E2-2013-78

E. Ivanov[1], S. Sidorov[2]

DEFORMED SUPERSYMMETRIC MECHANICS

[1]E-mail: eivanov@theor.jinr.ru
[2]E-mail: sidorovstepan88@gmail.com

Иванов Е., Сидоров С. E2-2013-78
Деформация суперсимметричной механики

Построен новый тип $\mathcal{N} = 4$, $d = 1$ суперсимметричных систем с использованием суперполей на фактор-пространствах $SU(2|1)$. Соответствующие модели обладают мировой суперсимметрией, где стандартная $\mathcal{N} = 4, d = 1$ суперсимметрия модифицирована (деформирована) параметром m. В качестве примеров рассмотрены классические и квантовые модели для супермультиплетов (**1,4,3**) и (**2,4,2**) и установлена их взаимосвязь с некоторыми работами по нестандартным $d = 1$ суперсимметриям. В частности, $d = 1$ система со «слабой суперсимметрией» естественно воспроизводится из нового суперполевого подхода как подкласс (**1,4,3**) моделей. Суперполевые модели, определенные на косетах супергруппы $SU(2|2)$, соответствуют обобщению до $\mathcal{N} = 8, d = 1$ суперсимметрии.

Работа выполнена в Лаборатории теоретической физики им. Н. Н. Боголюбова ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна, 2013

Ivanov E., Sidorov S. E2-2013-78
Deformed Supersymmetric Mechanics

Motivated by a recent interest in curved rigid supersymmetries, we construct a new type of $\mathcal{N} = 4$, $d = 1$ supersymmetric systems by employing superfields defined on the cosets of the supergroup $SU(2|1)$. The relevant worldline supersymmetry is a deformation of the standard $\mathcal{N} = 4, d = 1$ supersymmetry by a mass parameter m. As instructive examples we consider at the classical and quantum levels the models associated with the supermultiplets (**1,4,3**) and (**2,4,2**) and find out interesting interrelations with some previous works on nonstandard $d = 1$ supersymmetry. In particular, the $d = 1$ systems with «weak supersymmetry» are naturally reproduced within our $SU(2|1)$ superfield approach as a subclass of the (**1,4,3**) models. A generalization to the $\mathcal{N} = 8, d = 1$ case implies the supergroup $SU(2|2)$ as a candidate deformed worldline supersymmetry.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna, 2013

1. INTRODUCTION

Recently, there was an outburst of interest in rigid supersymmetric theories in diverse dimensions, such that the relevant supersymmetry groups include, as the bosonic subgroups, the groups of motion of some curved spaces [1]. This should be contrasted with the standard rigid supersymmetric theories in which the bosonic invariance subgroup is the Poincaré group, the group of motion of the flat Minkowski space. There is the hope that the study of the new class of theories will give rise to a further progress in understanding the generic gauge/gravity correspondence.

The simplest Poincaré supergroup is the $d = 1$ one,

$$\{Q^m, Q^n\} = 2\delta^{mn} H, \quad [H, Q^m] = 0, \quad m = 1, \dots \mathcal{N}, \tag{1.1}$$

where Q^m are $\mathcal{N}$ real supercharges and H is the time-translation generator. The associated systems are models of supersymmetric quantum mechanics (SQM) [2], with H being the relevant Hamiltonian. The SQM models, including their versions with extended $\mathcal{N} > 2$ supersymmetry, have a lot of applications in various physical and mathematical domains. It is tempting to construct SQM models based on some curved versions of the $d = 1$ Poincaré supersymmetry. They could be considered as the $d = 1$ analogs of the higher-dimensional supersymmetric models just mentioned and, in some cases, follow from the latter via dimensional reduction.

