq. (5.3), the behavior of the solutions ${}_sS^{\pm}_{\omega,E,m,\sigma_a,\sigma_b,\sigma_c}$ around the corresponding singular points $z = z_+(\theta_S) = z_-(\theta_N) = 0$ is defined by the dominant factor $\left(z_\pm\right)^{\sigma_b b_\pm/2}$. All other factors in (5.3) are regular around these points. The same solutions are in general singular around the corresponding singular points $z = z_+(\theta_N) = z_-(\theta_S) = 1$. Hence, at this point we have a complete analogy with the case of TRE.

Only two of the sixteen solutions (5.3) are linearly independent. Nevertheless, it is important to know all of them, since for various purposes one can use different pairs of independent local solutions, see below. If one chooses some two linearly independent solutions, then one can represent the other fourteen using this basis. Unfortunately, at present the form of the corresponding coefficients is completely unknown.

We can establish simple relations between some of the different solutions (5.3) in proper domains of the parameters s and m, if we divide the whole plane $\{s,\, m\}$ into four sectors. In each of them we choose the solutions with the same *regular* asymptotic behavior around the corresponding pole as follows:

I. Sector $s \geqslant 0$, $|m| \leqslant |s|$:

$$
\begin{aligned}
{}_sS^{+\mathrm{reg}}_{\omega,E,m}(\theta) &= {}_sS^{+}_{\omega,E,m,+++} = {}_sS^{+}_{\omega,E,m,-++} = {}_sS^{+}_{\omega,E,m,++-} = \\
&= {}_sS^{+}_{\omega,E,m,-+-} \underset{\theta\to\pi}{\sim} \left(\cos\frac{\theta}{2}\right)^{s-m}, \qquad (5.5a)
\end{aligned}
$$

$$
\begin{aligned}
{}_sS^{-\mathrm{reg}}_{\omega,E,m}(\theta) &= {}_sS^{-}_{\omega,E,m,+++} = {}_sS^{-}_{\omega,E,m,-++} = {}_sS^{-}_{\omega,E,m,++-} = \\
&= {}_sS^{-}_{\omega,E,m,-+-} \underset{\theta\to 0}{\sim} \left(\sin\frac{\theta}{2}\right)^{s+m}. \qquad (5.5b)
\end{aligned}
$$

II. Sector $m \leqslant 0$, $|s| \leqslant |m|$:

$$
{}_sS^{+\mathrm{reg}}_{\omega,E,m}(\theta) = {}_sS^{+}_{\omega,E,m,+++} = {}_sS^{+}_{\omega,E,m,-++} = {}_sS^{+}_{\omega,E,m,++-} = \\
= {}_sS^{+}_{\omega,E,m,-+-} \underset{\theta\to\pi}{\sim} \left(\cos\frac{\theta}{2}\right)^{s-m}, \quad (5.6a)
$$

$$
{}_sS^{-\mathrm{reg}}_{\omega,E,m}(\theta) = {}_sS^{-}_{\omega,E,m,---} = {}_sS^{-}_{\omega,E,m,+--} = {}_sS^{-}_{\omega,E,m,--+} = \\
= {}_sS^{-}_{\omega,E,m,+-+} \underset{\theta\to 0}{\sim} \left(\sin\frac{\theta}{2}\right)^{-s-m}. \quad (5.6b)
$$

III. Sector $s \leqslant 0$, $|m| \leqslant |s|$:

$$
{}_sS^{+\mathrm{reg}}_{\omega,E,m}(\theta) = {}_sS^{+}_{\omega,E,m,---} = {}_sS^{+}_{\omega,E,m,+--} = {}_sS^{+}_{\omega,E,m,--+} = \\
= {}_sS^{+}_{\omega,E,m,+-+} \underset{\theta\to\pi}{\sim} \left(\cos\frac{\theta}{2}\right)^{-s+m}, \quad (5.7a)
$$

$$
{}_sS^{-\mathrm{reg}}_{\omega,E,m}(\theta) = {}_sS^{-}_{\omega,E,m,---} = {}_sS^{-}_{\omega,E,m,+--} = {}_sS^{-}_{\omega,E,m,--+} = \\
= {}_sS^{-}_{\omega,E,m,+-+} \underset{\theta\to 0}{\sim} \left(\sin\frac{\theta}{2}\right)^{-s-m}. \quad (5.7b)
$$

IV. Sector $m \geqslant 0$, $|s| \leqslant |m|$:

$$
{}_sS^{+\mathrm{reg}}_{\omega,E,m}(\theta) = {}_sS^{+}_{\omega,E,m,---} = {}_sS^{+}_{\omega,E,m,+--} = {}_sS^{+}_{\omega,E,m,--+} = \\
= {}_sS^{+}_{\omega,E,m,+-+} \underset{\theta\to\pi}{\sim} \left(\cos\frac{\theta}{2}\right)^{-s+m}, \quad (5.8a)
$$

$$
{}_sS^{-\mathrm{reg}}_{\omega,E,m}(\theta) = {}_sS^{-}_{\omega,E,m,+++} = {}_sS^{-}_{\omega,E,m,-++} = {}_sS^{-}_{\omega,E,m,++-} = \\
= {}_sS^{-}_{\omega,E,m,-+-} \underset{\theta\to 0}{\sim} \left(\sin\frac{\theta}{2}\right)^{s+m}. \quad (5.8b)
$$

Note that in each sector the four solutions in the above relations of type (a), or in the above relations of type (b) are equal, since the local *regular* solution around any regular singular point of the TAE is unique.

In the case of integer spin weights $s = 0, \pm 1, \pm 2$ there exists an additional complication. The confluent Heun functions $\mathrm{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$ are not defined when β is a negative integer [13]. Therefore, if $\beta = \sigma_b b_\pm < 0$ is a negative integer, we must write down the corresponding solutions in the form

$$
{}_sS^{\pm}_{\omega,E,m,\sigma_a,\sigma_b,\sigma_c} = \\
= e^{\sigma_a \frac{a_\pm z_\pm}{2}} z_\pm^{\sigma_b \frac{b_\pm}{2}} z_\mp^{\sigma_c \frac{c_\pm}{2}} \, \underline{\mathrm{HeunC}}\,(\sigma_a a_\pm, \sigma_b b_\pm, \sigma_c c_\pm, d_\pm, n_\pm, z_\pm). \quad (5.9)
$$

For this purpose we define the *concomitant* confluent Heun function*

$$\underline{\mathrm{HeunC}}\,(\alpha,\,\beta,\,\gamma,\,\delta,\,\eta,\,z) =$$
$$= z^{-\beta}\mathrm{HeunC}\,(\alpha,\,-\beta,\,\gamma,\,\delta,\,\eta,\,z)\int \frac{e^{-\alpha\zeta}\zeta^{\beta-1}(1-\zeta)^{-\gamma-1}}{\left(\mathrm{HeunC}\,(\alpha,\,-\beta,\,\gamma,\,\delta,\,\eta,\,\zeta)\right)^2}d\zeta. \quad (5.10)$$

This function is well defined for negative integer $\beta = \sigma_b b_\pm < 0$, together with the confluent function $\mathrm{HeunC}\,(\alpha,\,-\beta,\,\gamma,\,\delta,\,\eta,\,z)$. In this case, the function $z^{-\beta}\,\mathrm{HeunC}\,(\alpha,\,-\beta,\,\gamma,\,\delta,\,\eta,\,z)$ represents the local regular solution to the confluent Heun equation (1.5) around the singular point $z=0$ and the concomitant confluent function $\underline{\mathrm{HeunC}}\,(\alpha,\,\beta,\,\gamma,\,\delta,\,\eta,\,z)$ represents a second linearly independent local solution, which is singular around this point.

It can be shown that for negative integer β the concomitant confluent Heun function has the form

$$\underline{\mathrm{HeunC}}\,(\alpha,\,\beta,\,\gamma,\,\delta,\,\eta,\,z) = \sum_{n=1}^{|\beta|}\frac{c_n}{z^n} + h_1(z) + h_2(z)\ln(z), \quad \text{all} \quad c_n \neq 0. \quad (5.11)$$

Here $h_{1,2}(z)$ denote two definite functions of the complex variable z which are analytic in the vicinity of the point $z=0$. In the problem at hand $|\beta| = |\beta_\pm| = |s \mp m|$. The logarithmic term is present in the concomitant confluent Heun function when $|\beta| = 0$, too, but then we have no poles in the solution (5.11). For $|\beta| = 0$ its form otherwise is similar to (5.11). One can reach the last results using general analytical methods described, for example, in [24].