One possible way to define such generalized SQM models is suggested by the simplest nontrivial $d = 1$ Poincaré superalgebra, the $\mathcal{N} = 2$ one. Introducing complex generators

$$Q = \frac{1}{\sqrt{2}}(Q^1 + iQ^2)\,, \quad \bar{Q} = \frac{1}{\sqrt{2}}(Q^1 - iQ^2),$$

the superalgebra (1.1) for $\mathcal{N} = 2$ can be rewritten as

$$\{Q, \bar{Q}\} = 2H, \quad Q^2 = \bar{Q}^2 = 0, \quad [H, Q] = [H, \bar{Q}] = 0. \tag{1.2}$$

It is instructive to add the commutators with the generator J of the group $O(2) \sim U(1)$ which is the automorphism group of (1.1) for $\mathcal{N} = 2$:

$$[J, Q] = Q, \quad [J, \bar{Q}] = -\bar{Q}, \quad [H, J] = 0. \tag{1.3}$$

On the one hand, the relations (1.2) and (1.3) define the $\mathcal{N} = 2,\ d = 1$ Poincaré superalgebra. On the other hand (and this fact is less known), these relations are recognized as defining the superalgebra $u(1|1)$, with H being the relevant central charge generator. After factoring out the $U(1)$ generator J, we are left with the superalgebra $su(1|1)$ (1.2).

This twofold interpretation of $\mathcal{N} = 2,\ d = 1$ Poincaré superalgebra suggests two ways of extending it to higher-rank $d = 1$ supersymmetries.

The first one is the straightforward extension

$$(\mathcal{N} = 2,\ d = 1) \Rightarrow (\mathcal{N} > 2,\ d = 1 \text{ Poincaré}), \tag{1.4}$$

where the general $\mathcal{N}, d = 1$ Poincaré superalgebra is defined by the relations (1.1). Except for $\mathcal{N} = 2$, these algebras cannot be identified with any simple or semi-simple superalgebras (though can still be recovered through contractions and/or truncations of such superalgebras). The possible extra bosonic generators are those of the automorphism group (it is $O(\mathcal{N})$ in the general case) and/or central (or «semi-central») charge generators which commute with the supercharges.

Another, less evident opportunity corresponds to the following chain of embeddings:

$$(\mathcal{N} = 2,\ d = 1) \equiv u(1|1)\ \subset\ su(2|1)\ \subset\ su(2|2)\ \subset\ \ldots \tag{1.5}$$

The characteristic feature of this sort of extensions is that the relevant superalgebras necessarily contain, besides an analog of the Hamiltonian H, also additional bosonic generators which form some internal symmetry subgroups commuting with the «would-be Hamiltonian». They appear in the closure of the supercharges, and *do not commute* with the latter (as opposed, e.g., to the central charges in the super Poincaré algebras). Though the chain (1.5) is certainly nonunique, in the sense that one could imagine some other extensions of $u(1|1)$ among its links, the superalgebras written down in (1.5) are distinguished in that they seem to be minimal deformations of the $\mathcal{N} = 4$ and $\mathcal{N} = 8$ one-dimensional Poincaré superalgebras: they go over into the latter, when taking the contraction limit in some dimensionful parameter.

The supergroup $SU(2|1)$ as the alternative of the standard worldline $\mathcal{N} = 4, d = 1$ supersymmetry in SQM models already appeared in literature under the name «weak supersymmetry» [3] (see also [4,5]), though no explicit identification of the latter with $SU(2|1)$ was made and no systematic methods of constructing such new SQM models were given. The basic aim of the present paper is to develop such methods, which would be applicable not only to $SU(2|1)$, but also to the case of the supergroup $SU(2|2)$ and, hopefully, to other interesting examples of this type.

We construct the worldline superfield approach to $SU(2|1)$ and demonstrate that most of the off-shell multiplets of $\mathcal{N} = 4, d = 1$ supersymmetry have the well-defined $SU(2|1)$ analogs. In particular, the models considered in [3] are based on the $SU(2|1)$ multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ and we reproduce these models from our superfield approach. Some peculiarities of their quantum spectra find a natural explanation in the framework of the $SU(2|1)$ representation theory [6], based on the property that the relevant Casimir operators have a notable expression in terms of the Hamiltonian. This supergroup has also invariant chiral subspaces which are natural carriers of the chiral superfields encompassing off-shell multiplets $(\mathbf{2},\mathbf{4},\mathbf{2})$, for which we also construct general superfield and component actions. An interesting new feature of these actions is the inevitable presence of the bosonic $d=1$ Wess–Zumino terms of the first order in time derivative, parallel with the standard second-order kinetic terms. We also show that $SU(2|1)$ admits a supercoset which is an analog of the harmonic analytic superspace of the standard $\mathcal{N}=4, d=1$ supersymmetry [7]. This means that one can define $SU(2|1)$ analogs of the «root» $\mathcal{N}=4$ multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ and of the multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ by embedding them into the appropriate harmonic analytic superfields. Detailed analysis of these and related issues (including, e.g., the appropriate generalization of the $d=1$ superfield gauging procedure [8]) will be given elsewhere.

The paper is organized as follows. The $SU(2|1)$ superspace is constructed in Sec. 2. The study of the $SU(2|1)$ SQM models based on the multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ is performed in Secs. 3 and 4. The similar study for the $(\mathbf{2},\mathbf{4},\mathbf{2})$ multiplets is the subject of Secs. 5 and 6. The summary and some problems for the future analysis are the contents of Sec. 7. In Appendix, some technical details are collected.

2. $SU(2|1)$ SUPERSPACE

2.1. The $su(2|1)$ Algebra. We start with the following form of the (central-extended) superalgebra $su(2|1)$:

$$\begin{aligned}
&\{Q^i,\bar{Q}_j\} = 2m\left(I^i_j - \delta^i_j F\right) + 2\delta^i_j H, \qquad \left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j,\\
&\left[I^i_j,\bar{Q}_l\right] = \frac{1}{2}\delta^i_j\bar{Q}_l - \delta^i_l\bar{Q}_j, \qquad \left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\delta^i_j Q^k,\\
&\left[F,\bar{Q}_l\right] = -\frac{1}{2}\bar{Q}_l\,, \qquad \left[F, Q^k\right] = \frac{1}{2}Q^k.
\end{aligned} \tag{2.1}$$

All other (anti)commutators are vanishing.

The dimensionless generators I^i_j and F generate $U(2)$ symmetry, while the mass-dimension generator H commutes with everything and so can be interpreted as the central charge generator. In the quantum-mechanical realization of $SU(2|1)$ we will be interested in, this generator becomes just the canonical Hamiltonian, while in the superspace realization it is interpreted as the time-translation generator. The mass parameter m is arbitrary and it is introduced in order to separate the generator H from the internal symmetry generator F which possesses non-trivial commutation relations with the fermionic generators. It can be treated as the contraction parameter: sending $m \to 0$ leads to the standard $\mathcal{N}=4$, $d=1$ Poincaré superalgebra. In the limit $m=0$, the generators I^i_j and F become the $U(2)$ automorphism generators of this $\mathcal{N}=4$, $d=1$ superalgebra. It is worth mentioning that the full automorphism group of the flat $\mathcal{N}=4$, $d=1$ is $SO(4) \sim SU(2) \times SU(2)$ and, after reduction, F is recognized as belonging just to the second $SU(2)$ factor in this product. At $m \neq 0$, only the generator F is present, as the only counterpart of the second automorphism $SU(2)$ of the $m=0$ case.

Note that the substitutions

$$m \to -m,\ Q^i \to \bar{Q}^i,\ \bar{Q}_j \to -Q_j,\ I^i_j \to \varepsilon_{jk}\varepsilon^{il} I^k_l,\ H \to H,\ F \to -F \tag{2.2}$$

leave the superalgebra (2.1) intact, i.e., they constitute an automorphism of (2.1). This means that the cases of positive and negative m are in fact equivalent, and so in what follows, we can limit our consideration to $m > 0$.