6. A NEW CLASSIFICATION OF THE SOLUTIONS TO THE TAE BASED ON THE δ_N-CONDITION. NOVEL δ_N-ANGULAR SOLUTIONS

For solutions ${}_sS^{\pm}_{\omega,E,m,\sigma_a,\sigma_b,\sigma_c}$ (5.3) to TAE the δ_N-condition reads:

$$0 = \mp\, m\frac{\sigma_b - \sigma_c}{2} + N + 1 + \left(\sigma_a + \frac{\sigma_b + \sigma_c}{2}\right)s. \quad (6.1)$$

We call *angular δ_N-solutions* the solutions defined via the δ_N-condition (6.1). To some extent these solutions are similar to the radial δ_N-solutions introduced

*Note that for any value of the parameter β, when the confluent Heun function in the right-hand side of Eq. (5.10) is well defined, its left-hand side represents a second, linearly independent solution of the confluent Heun equation.

in Sec. 3. A set of important new mathematical properties of the angular δ_N-solutions can be found in [16, 17]. In [17] it is shown that these solutions define the most general class of solutions to the TAE for which properly generalized Teukolsky–Starobinsky identities exist.

Comparing equation (6.1) with the corresponding one for the TRE — (3.1), we see both essential differences and similarities. For the coefficients in equation (6.1), which are analogous to the ones in (3.1), one obtains

$$\mathcal{L}^{\pm}_{\sigma_a,\sigma_b,\sigma_c} \equiv 0, \quad \mathcal{M}^{\pm}_{m,\sigma_a,\sigma_b,\sigma_c} = \mp m(\sigma_b - \sigma_c)\frac{1}{2},$$
$${}_s\mathcal{N}^{\pm}_{\sigma_a,\sigma_b,\sigma_c} = N + 1 + \left(\sigma_a + \frac{\sigma_b + \sigma_c}{2}\right) s.$$

Hence:

i) The coefficients $\mathcal{L}^{\pm}_{\sigma_a,\sigma_b,\sigma_c}$ vanish identically, in contrast to the coefficients $\mathcal{L}^{\pm}$ in equation (3.1). Consequently, there are no cases in which the condition (6.1) can fix the frequencies ω.

ii) The form of the coefficients $\mathcal{M}^{\pm}$ of both equations (3.1) and (6.1) is the same only for $a^2/M^2 = 1/2$.

iii) The coefficients $\mathcal{N}^{\pm}$ of both equations are of the same form.

We obtain two different cases depending on the coefficient $(\sigma_b - \sigma_c)$ in front of the azimuthal number m:

1. The first class angular δ_N-solutions with $\sigma_c = \sigma_b$ and $\sigma_a = \sigma_b = \sigma_c = -\sigma$. As a result, Eq. (6.1) fixes the degree of the second polynomial condition $\Delta_{N+1} = 0$ in the same form as equation (3.2)*:

$${}_sN + 1 = 2|s| \geqslant 1 \quad \text{for} \quad |s| \geqslant 1/2. \tag{6.2}$$

Then

A. In the case of half-integer spins $1/2$, $3/2$ for any value of m we have angular δ_N-solutions ${}_sS^{\pm}_{\omega,E,m}(z_\pm) \equiv {}_sS^{\pm}_{\omega,E,m,-\sigma,-\sigma,-\sigma}(z_\pm)$:

$${}_sS^{\pm}_{\omega,E,m}(z_\pm) = e^{\mp 2\sigma a\omega z_\pm}\,(z_\pm)^{\frac{\pm\sigma m - |s|}{2}}\,(z_\mp)^{\frac{\mp\sigma m - |s|}{2}} \times$$
$$\times\,\mathrm{HeunC}\left(\mp 4\sigma a\omega,\, \pm\sigma m - |s|,\, \mp\sigma m - |s|,\, \pm 4a\omega s,\right.$$
$$\left.\frac{m^2 + s^2}{2} \mp 2sa\omega - a^2\omega^2 - E,\, z_\pm\right). \tag{6.3}$$

*The alternative case $\sigma_b = \sigma_c = -\sigma_a$ leads to a noninteresting relation $N + 1 = 0$.

Hence, one obtains the behavior of solutions (6.3) around the corresponding singular points $z_\pm = 0$:

$$
{}_sS^{\pm}_{\omega,E,m}(z_\pm) \sim \left(z_\pm\right)^{\frac{\pm\sigma m-|s|}{2}} \quad \text{for} \quad z_\pm \to 0. \tag{6.4}
$$

The behavior of the solutions (6.3) around the singular points $z_\pm = 1 (\Leftrightarrow z_\mp = 0)$ is more complicated. To study this behavior, we use the expansion of the solutions (6.3) with respect to the basis of the two linearly-independent local solutions (5.9) around the points $z_\mp = 0$, well defined for $|s| = 1/2,\, 3/2$:

$$
\begin{aligned}
{}_sS^{\pm}_{\omega,E,m}(z_\pm) &= {}_s\Gamma^{\pm}_1(\omega,E,m)\, e^{\pm 2\sigma a\omega z_\mp}\,(z_\mp)^{\frac{\mp\sigma m-|s|}{2}}\,(z_\pm)^{\frac{\pm\sigma m-|s|}{2}} \times \\
&\times \mathrm{HeunC}\left(\pm 4\sigma a\omega,\, \mp\sigma m-|s|,\, \pm\sigma m-|s|,\, \mp 4\sigma a\omega|s|,\, \frac{m^2+s^2}{2} \pm 2\sigma a\omega|s| - \right. \\
&\left. - a^2\omega^2 - E,\, z_\mp\right) + {}_s\Gamma^{\pm}_2(\omega,\, E,\, m) e^{\pm 2\sigma a\omega z_\mp}\,(z_\mp)^{\frac{\pm\sigma m+|s|}{2}}\,(z_\pm)^{\frac{\pm\sigma m-|s|}{2}} \times \\
&\times \mathrm{HeunC}\left(\pm 4\sigma a\omega,\, \pm\sigma m+|s|,\, \pm\sigma m-|s|,\, \mp 4\sigma a\omega|s|,\, \frac{m^2+s^2}{2} \pm 2\sigma a\omega|s| - \right. \\
&\left. - a^2\omega^2 - E,\, z_\mp\right).
\end{aligned} \tag{6.5}
$$

Note that the first term in Eq. (6.5) is a δ_N-solution with value of the integer ${}_sN$ given by Eq. (6.2), while the second one is not a δ_N-solution for half-integer spins.

From Eq. (6.5) one obtains the behavior of the solutions (6.3) around the corresponding singular points $z_\pm = 1$:

$$
\begin{aligned}
{}_sS^{\pm}_{\omega,E,m}(z_\pm) &\sim {}_s\Gamma^{\pm}_1(\omega,\, E,\, m)\,(1-z_\pm)^{\frac{\mp\sigma m-|s|}{2}} + \\
&+ {}_s\Gamma^{\pm}_2(\omega,\, E,\, m)\,(1-z_\pm)^{\frac{\pm\sigma m+|s|}{2}} \quad \text{for} \quad z_\pm \to 1.
\end{aligned} \tag{6.6}
$$

From Eqs. (6.4) and (6.6) we obtain the following results:

i) Since $|s|$ is a half-integer and m — an integer, the solutions (6.3) are regular around the corresponding points $z_\pm = 0$, if and only if $\sigma\sigma_m = \pm 1$ ($\sigma_m = \mathrm{sign}(m)$) and $|m| > |s|$, and singular, otherwise.

ii) For $|s|$ — a half-integer and m — an integer we have either $(\mp\sigma m - |s|) < 0$, or $(\pm\sigma m + |s|) < 0$. Then the solutions (6.3) are singular around the corresponding points $z_\pm = 1$ in the general case, i.e., when ${}_s\Gamma^{\pm}_1(\omega,\, E,\, m) \neq 0$ and ${}_s\Gamma^{\pm}_2(\omega,\, E,\, m) \neq 0$.

iii) The solutions (6.3) are regular simultaneously around the two corresponding singular points $z_\pm = 0$ and $z_\pm = 1$, if and only if according to i) $\sigma\sigma_m = \pm 1$, $|m| > |s|$ and in addition ${}_s\Gamma_1^\pm(\omega,\, E,\, m) = 0$.

iv) There exist several cases in which the solutions (6.3) are regular around one pole and singular around the other one. For brevity, we shall skip here their detailed description.

B. In the case of integer spins 1, 2 the set of δ_N-solutions (6.3) consists of ones with *integer* parameters $\beta = \sigma_b b_\pm$ of both signs. According to Eq. (5.11), for integer $\sigma_b b_\pm < 0$ in the solutions we have logarithmic terms. Such solutions are infinitely-valued functions. To avoid this physically not admissible case, one must impose the additional requirement $\sigma_b b_\pm \geqslant 0$. As a result, one obtains $\sigma\sigma_m = \pm 1$, $\sigma_b b_\pm = |m| - |s| \geqslant 0$ and $\sigma_c c_\pm = -|m| - |s| < 0$. Thus, the angular δ_N-solutions of the first class with integer spin correspond to sectors II (5.6) and IV (5.8) and acquire the form

$$
\begin{aligned}
{}_sS^\pm_{\omega,E,m}(z_\pm) &= e^{-2\sigma_m a\omega z_\pm} \left(z_\pm\right)^{\frac{|m|-|s|}{2}} \left(z_\mp\right)^{\frac{-|m|-|s|}{2}} \times \\
&\times \mathrm{HeunC}\Bigg(-4\sigma_m a\omega,\, |m|-|s|,\, -|m|-|s|,\, 4\sigma_m a\omega|s|,\, \frac{m^2+s^2}{2} - 2\sigma_m a\omega|s| - \\
&\qquad - a^2\omega^2 - E,\, z_\pm\Bigg). \qquad (6.7)
\end{aligned}
$$

Applying Eq. (6.4) to this case we see that the solutions (6.7) are regular around the singular points $z_\pm = 0$. Their behavior around the singular points $z_\pm = 1 (\Leftrightarrow z_\mp = 0)$ is more complicated. We can study this behavior using the expansion of the solutions (6.7) with respect to the basis of the two linearly-independent local solutions (5.9) around the points $z_\mp = 0$, which are well defined for $\sigma\sigma_m = \pm 1$, $\sigma_b b_\pm = |m| - |s| \geqslant 0$ and $\sigma_c c_\pm = -|m| - |s| < 0$:

$$
\begin{aligned}
{}_sS^\pm_{\omega,E,m}(z_\pm) &= {}_s\Gamma_1^\pm(\omega,\, E,\, m) e^{2\sigma_m a\omega z_\mp} \left(z_\mp\right)^{\frac{-|m|-|s|}{2}} \left(z_\pm\right)^{\frac{|m|-|s|}{2}} \times \\
&\times \underline{\mathrm{HeunC}}\Bigg(4\sigma_m a\omega,\, -|m|-|s|,\, |m|-|s|,\, -4\sigma_m a\omega|s|,\, \frac{m^2+s^2}{2} + 2\sigma_m a\omega|s| - \\
&\quad - a^2\omega^2 - E,\, z_\mp\Bigg) + {}_s\Gamma_2^\pm(\omega,\, E,\, m) e^{2\sigma_m a\omega z_\mp} \left(z_\mp\right)^{\frac{|m|+|s|}{2}} \left(z_\pm\right)^{\frac{|m|-|s|}{2}} \times \\
&\quad \times \mathrm{HeunC}\Bigg(4\sigma_m a\omega,\, |m|+|s|,\, |m|-|s|,\, -4\sigma_m a\omega|s|,\, \frac{m^2+s^2}{2} + \\
&\qquad + 2\sigma_m a\omega|s| - a^2\omega^2 - E,\, z_\mp\Bigg). \qquad (6.8)
\end{aligned}
$$

Now it is clear that in the general case, when ${}_s\Gamma_1^{\pm}(\omega,\, E,\, m) \neq 0$, the solutions (6.7) are singular around the corresponding points $z_{\pm} = 1$ and in addition — infinite valued, because of the poles and of the logarithmic terms in the concomitant confluent Heun function in Eq. (6.8), as well as because of the singular factor $(z_{\mp})^{\frac{-|m|-|s|}{2}}$. One can remove at once all these singularities imposing the condition ${}_s\Gamma_1^{\pm}(\omega,\, E,\, m) = 0$.

Unfortunately, the explicit form of the connection constants ${}_s\Gamma_{1,2}^{\pm}(\omega,\, E,\, m)$ is completely unknown. At present, this is one of the main unsolved problems in the theory of the confluent Heun functions.

Another way to avoid the logarithmic terms in the solutions (6.7), (6.8) is to impose the Δ_{N+1}-condition, reducing this way confluent Heun's functions to polynomials. We consider in detail these two possibilities in the next Secs. 7 and 8.

2. The second class angular δ_N-solutions: $\sigma_b = -\sigma_c$. Now we obtain

$$ {}_sN_{m,\sigma_a,\sigma_b} + 1 = \pm\, m\sigma_b - \sigma_a s \geqslant 1. \tag{6.9}$$

Then

A. For half-integer spin δ_N-solutions are ${}_sS^{\pm}_{\omega,E,m,\sigma_a,\sigma_b,-\sigma_b}(z_{\pm})$ with m restricted in the semi-infinite intervals $\pm m\sigma_b \geqslant \sigma_a s + 1$.

B. For integer spins the additional requirement $\sigma_b(s \mp m) \geqslant 0$ and (6.9) yield the solutions

$$ {}_sS^{\pm}_{\omega,E,m}(z_{\pm}) = e^{\mp 2\sigma a\omega z_{\pm}}\, (z_{\pm})^{\frac{|s|\mp\sigma m}{2}}\, (z_{\mp})^{\frac{-|s|\mp\sigma m}{2}} \times \\ \times \operatorname{HeunC}\Bigg(\mp 4\sigma a\omega,\, |s| \mp \sigma m,\, -|s| \mp \sigma m,\, \pm 4\sigma a\omega|s|,\, \frac{m^2+s^2}{2} \mp 2\sigma_m a\omega|s| - \\ - a^2\omega^2 - E,\, z_{\pm}\Bigg), \tag{6.10}$$

with m restricted in the asymmetric finite intervals $1 - |s| \leqslant \pm\sigma m \leqslant |s|$ ($\Rightarrow |m| \leqslant |s| - (1 \mp \sigma\sigma_m)/2 \leqslant |s|$), which correspond to sectors I (5.5) and IV (5.7). For brevity, here we will not discuss the behavior of the solutions (6.10) around the second regular singular points $z_{\mp}$.

As seen, in the case of the TAE the only role of the δ_N-condition is to relate the degree N of the Δ_{N+1}-condition with the spin-weight s and the azimuthal number m and to select the proper solutions.

Note that up to now only regular solutions to the TAE, which obey the condition (6.2) have been studied and used in the literature [2, 3, 6]. In Sec. 7 we develop a new approach to the regular solutions, based on confluent Heun's functions. The nonregular angular δ_N-solutions, subject to the condition (6.2), and the infinite series of solutions, subject to the condition (6.9), are introduced and considered for the first time in the present work.

7. REGULAR SOLUTIONS OF THE TAE

The spectral condition ${}_s\Gamma_1^{\pm}(\omega,\, E,\, m) = 0$ ensures the regularity of the solutions (6.7). It cannot be used directly, since the explicit form of the connection constant ${}_s\Gamma_1^{\pm}(\omega,\, E,\, m)$ is not known. Therefore, we are forced to use a roundabout way to find the regular solutions to the TAE.

Suppose we have a solution ${}_sS^{+\mathrm{reg}}_{\omega,E,m}(\theta)$ which is regular around the S-pole ($\theta_S = \pi$) and another solution ${}_sS^{-\mathrm{reg}}_{\omega,E,m}(\theta)$ which is regular around the N-pole ($\theta_N = 0$). We will have a solution ${}_sS^{\mathrm{REG}}_{\omega,E,m}(\theta)$, regular everywhere in the interval $\theta \in [0,\, \pi]$, if and only if ${}_sS^{+\mathrm{reg}}_{\omega,E,m}(\theta) = \mathrm{const} \times {}_sS^{-\,\mathrm{reg}}_{\omega,E,m}(\theta)$, i.e., if the Wronskian vanishes: $\mathrm{W}\left[{}_sS^{+\mathrm{reg}}_{\omega,E,m}(\theta),\, {}_sS^{-\mathrm{reg}}_{\omega,E,m}(\theta)\right] = 0$. This condition determines the constant E in the form $E = E(a\omega,\, s,\, m,\, l)$, l being an integer. The Wronskian will vanish for any $\theta \in [0,\, \pi]$, if it is zero for some $\theta_0 \in (0, \pi)$.