2.2. Coset Superspace. The supergroup $SU(2|1)$ can be realized by left shifts on a few coset supermanifolds. The supercosets which have appeared so far in diverse variants of the super Landau problem with $SU(2|1)$ as the target space supersymmetry include $SU(2|1)/U(1|1)$ ($(\mathbf{2}|\mathbf{2})$ dimensional supersphere, with the sphere S^2 as the bosonic submanifold) [9], $SU(2|1)/[U(1) \times U(1)]$ ($(\mathbf{2}|\mathbf{4})$ dimensional superflag, again with S^2 as the bosonic submanifold) [10] and $SU(2|1)/U(2)$ (purely odd coset of the dimension $(\mathbf{0}|\mathbf{4})$) [11,12]. One could also consider, e.g., the supercoset $SU(2|1)/U(1)$ with S^3 as the bosonic submanifold and the full $SU(2|1)$ group manifold as a superextension of $S^1 \times S^3$ (or $R^1 \times S^3$). In all these realizations the coset parameters are regarded as some worldline fields, in accordance with the treatment of $SU(2|1)$ as a nonlinearly realized internal supersymmetry. The relevant Hamiltonians are purely external: they commute with all $SU(2|1)$ generators, but never come out in the closure of the latter.

Here we will be interested in the $SU(2|1)$ coset of the entirely different type. It is a direct analog of the standard $\mathcal{N}=4$, $d=1$ superspace [7, 13], with the coset parameters being identified with the coordinates, not with the fields. The fields will finally appear as the components of the appropriate superfields given on this supercoset. The splitting of the $U(2)$ singlet generator in (2.1) into the H and F parts plays the crucial role for the possibility to define such a coset supermanifold in the self-consistent way. We place the $U(2)$ generators into the stability subgroup and are left with H, Q_i and $\bar{Q}^i$ as the coset generators

$$\frac{SU(2|1)}{SU(2) \times U(1)} \sim \frac{\{Q^i, \bar{Q}_j, H, I^i_j, F\}}{\{I^i_j, F\}}. \tag{2.3}$$

The corresponding superspace coordinates $\{t, \theta_i, \bar{\theta}^j\}$ are then identified with the parameters associated with these coset generators. An element of this supercoset can be conveniently parametrized as

$$g = \exp\Big(itH + i\tilde{\theta}_i Q^i - i\tilde{\bar{\theta}}^j \bar{Q}_j\Big), \tag{2.4}$$

where*

$$\tilde{\theta}_i = \left[1 - \frac{2m}{3}\left(\bar{\theta}\cdot\theta\right)\right]\theta_i. \tag{2.5}$$

2.3. Cartan Forms. Prior to giving how $SU(2|1)$ is realized on the superspace coordinates, it is convenient to calculate the left-covariant Cartan one-forms. They are defined by the standard relation

$$\Omega := g^{-1}dg = e^{-B}d\,e^{B} + i\,dt\,H = i\Delta\theta_i Q^i - i\Delta\bar{\theta}^j\bar{Q}_j + i\Delta h_i^j\, I_j^i + i\Delta\hat{h}\,F + i\Delta t\,H\,, \tag{2.6}$$

where

$$B := i\left(\tilde{\theta}_i Q^i - \tilde{\bar{\theta}}^j\bar{Q}_j\right). \tag{2.7}$$

Using the nilpotency property of the fermionic coordinates and the (anti)commutation relations (2.1), it is straightforward to find the explicit expressions for the Cartan forms:

$$\begin{aligned}
\Delta\theta_i &= d\theta_i + m\left(d\theta_l\bar{\theta}^l\theta_i - d\theta_i\bar{\theta}^k\theta_k\right) + \frac{m^2}{4}d\theta_i\left(\bar{\theta}\cdot\theta\right)^2,\\
\Delta\bar{\theta}^j &= d\bar{\theta}^j - m\left(d\bar{\theta}^l\theta_l\bar{\theta}^j - d\bar{\theta}^j\theta_k\bar{\theta}^k\right) + \frac{m^2}{4}d\bar{\theta}^j\left(\bar{\theta}\cdot\theta\right)^2,\\
\Delta t &= dt + i\left(d\theta_i\bar{\theta}^i + d\bar{\theta}^i\theta_i\right)\left[1 - 2m\left(\bar{\theta}\cdot\theta\right)\right],\\
\Delta\hat{h} &= -im\left(d\theta_i\bar{\theta}^i + d\bar{\theta}^i\theta_i\right)\left[1 - 2m\left(\bar{\theta}\cdot\theta\right)\right],\\
\Delta h_i^j &= im\left(d\theta_i\bar{\theta}^j + d\bar{\theta}^j\theta_i - \frac{\delta_i^j}{2}\left(d\theta_l\bar{\theta}^l + d\bar{\theta}^l\theta_l\right)\right)\left(1 - \frac{3m}{2}\left(\bar{\theta}\cdot\theta\right)\right)\\
&\quad - \frac{im^2}{2}\left(d\theta_l\bar{\theta}^l + d\bar{\theta}^l\theta_l\right)\left(\bar{\theta}^j\theta_i - \frac{\delta_i^j}{2}\left(\bar{\theta}\cdot\theta\right)\right).
\end{aligned} \tag{2.8}$$

2.4. Transformation Properties. The transformation properties of the superspace coordinates under the left shifts with the parameters ϵ_i and $\bar{\epsilon}^j$, as well as the induced infinitesimal transformations belonging to the stability subgroup (I_j^i, F), can be found from the general formula

$$\left(1 + i\epsilon_i Q^i - i\bar{\epsilon}^i\bar{Q}_i\right)g = g'\,h\,, \tag{2.9}$$

*We use the convention $\left(\bar{\theta}\cdot\theta\right) = \bar{\theta}^k\theta_k$.

where

$$h = 1 + \left(i\delta h_i^j \, I_j^i + i\delta\hat{h} \, F \right). \tag{2.10}$$

Equations (2.9) and (2.10) are equivalent to the relation

$$g^{-1} \left(i\epsilon_i Q^i - i\bar{\epsilon}^i \bar{Q}_i \right) g = g^{-1}\delta g + i\delta h_i^j \, I_j^i + i\delta\hat{h} \, F. \tag{2.11}$$

Taking into account that $g^{-1}\delta g$ is given by the same formulas (2.6)–(2.8) with δ in place of d, it is easy to find the basic ϵ transformations of the superspace coordinates

$$\begin{aligned} &\delta\theta_i = \epsilon_i + 2m\,(\bar{\epsilon}\cdot\theta)\,\theta_i\,, \qquad \delta\bar{\theta}^j = \bar{\epsilon}^i - 2m\left(\epsilon\cdot\bar{\theta}\right)\bar{\theta}^i, \\ &\delta t = i\left[\left(\epsilon\cdot\bar{\theta}\right) + (\bar{\epsilon}\cdot\theta)\right] \end{aligned} \tag{2.12}$$

and the induced $u(2)$ elements

$$\begin{aligned} \delta\hat{h} &= -im\left[\left(\epsilon\cdot\bar{\theta}\right) + (\bar{\epsilon}\cdot\theta)\right], \\ \delta h_i^j &= im\left(\epsilon_i\bar{\theta}^j + \bar{\epsilon}^j\theta_i - \frac{\delta_i^j}{2}\left[\left(\epsilon\cdot\bar{\theta}\right) + (\bar{\epsilon}\cdot\theta)\right]\right)\left(1 - \frac{m}{2}\left(\bar{\theta}\cdot\theta\right)\right) \\ &\quad - \frac{3im^2}{2}\left[\left(\epsilon\cdot\bar{\theta}\right) + (\bar{\epsilon}\cdot\theta)\right]\left(\bar{\theta}^j\theta_i - \frac{\delta_i^j}{2}\left(\bar{\theta}\cdot\theta\right)\right). \end{aligned} \tag{2.13}$$

The integration measure defined as

$$\mu := dtd^2\theta\, d^2\bar{\theta}\left[1 + 2m\left(\bar{\theta}\cdot\theta\right)\right] \tag{2.14}$$

is invariant under these transformations, $\delta\mu = 0$.