To utilize this idea for all values of the parameters s and m, we have to divide the whole plane $\{s,\, m\}$ into four sectors and to choose the solutions ${}_sS^{\pm\mathrm{reg}}_{\omega,E,m}(\theta)$ defined by Eqs. (5.5)–(5.8).

The spectral condition makes equal the solutions of group (a) and the solutions of group (b) in each sector. It can be written in different equivalent forms combining in pairs one solution from the group (a) and another one from the group (b). Below we give the simplest form of this condition in each sector, written here for the first time in terms of confluent Heun's function HeunC and its derivative HeunC′. The set of all conditions (7.1) defines the separation constant in the whole plane $\{s,\, m\}$ in the form $E = (m^2 + s^2)/2 - a^2\omega^2 + \varepsilon(a\omega,\, m,\, s)$. The new parameter $\varepsilon(a\omega,\, m,\, s)$ is to be found from the following transcendental equations:

$$\frac{\mathrm{HeunC}'\left(\pm 4a\omega,\, s+m,\, s-m,\, -4a\omega s,\, +2\omega a s-\varepsilon,\, \left(\sin\dfrac{\theta}{2}\right)^2\right)}{\mathrm{HeunC}\left(\pm 4a\omega,\, s+m,\, s-m,\, -4a\omega s,\, +2\omega a s-\varepsilon,\, \left(\sin\dfrac{\theta}{2}\right)^2\right)} +$$

$$+\frac{\mathrm{HeunC}'\left(\mp 4a\omega,\, s-m,\, s+m,\, +4a\omega s,\, -2a\omega s-\varepsilon,\, \left(\cos\dfrac{\theta}{2}\right)^2\right)}{\mathrm{HeunC}\left(\mp 4a\omega,\, s-m,\, s+m,\, +4a\omega s,\, -2a\omega s-\varepsilon,\, \left(\cos\dfrac{\theta}{2}\right)^2\right)} =$$

$$= 0 \quad \text{— in sector I,} \qquad (7.1\mathrm{a})$$

$$\frac{\mathrm{HeunC}'\left(\pm 4a\omega,\, -s-m,\, s-m,\, -4a\omega s,\, +2\omega a s-\varepsilon,\, \left(\sin\frac{\theta}{2}\right)^2\right)}{\mathrm{HeunC}\left(\pm 4a\omega,\, -s-m,\, s-m,\, -4a\omega s,\, +2\omega a s-\varepsilon,\, \left(\sin\frac{\theta}{2}\right)^2\right)}+$$
$$+\frac{\mathrm{HeunC}'\left(\mp 4a\omega,\, s-m,\, -s-m,\, +4a\omega s,\, -2a\omega s-\varepsilon,\, \left(\cos\frac{\theta}{2}\right)^2\right)}{\mathrm{HeunC}\left(\mp 4a\omega,\, s-m,\, -s-m,\, +4a\omega s,\, -2a\omega s-\varepsilon,\, \left(\cos\frac{\theta}{2}\right)^2\right)}=$$
$$=0 \quad \text{— in sector II,} \tag{7.1b}$$

$$\frac{\mathrm{HeunC}'\left(\pm 4a\omega,\, -s-m,\, -s+m,\, -4a\omega s,\, +2\omega a s-\varepsilon,\, \left(\sin\frac{\theta}{2}\right)^2\right)}{\mathrm{HeunC}\left(\pm 4a\omega,\, -s-m,\, -s+m,\, -4a\omega s,\, +2\omega a s-\varepsilon,\, \left(\sin\frac{\theta}{2}\right)^2\right)}+$$
$$+\frac{\mathrm{HeunC}'\left(\mp 4a\omega,\, -s+m,\, -s-m,\, +4a\omega s,\, -2a\omega s-\varepsilon,\, \left(\cos\frac{\theta}{2}\right)^2\right)}{\mathrm{HeunC}\left(\mp 4a\omega,\, -s+m,\, -s-m,\, +4a\omega s,\, -2a\omega s-\varepsilon,\, \left(\cos\frac{\theta}{2}\right)^2\right)}=$$
$$=0 \quad \text{— in sector III,} \tag{7.1c}$$

$$\frac{\mathrm{HeunC}'\left(\pm 4a\omega,\, s+m,\, -s+m,\, -4a\omega s,\, +2\omega a s-\varepsilon,\, \left(\sin\frac{\theta}{2}\right)^2\right)}{\mathrm{HeunC}\left(\pm 4a\omega,\, s+m,\, -s+m,\, -4a\omega s,\, +2\omega a s-\varepsilon,\, \left(\sin\frac{\theta}{2}\right)^2\right)}+$$
$$+\frac{\mathrm{HeunC}'\left(\mp 4a\omega,\, -s+m,\, s+m,\, +4a\omega s,\, -2a\omega s-\varepsilon,\, \left(\cos\frac{\theta}{2}\right)^2\right)}{\mathrm{HeunC}\left(\mp 4a\omega,\, -s+m,\, s+m,\, +4a\omega s,\, -2a\omega s-\varepsilon,\, \left(\cos\frac{\theta}{2}\right)^2\right)}=$$
$$=0 \quad \text{— in sector IV} \tag{7.1d}$$

valid simultaneously for all values of $\theta \in (0, \pi)$. Thus, the two-singular-points boundary problem for the TAE is solved. It yields a countable set of values $E(a\omega, m, s, l)$ numbered by some integer l. Due to the symmetries (5.4) of the solutions to the TAE, the different relations (7.1a) and (7.1c), or (7.1b) and (7.1d) give similar results. More precisely $E(a\omega, m, -s, l) = E(a\omega, m, s, l)$ and $E(-a\omega, -m, s, l) = E(a\omega, m, s, l)$.

An important consequence is that all the regular solutions obtained this way are angular δ_N-solutions with the same ${}_sN$ (6.2) in sectors II and IV, or with the same ${}_sN_{m,\sigma_a,\sigma_b}$ (6.9) — in sectors I and III. This is because between the solutions (5.6) and (5.8) we certainly have δ_N-solutions: ${}_sS^{\pm}_{\omega,E,m,---}$, for $s > 0$, and ${}_sS^{\pm}_{\omega,E,m,+++}$, for $s < 0$. Between the solutions (5.5) and (5.7) δ_N-solutions are ${}_sS^{-}_{\omega,E,m,\mp\pm\mp}$, for $m > 0$, and ${}_sS^{+}_{\omega,E,m,\mp\pm\mp}$, for $m < 0$. As a result of uniqueness of the regular solutions with given values of the parameters, all regular solutions inherit the δ_N-property. Hence, all regular solutions of the TAE obey the Teukolsky–Starobinsly identities [17].

Let us consider the limit $a\omega \to 0$ of the regular solutions to the TAE. Since

$$\begin{aligned}\mathrm{HeunC}(0, \beta, \gamma, 0, \eta, z) = (1-z)^{\beta+\gamma+1+\sqrt{\beta^2+\gamma^2+1-4\eta}}\times\\ \times\ {}_2F_1\Bigg(\frac{\beta+\gamma+1+\sqrt{\beta^2+\gamma^2+1-4\eta}}{2},\\ \frac{\beta+\gamma+1-\sqrt{\beta^2+\gamma^2+1-4\eta}}{2}; \beta+1; z\Bigg), \quad (7.2)\end{aligned}$$

in this limit the Heun functions in Eqs. (5.5)–(5.8) and (7.1) can be reduced to the Gauss hypergeometric ones. Then, using the well-known properties of the Gauss hypergeometric function ${}_2F_1$ one can derive from Eqs. (7.1) with $a\omega = 0$ the spectrum $E(0, s, l, m) = l(l+1)$, $l = l(s, m, \bar{l}) = \max(|m|, |s|) + \bar{l}$, $\bar{l} = 0, 1, 2, \ldots$ The values of the separation constant $E(0, s, l, m)$ in this case are real. The numerical analysis of Eqs. (7.1) written directly in terms of confluent Heun's functions confirms this standard result for the limit $a\omega = 0$. The corresponding regular confluent Heun functions in ${}_sS^{\mathrm{REG}}_{\omega=0,l,m}(\theta)$ in this case are reduced to Jacobi's polynomials.

The solutions $E(a\omega, s, l, m)$ for small $a\omega$ have been studied many times [2, 6] in the form of Taylor's series expansion $E(a\omega, s, l, m) = l(l+1) + \sum_{j=1}^{\infty} E_{j,s,l,m}(a\omega)^j$ without use of Eqs. (7.1) and without utilizing the Heun functions. A little bit surprising thing is that the solutions ${}_sS^{\mathrm{REG}}_{\omega,l,m}(\theta)$ with $a\omega \neq 0$, regular at both poles, are not polynomial and can be represented as an infinite series with respect to Jacobi's polynomials. Here we describe the regular solutions to the TAE in terms of confluent δ_N-Heun's functions for the first time.

8. POLYNOMIAL SOLUTIONS OF THE TAE

The polynomial solutions to the TAE are a special subclass of the angular δ_N-solutions studied in Sec. 6, since both of the two conditions (7.1) are valid for them. Being a polynomial in z, the HeunC-factor is regular at both regular singular points $\theta = 0, \pi$. Then the singularities of the polynomial solutions around the poles are defined completely by the factors $(z_\pm)^{\sigma_b b_\pm/2}$ and $(z_\mp)^{\sigma_c c_\pm/2}$ in Eq. (5.3). Thus:

A. In the case of half-integer spins $|s| = 1/2,\, 3/2$ from Eq. (6.3) we see that the singularities are defined by the factors $\left(\cos\dfrac{\theta}{2}\right)^{\sigma m - |s|}$ and $\left(\sin\dfrac{\theta}{2}\right)^{-\sigma m - |s|}$. Hence:

For $|s| = 1/2$ we have two singularities on both poles — for $|m| = 0$, or a singularity only on one of the poles — when $|m| > 0$.

For $|s| = 3/2$ we have two singularities on both poles — for $|m| = 0, 1$, or a singularity only on one of the poles — when $|m| > 1$.

B. In the case of integer spins $|s| = 1,\, 2$ from Eq. (6.7) we see that the singularities are defined by the factors $\left(\cos\dfrac{\theta}{2}\right)^{\pm|m| - |s|}$ and $\left(\sin\dfrac{\theta}{2}\right)^{\mp|m| - |s|}$. Hence:

For $|s| = 1$ we have two singularities on both poles — for $|m| = 0$, or a singularity only on one of the poles — when $|m| > 0$.

For $|s| = 2$ we have two singularities on both poles — for $|m| = 0,\, 1$, or a singularity only on one of the poles — when $|m| > 1$.

As a result, we see that in any case the polynomial solutions are singular at least around one of the poles.

Using relations (1.6b) and (5.2) we obtain the general formula for the constant E in the form

$$E^\pm = \mu^\pm - a\omega^2 \mp 2\sigma_a \left(1 \mp \sigma_b m + (\sigma_a + \sigma_b)s\right) a\omega + \\ + \frac{\sigma_b - \sigma_c}{2} m \left(\sigma_b m \mp 1\right) + \frac{\sigma_b + \sigma_c}{2} s \left(\sigma_b s + 1\right). \quad (8.1)$$

Further analysis shows that we have again two classes of polynomial solutions to the TAE, as in the cases of the TRA, but their structure in some cases may be different.

8.1. First Class of Polynomial Solutions to the TAE. These are the solutions ${}_sS^{\pm}_{\omega,E,m,-\sigma,-\sigma,-\sigma}$. For them the condition (6.2) is fulfilled independently of the values of the integer m, but for integer $|s|$ the specific requirement $\sigma_b b_\pm \geqslant 0$ yields the restriction $|m| \geqslant |s|$. As in the case of the first class polynomial solutions to the TRA — Sec. 4, the value $s = 0$ is eliminated by (6.2). Hence, we have an infinite series of the first class polynomial solutions to the TAE for all admissible values of s and m. Preserving the style accepted in the previous sections we denote the polynomial solutions to TAE of the first class as ${}_sS^{\pm}_{\omega,E,m} = {}_sS^{\pm}_{\omega,E,m,-\sigma,-\sigma,-\sigma}$.

For them the Δ_{N+1}-condition reads $\Delta_{2|s|}(\mu) = 0$ and has $2|s|$-in-number solutions ${}_s\mu^{\pm}_{\omega,k,m}$. From formulae (8.1) one obtains

$$ {}_sE^{\pm}_{\omega,k,m} = {}_s\mu^{\pm}_{\omega,k,m} + |s|(|s|-1) - a\omega(a\omega - 2m) \mp 2\sigma(2|s|-1)a\omega, \quad (8.2) $$

where $k = 1, \ldots, 2|s|$, $s = \pm1/2, \pm1, \pm3/2, \pm2$ and for integer $|s|$ in addition $|m| \geqslant |s|$.

Solving the Δ_{N+1}-condition, we obtain for the different values of $|s|$:

$$ {}_sE^{\pm}_{\omega,m} = -a^2\omega^2 + 2a\omega m - \frac{1}{4}: \quad \text{for} \quad |s| = \frac{1}{2}; \quad (8.3) $$

$$ {}_sE^{\pm}_{\omega,k,m} = -a^2\omega^2 + 2a\omega\left(m - (-1)^k\sqrt{1 - m/a\omega}\right): \quad (8.4) $$
$$ \text{for} \quad |s| = 1, \; |m| \geqslant 1, \; k = 1, 2. $$

The values (8.3) and (8.4) of the separation constant E obtained for the first class polynomial solutions to the TAE are the same as the corresponding values (4.2) and (4.3) for the first class polynomial solutions to the TRE. Important consequences of this unexpected fact are considered in a separate article [25].

For the gravitational waves ($|s| = 2$) the quantities ${}_s\mu^{\pm}_{\omega,k,m}$ are solutions of the algebraic equations of the fourth degree $\Delta^{\pm}_4(\mu) = 0$. We do not need here the exact form of these roots. It is quite complicated. Below we present only the form of the separation constant E for the TAE obtained making use of the Taylor series expansions of the roots around the point $a\omega = 0$.

Thus, we obtain for $|s| = 2$ and $k = 1, 2$ the following eight values:

$$ {}_sE^{\pm}_{\omega,k,m} = 2 - 4ma\omega - i(-1)^k 12\sqrt{(m-1)m(m+1)}(a\omega)^{3/2} + $$
$$ + 6\left(m^2 - \frac{7}{6}\right)(a\omega)^2 + \mathcal{O}_{5/2}(a\omega), \quad (8.5) $$

and for $|s| = 2$, $m \neq 0$, $k = 3, 4$ another eight values:

$$ {}_sE^{\pm}_{\omega,k,m} = -(-1)^k 4\sqrt{ma\omega}\left(1 + \left(3m - \frac{2}{m}\right)a\omega + \mathcal{O}_2(a\omega)\right) + $$
$$ + 8ma\omega - 6\left(m^2 - \frac{5}{6}\right)(a\omega)^2 + \mathcal{O}_3(a\omega). \quad (8.6) $$

As seen, for gravitational waves of the first polynomial class the values (8.5) and (8.6) of the corresponding constants E differ substantially from the analogous values (4.4) and (4.5) of the constants E obtained for the TRE in Subsec. 4.1.1. This is in sharp contrast to the case of neutrino waves ($|s| = 1/2$) of the first polynomial class and to the case of electromagnetic waves ($|s| = 1$) of this kind.

It can be shown that this phenomenon reflects the difference between the Starobinsky constants for solutions with spin 2 to the TAE and for solutions with the same spin 2 to the TRE [2, 3, 17]. The solutions to the TAE and to the TRE with the same spin $1/2$ or 1 have the same Starobinsky constants.

Despite the above essential difference, the first class polynomial solutions to the TAE and to the TRE with spin 2 have similar qualitative properties, discussed at the end of Subsec. 4.1.1.

8.2. Second Class of Polynomial Solutions to the TAE. We have a finite number of second class polynomial solutions to the TAE for which the relation $\sigma_c = -\sigma_b$ holds. For brevity, we list here only the ones of integer spin 1 and 2. For them the conditions $N \geqslant 0$ and $\sigma_b b_\pm \geqslant 0$ must be satisfied simultaneously, yielding the requirement $|m| \leqslant |s| - (1 \mp \sigma\sigma_m)/2 \leqslant |s|$ — almost opposite to the analogous requirement $|m| \geqslant |s|$ for the polynomial solutions of the first class. Altogether there exist only the following 24 polynomial solutions of the second class ${}_sS^{\pm}_{\omega,E,m,\mp,\pm,\mp}$ with spin 1 and 2:

$$\begin{aligned}
{}_sS^{+}_{\omega,E,m,-,+,-}:&\quad s=+2,\ m=-1,\,0,\,1,\,2;\quad s=+1,\ m=0,1;\\
{}_sS^{+}_{\omega,E,m,+,-,+}:&\quad s=-2,\ m=-2,\,-1,\,0,\,1;\ s=-1,\ m=-1,\,0;\\
{}_sS^{-}_{\omega,E,m,-,+,-}:&\quad s=+2,\ m=-2,\,-1,\,0,\,1;\ s=+1,\ m=-1,\,0;\\
{}_sS^{-}_{\omega,E,m,+,-,+}:&\quad s=-2,\ m=-1,\,0,\,1,\,2;\quad s=-1,\ m=0,\,1.
\end{aligned} \tag{8.7}$$

The relation between the constants E and ω follows from (8.1), when μ in it is replaced by the solutions of the Δ_{N+1}-condition in the form $\Delta^{\pm}_{|s\pm m|}(\mu) = 0$. Here we omit these relations.

9. THE 256 CLASSES OF EXACT SOLUTIONS TO THE TEUKOLSKY MASTER EQUATION AND THEIR SINGULARITIES

Combining solutions to the TRE and to the TAE studied in the previous sections we can construct the following 256 classes of exact solutions to the TME

$$\begin{aligned}
&{}_s\mathcal{K}^{\pm,\pm}_{\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma,\sigma_a,\sigma_b,\sigma_c}(t,\,r,\,\theta,\,\varphi) =\\
&\qquad = e^{-i\omega t}e^{im\varphi}\,{}_sR^{\pm}_{\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma}(r;\,r_+,\,r_-)\,{}_sS^{\pm}_{\omega,E,m,\sigma_a,\sigma_b,\sigma_c}(\theta).
\end{aligned} \tag{9.1}$$

For specific physical problems one has to impose specific additional conditions, like stability conditions, boundary conditions, casuality conditions, specific fixing of the in-out properties, regularity conditions, etc. Thus, one selects some specific combinations of solutions to the TRE and to the TAE in Eq. (9.1) and derives the spectrum of the separation constants ω and E in the given problem.

For example, choosing solutions to the TRE which enter both the event horizon and the 3D-space infinity we study the Kerr black holes [2, 3]. If in addition we choose regular solution to the TAE, we will obtain the standard QNM of the Kerr black holes. Choosing other solutions to the TRE and/or to the TAE in Eq. (9.1) we may hope to describe other physical objects and phenomena, see, for example, [15, 19].

The solutions (9.1) do not necessarily have a direct physical meaning. Instead, some linear combination of the specific solutions, which obey proper boundary conditions, is to describe the Nature. In general the solutions (9.1) have to be considered as auxiliary mathematical objects — (maybe singular) kernels of integral representations (1.3) of the physical solutions. The choice of the corresponding amplitudes ${}_sA_{\omega,E,m,\sigma_\alpha,\sigma_\beta,\sigma_\gamma,\sigma_a,\sigma_b,\sigma_c}$ will fix completely the physical model and can ensure the convergence of the integrals and discrete sums to physically acceptable solutions. We will study this complicated issue in the next Sec. 10.

10. CONSTRUCTION OF BOUNDED LINEAR COMBINATIONS OF POLYNOMIAL SOLUTIONS TO THE TAE

We have seen in Sec. 8 that the polynomial solutions to the TAE are singular and unbounded with respect to the angle θ around the N- and S-poles. These solutions produce a singular kernel in the integral representation (1.3). It is important to know whether it is possible to have bounded with respect to the angle $\theta \in [0, \pi]$ solutions ${}_s\Psi(t, r, \theta, \varphi)$ defined by Eq. (1.3), despite the singular character of the kernel in it. The answer to this question is a quite nontrivial issue. Here we reach a positive answer for perturbations of spin $1/2$ in several steps.

Let us consider the simplest case of double polynomial solutions of the first class to the TME with spin $1/2$ and $s = \sigma/2$. For them we have an essential simplification, since according to Eqs. (3.2) and (6.2) ${}_sN = 0$. Hence, the HeunC-factors in both the radial and the angular polynomial solutions are equal to $\mathrm{const} \equiv 1$. The value of the separation constant $E = -a^2\omega^2 + 2a\omega m - \dfrac{1}{4}$ is uniquely defined in both cases by Eqs. (4.2) and (8.3). Hence, the integration over the constant E in (1.3) produces only one term with this fixed value. As a result, the corresponding singular kernel (9.1) is*:

$$
{}_{\frac{\sigma}{2}}\mathcal{K}_{\omega,E,m}(t,\,r,\,\theta,\,\varphi) = \delta\left(E + a^2\omega^2 - 2ma\omega + 1/4\right)\Delta^{\frac{1+\sigma}{4}} e^{-i\omega T_\sigma}\,\frac{(W_\sigma)^m}{\sqrt{\sin\theta}}, \tag{10.1}
$$

where $T_\sigma = t + \sigma\,(r_* - ia\cos\theta)$, $W_\sigma = e^{i\phi_\sigma}\cot\dfrac{\theta_\sigma}{2}$, $\phi_\sigma = \varphi + \dfrac{\sigma}{p}\ln\left|\dfrac{r-r_+}{r-r_-}\right|$,

and $\theta_\sigma = \theta$, if $\sigma = +1$, or $\theta_\sigma = \pi - \theta$, if $\sigma = -1$. The complex variable W_σ defines a stereographic projection of the two-sphere $\mathbb{S}^{(2)}_{\phi_\sigma,\theta_\sigma}$ on the compactified complex plane $\tilde{\mathbb{C}}_{W_\sigma}$. Its use is critical for further analysis of the problem.

Taking the trivial integral on the variable E, one obtains from the representation (1.3) and Eq. (10.1)

$$
\begin{aligned}
{}_{\frac{\sigma}{2}}\Psi(t,\,r,\,\theta,\,\varphi) = \Delta(r)^{-\frac{1+\sigma}{4}}\sqrt{\left(|W_\sigma| + |W_\sigma|^{-1}\right)/2}\,\times \\
\times\sum_{m=-\infty}^{\infty}\left(\frac{1}{2\pi}\int\limits_{\mathcal{L}_\omega} d\omega e^{-i\omega T_\sigma}\;{}_{\frac{\sigma}{2}}A_{\omega,m}\right)(W_\sigma)^m .
\end{aligned} \tag{10.2}
$$

Since in this case we have no other restriction on the frequencies ω, different from the stability requirement $\Im(\omega) < 0$, the otherwise arbitrary integration contour $\mathcal{L}_\omega \in \mathbb{C}_\omega$ in (10.2) must lie in the lower complex half-plane. Suppose that the amplitudes ${}_{\frac{\sigma}{2}}A_{\omega,m}$ and the contour $\mathcal{L}_\omega$ are chosen in such way that for all $m \in \mathbb{Z}$ there exist well-defined integrals

$$
\frac{1}{2\pi}\int\limits_{\mathcal{L}_\omega} d\omega e^{-i\omega T_\sigma}\;{}_{\frac{\sigma}{2}}A_{\omega,m} = {}_{\frac{\sigma}{2}}\mathfrak{A}_m(T_\sigma). \tag{10.3}
$$

Then

$$
\begin{aligned}
{}_{\frac{\sigma}{2}}\Psi(t,\,r,\,\theta,\,\varphi) = \Delta(r)^{-\frac{1+\sigma}{4}}\sqrt{\left(|W_\sigma| + |W_\sigma|^{-1}\right)/2}\,\times \\
\times\sum_{m=-\infty}^{\infty}{}_{\frac{\sigma}{2}}\mathfrak{A}_m\left(T_\sigma\right)(W_\sigma)^m .
\end{aligned} \tag{10.4}
$$

*To simplify formula (10.1), we have omitted some constant factors in the corresponding solutions to the TRE and TAE, which do not depend continuously on the *real* variables r and θ, but may have different values outside the event horizon, in the domain between the event horizon and the Cauchy horizon and inside the Cauchy horizon. This is a legal operation, since one can include these factors in the amplitudes ${}_sA_{\dots}$ in the representation (1.3).

Suppose, in addition, that in some ring domain $|W_\sigma| \in (|W|', |W|'')$, $0 < |W|' < |W|'' < \infty$ the sum $\sum_{m=-\infty}^{\infty} {}_{\frac{\sigma}{2}}\mathfrak{A}_m\left(T_\sigma\right)\left(W_\sigma\right)^m = {}_{\frac{\sigma}{2}}\mathfrak{A}\left(T_\sigma,\, W_\sigma\right)$ represents a convergent Laurent series of some analytic function ${}_{\frac{\sigma}{2}}\mathfrak{A}\left(T_\sigma,\, W_\sigma\right)$. For this purpose the coefficients ${}_{\frac{\sigma}{2}}\mathfrak{A}_m\left(T_\sigma\right)$ in Eq. (10.3) for $m > 0$ and, independently, for $m < 0$ must satisfy some of the well-known criteria for convergence of the corresponding series. Thus, we finally obtain a solution to the TME with spin $1/2$ which depends on an arbitrary analytic function ${}_{\frac{\sigma}{2}}\mathfrak{A}\left(T_\sigma,\, W_\sigma\right)$ of the two variables T_σ and W_σ:

$$ {}_{\frac{\sigma}{2}}\Psi(t,\, r,\, \theta,\, \varphi) = \Delta(r)^{-\frac{1+\sigma}{4}} \sqrt{\left(|W_\sigma| + |W_\sigma|^{-1}\right)/2}\, {}_{\frac{\sigma}{2}}\mathfrak{A}\left(T_\sigma,\, W_\sigma\right). \qquad (10.5) $$

Returning to the Boyer–Lindquist variables one can check directly that (10.5) is indeed a general solution to TME with spin $1/2$. The explicit form of the variable T_σ shows that outside the event horizon these solutions describe *one-way-running* waves: outgoing to space infinity running waves — for $\sigma = -1$ and incoming from space infinity running waves — for $\sigma = +1$.

Now it is easy to remove the singularities from the z axis, i.e., on the poles $\theta = 0,\, \pi$. For example, let us choose ${}_{\frac{\sigma}{2}}\mathfrak{A}\left(T_\sigma,\, W_\sigma\right) = 1/\sqrt{\left(W_\sigma + W_\sigma^{-1}\right)/2}$. Then ${}_{\frac{\sigma}{2}}\Psi(t,\, r,\, \theta,\, \varphi) = \Delta(r)^{-\frac{1+\sigma}{4}}/\sqrt{1 - \sin^2\phi_\sigma \sin^2\theta_\sigma}$ has no singularities on the poles $\theta_\sigma = 0,\, \pi$, but this way we have worked out two new singular lines $\phi_\sigma = \varphi + \frac{\sigma}{p}\ln\left|\dfrac{r - r_+}{r - r_-}\right| = \pm\pi/2$ on the equatorial plane $\theta = \pi/2$. Hence, this way the singular line of the solution has been only deformed and translated to a new position. The same happens if we choose the more general function ${}_{\frac{\sigma}{2}}\mathfrak{A}\left(T_\sigma,\, W_\sigma\right) = 1/\sqrt{\left(a(T_\sigma)W_\sigma + b(T_\sigma)W_\sigma^{-1} + c(T_\sigma)\right)/2}$. In this case, the singular z axis will be deformed, translated and doubled to the non-static singular lines $\phi_\sigma = \varphi + \dfrac{\sigma}{p}\ln\left|\dfrac{r - r_+}{r - r_-}\right| = \phi_{1,2} = \arg(W_{1,2})$ on the (in general) moving cones $\theta = \theta_{1,2} = \arctan\left(|W_{1,2}|^{-1}\right)$, where $W_{1,2}$ are the two roots of the equation $a(T_\sigma)W_\sigma + b(T_\sigma)W_\sigma^{-1} + c(T_\sigma) = 0$. Here we have chosen a special form of the function ${}_{\frac{\sigma}{2}}\mathfrak{A}\left(T_\sigma,\, W_\sigma\right)$ which yields finite nonzero values of the solution on the poles $\theta = 0,\, \pi$.

It is possible to chose the function ${}_{\frac{\sigma}{2}}\mathfrak{A}\left(T_\sigma,\, W_\sigma\right)$ with the denominator which is a sum of polynomials of higher degree with respect to variables W_σ and W_σ^{-1}. Then the solution (10.5) equals zero at the N- and S-poles and we can work out an arbitrary number of singular lines of the solution related to the zeros of the denominator. At first glance, this possibility may not seem to be interesting for the physical applications, since on the singular lines the linear perturbation theory in use is not applicable. We mention it here just to have a clear mathematical picture. It is interesting to study the same situation in the whole nonlinear theory and to know whether in it the singular lines may be replaced by regular ones. If so, the perturbation theory under consideration indicates a possible complicated structure of the exact radiation field on the Kerr background.

The most important question for a correct application of the linear perturbation theory under consideration, is whether one can find a regular analytical function ${}_{\frac{\sigma}{2}}\mathfrak{A}\left(T_\sigma,\, W_\sigma\right)$ without singularities in the complex plane $\mathbb{C}_{W_\sigma}/\{0,\,\infty\}$, i.e., with the points $W_\sigma = 0$ and $W_\sigma = \infty$ punctured and which, in addition, can remove the unbounded increase of the solutions due to the singularities of the factor $\sqrt{\left(|W_\sigma| + |W_\sigma|^{-1}\right)/2}$ in (10.5). We give a positive answer to this question constructing two explicit examples:

1. Using the basic equality $\sum\limits_{m=-\infty}^{\infty} W^m I_m(z) = \exp\left(\dfrac{1}{2}\left(W + W^{-1}\right)z\right)$ for the modified Bessel functions $I_m(z)$ [26] we choose the coefficients in (10.4) in the specific form ${}_{\frac{\sigma}{2}}\mathfrak{A}_m\left(T_\sigma\right) = \exp\left(-\dfrac{\bar{\sigma}}{2}\omega^2 T_\sigma^2\right) I_m(\omega T_\sigma)$, where $\omega = \omega_R + i\omega_I$ is a fixed frequency and $\bar{\sigma} = \mathrm{sign}\left(|\omega_R| - |\omega_I|\right)$. Then

$$
\begin{aligned}
{}_{\frac{\sigma}{2}}\Psi_\omega(t,\, r,\, \theta,\, \varphi) = \Delta(r)^{-\frac{1+\sigma}{4}} \sqrt{\left(|W_\sigma| + |W_\sigma|^{-1}\right)/2}\times \\
\times \exp\left(-\frac{\bar{\sigma}}{2}\omega^2 T_\sigma^2\right) \exp\left(\frac{1}{2}\left(W_\sigma + W_\sigma^{-1}\right)\omega T_\sigma\right) \quad (10.6)
\end{aligned}
$$

is a stable solution, since by construction it goes to zero when $t \to +\infty$. It is not difficult to obtain its limit when $\theta_\sigma \to 0,\, \pi$ in the form

$$
\begin{aligned}
\lim_{\theta_\sigma \to 0,\pi}\left({}_{\frac{\sigma}{2}}\Psi_\omega(t,\, r,\, \theta,\, \varphi)\right) = \Delta(r)^{-\frac{1+\sigma}{4}} \exp\left(-\frac{\bar{\sigma}}{2}\omega^2 T_{\sigma;0,\pi}^2\right)\times \\
\times \lim_{\theta_\sigma \to 0,\pi}\left(\frac{1}{\sqrt{\sin\theta}} \exp\left(\frac{|\omega|\sqrt{(t+\sigma r_*)^2 + a^2}}{\sin\theta} e^{i\Upsilon_{\omega\sigma;0,\pi}}\right)\right). \quad (10.7)
\end{aligned}
$$

Here

$$
\begin{aligned}
\Upsilon_{\omega,\sigma;0,\pi} = \pm\left(\varphi + \frac{\sigma}{p}\ln\left|\frac{r-r_+}{r-r_-}\right| - \right. \\
\left. -\sigma\arctan\left(\frac{a}{t+\sigma r_*}\right)\right) + \arg(\omega), \quad \text{for} \quad \theta = 0, \quad \text{or} \quad \pi \quad (10.8)
\end{aligned}
$$

is the limit of the total phase of the term $\dfrac{1}{2}\left(W_\sigma + W_\sigma^{-1}\right)\omega T_\sigma$ and $T_{\sigma;0,\pi} = t + \sigma(r_* \mp ia)$. In Eq. (10.8) the sign $(+)$ corresponds to the limit $\theta_\sigma \to 0$ and the sign $(-)$ — to the limit $\theta_\sigma \to \pi$. Formula (10.7) shows that when $\Upsilon_{\omega,\sigma;0,\pi} \in \left(-\dfrac{\pi}{2},\, \dfrac{\pi}{2}\right)$ the solution ${}_{\frac{\sigma}{2}}\Psi_\omega(t,\, r,\, \theta,\, \varphi)$ is bounded everywhere in the interval $\theta \in [0,\, \pi]$, since in this case $\lim\limits_{\theta_\sigma \to 0,\pi}\left({}_{\frac{\sigma}{2}}\Psi_\omega(t,\, r,\, \theta,\, \varphi)\right) = 0$. Otherwise this limit diverges and the solution is singular and unbounded around the poles.

Actually, the value of the parameter $\Upsilon_{\omega,\sigma;0,\pi}$ is not defined from a geometrical point of view, because the value of the angle φ is completely arbitrary on the poles $\theta_\sigma = 0,\, \pi$. As a result, we can choose any value of the parameter $\Upsilon_{\omega,\sigma;0,\pi}$ without changing the geometrical points associated with the N- and S-poles of the sphere $\mathbb{S}^{(2)}_{\theta,\varphi}$. Since the different values of this parameter yield different solutions of the TAE, we see that under the boundary conditions at hand the corresponding differential operator is not self-adjoint [27], but its self-adjoint extensions do exist and can be fixed by suitable fixing of the free parameter $\Upsilon_{\omega,\sigma;0,\pi}$. An analogous phenomenon is well known for the potentials $V(x) \sim 1/x^2$, or $1/r^2$ in quantum mechanics [27]. Note that around the poles $\theta_\sigma = 0, \pi$ the potential in the TAE (1.1) has precisely the same behavior: ${}_sW_{\omega,E,m}(\theta) \sim 1/\theta^2$ for $\theta \to 0$, and ${}_sW_{\omega,E,m}(\theta) \sim 1/(\theta-\pi)^2$ for $\theta \to \pi$. In our case, the fixing of the parameter $\Upsilon_{\omega,\sigma;0,\pi} \in \left(-\dfrac{\pi}{2},\, \dfrac{\pi}{2}\right)$ makes the solutions (10.7) to the TME for spin $1/2$ smooth and bounded everywhere in the interval $\theta \in [0,\, \pi]$, i.e., physically acceptable.

2. Another solution, which is *finite* everywhere in the interval $\theta \in [0,\, \pi]$ but has an infinite number of bounded oscillations around the poles $\theta = 0,\, \pi$ can be obtained using the following equality for the Bessel functions $J_m(z)$: $\sum\limits_{m=-\infty}^{\infty} (-1)^m W^{2m} \left(J_m(z)\right)^2 = J_0\left(\left(W + W^{-1}\right) z\right)$ [26]. Now we choose the coefficients in (10.4) in the specific form ${}_{\frac{\sigma}{2}}\mathfrak{A}_{2m}\left(T_\sigma\right) = (-1)^m \exp\left(-\dfrac{\bar{\sigma}}{2}\omega^2 T_\sigma^2\right) \times \left(J_{2m}(\omega T_\sigma)\right)^2$ and ${}_{\frac{\sigma}{2}}\mathfrak{A}_{2m+1}\left(T_\sigma\right) = 0$ using the same notation as in the previous example. Then

$$
\begin{aligned}
{}_{\frac{\sigma}{2}}\Psi_\omega(t,\, r,\, \theta,\, \varphi) = \Delta(r)^{-\frac{1+\sigma}{4}} \sqrt{\left(|W_\sigma| + |W_\sigma|^{-1}\right)/2} \times \\
\times \exp\left(-\frac{\bar{\sigma}}{2}\omega^2 T_\sigma^2\right) J_0\left(\left(W_\sigma + W_\sigma^{-1}\right)\omega T_\sigma\right) \quad (10.9)
\end{aligned}
$$

is a stable solution to the TME with spin $1/2$. Taking into account the asymptotic expansion of the Bessel function $J_0(z) \sim \sqrt{\dfrac{2}{\pi z}} \cos\left(z - \pi/4\right)$ we obtain in the limits $\theta_\sigma \to 0,\, \pi$:

$$
\begin{aligned}
\lim_{\theta_\sigma \to 0,\pi} \left({}_{\frac{\sigma}{2}}\Psi_\omega(t,\, r,\, \theta,\, \varphi)\right) = \Delta(r)^{-\frac{1+\sigma}{4}} \exp\left(-\frac{\bar{\sigma}}{2}\omega^2 T_{\sigma;0,\pi}^2\right) \frac{1}{\sqrt{\pi\omega T_{\sigma;0,\pi}}} \times \\
\times \lim_{\theta_\sigma \to 0,\pi} \left(\cos\left(\frac{2|\omega|\sqrt{(t+\sigma r_*)^2 + a^2}}{\sin\theta} e^{i\Upsilon_{\omega\sigma;0,\pi}}\right)\right). \quad (10.10)
\end{aligned}
$$

As seen from Eq. (10.10), there exist only two choices of the free parameter: $\Upsilon_{\omega,\sigma;0,\pi} = 0,\, \pi$, for which the solutions (10.9) are finite everywhere in the interval $\theta \in [0,\, \pi]$ — a critical property for the use of the linear perturbation theory. Approaching these poles the solutions oscillate infinitely many times with bounded finite amplitudes. In this sense, the N- and S-poles remain singularities of the bounded solutions (10.9). Moreover, the gradients of the bounded solutions (10.9) are unbounded around the poles.

Obviously, superpositions of solutions (10.6), or (10.9) with different complex parameters ω, running in some (discrete or continuous) sets in $\mathbb{C}_\omega$, describe more general bounded solutions to the TAE with spin $1/2$.

One more remark. In the case $\sigma = +1$ the solutions (10.5) are unbounded on the horizons $r_\pm$ due to the factor $\Delta(r)^{-1/2}$. These *stationary* singularities cannot be removed by any choice of the function ${}_{\frac{\sigma}{2}}\mathfrak{A}\,(T_\sigma,\, W_\sigma)$, since it depends on the two variables T_σ and W_σ, not on the single one r. The variables t, r, θ enter in T_σ and the variables φ, r, θ enter in W_σ in a complex way. As a result, the variable r cannot be disentangled from the the function ${}_{\frac{\sigma}{2}}\mathfrak{A}\,(T_\sigma,\, W_\sigma)$ and one is not able to compensate the singularity due to the factor $\Delta(r)^{-1/2}$ which does not depend neither on the time t, nor on the angles φ and θ.

11. CONCLUSION

In the present work, we have demonstrated that the confluent Heun functions are an adequate and natural tool for a unified description of the linear perturbations to the gravitational field of the Kerr metric outside the horizons, as well as in the interior domains. These functions give us an effective tool for exact mathematical treatment of different boundary problems and corresponding physical phenomena. The same approach works, too, for the Schwarzschild metric [14, 17].

Large classes of exact solutions to the perturbation equations of the Kerr metric were described here for the first time. All possible types of solutions were classified uniformly in terms of confluent Heun's functions and confluent Heun's polynomials, using their specific properties. As we saw, the variety of the different solutions and possible spectra is much reacher than, for example, the variety of the corresponding solutions and spectra of the Hydrogen problem in quantum mechanics [27], solved in terms of the Gauss hypergeometric functions.

We have to stress especially the newly obtained singular polynomial solutions to the Teukolsky angular equation. These solutions can describe in the most natural way the collimation of radiated fields of all spins in the Kerr metric, using the perturbation theory. For spin $1/2$ we have proved that the singular kernels, constructed from polynomial solutions, can produce bounded solutions of the TME with very interesting physical properties: These solutions are able to describe correctly collimated one-way-running waves.

For spin 1 one can reach similar results in a more complicated way, since there we meet a new physical phenomenon — the electromagnetic superradiance [2, 3, 28]. For spin 2 we have no continuous spectrum of the TME and the problem needs a special treatment, too. We shall consider these important cases separately.

The solutions of some of the remaining basic mathematical problems, as well as some preliminary attempts for new specific physical applications of the obtained results can be found in [15–17, 19, 25].

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