From the general transformation law of the Cartan form Ω,

$$\Omega\,' = h\Omega h^{-1} - dh\, h^{-1}, \tag{2.15}$$

we find its infinitesimal transformation

$$\delta\Omega = \left[\left(i\delta h_i^j \, I_j^i + i\delta\hat{h} \, F\right), \Omega\right] - dh\, h^{-1}. \tag{2.16}$$

Thus all the component Cartan forms, except those belonging to the stability subalgebra (I_k^i, F), transform homogeneously in $SU(2|1)$.

The remaining $SU(2|1)$ transformations of the superspace coordinates are contained in the closure of the ϵ and $\bar{\epsilon}$ transformations. They can easily be found by computing the Lie brackets of (2.12).

Having found the superspace realization of the ϵ transformations, we can define the corresponding generators as the appropriate differential operators:

$$\left(\delta\theta_i, \delta\bar{\theta}^i, \delta t\right) = i\left[\epsilon_i Q^i - \bar{\epsilon}^j \bar{Q}_j, \left(\theta_i, \bar{\theta}^i, t\right)\right], \tag{2.17}$$

whence

$$Q^i = -i\frac{\partial}{\partial\theta_i} + 2im\bar\theta^i\bar\theta^j\frac{\partial}{\partial\bar\theta^j} + \bar\theta^i\frac{\partial}{\partial t}\,,$$
$$\bar Q_j = i\frac{\partial}{\partial\bar\theta^j} + 2im\theta_j\theta_k\frac{\partial}{\partial\theta_k} - \theta_j\frac{\partial}{\partial t}. \tag{2.18}$$

Their anticommutators yield the superspace realization of the bosonic generators I^i_j, F, H:

$$I^i_j = \left(\bar\theta^i\frac{\partial}{\partial\bar\theta^j} - \theta_j\frac{\partial}{\partial\theta_i}\right) - \frac{\delta^i_j}{2}\left(\bar\theta^k\frac{\partial}{\partial\bar\theta^k} - \theta_k\frac{\partial}{\partial\theta_k}\right),$$
$$H = i\partial_t, \qquad F = \frac{1}{2}\left(\bar\theta^k\frac{\partial}{\partial\bar\theta^k} - \theta_k\frac{\partial}{\partial\theta_k}\right). \tag{2.19}$$

It is a direct exercise to check that the operators (2.18) and (2.19) indeed form the $su(2|1)$ algebra (2.1) with respect to (anti)commutation.

Note that the same differential operators (taken with the minus sign) realize $SU(2|1)$ on the superfields having no external $U(2)$ indices, i.e., on the $U(2)$ scalar superfields. To construct the realization of $SU(2|1)$ on the superfields forming nontrivial $U(2)$ multiplets, one should extend (2.18) and (2.19) by the matrix parts $\delta h^j_i\tilde I^i_j$ and $\delta\hat h\tilde F$, with the parameters ϵ_i, $\bar\epsilon^i$ being separated. Here, $\tilde I^i_j$ and $\tilde F$ are matrix generators of the $U(2)$ representation by which the given superfield is rotated with respect to its external indices.

2.5. Covariant Derivatives. The covariant derivatives of some superfield $\Phi_B(t,\theta_i,\bar\theta^j)$, where B is the index of some $U(2)$ representation, can be found from the general expression for its covariant differential

$$\mathcal{D}\Phi_A := d\Phi_A + \left[i\Delta h^j_i\,\tilde I^i_j + i\Delta\hat h\,\tilde F\right]_A^B\Phi_B \equiv$$
$$\equiv \left[\Delta\theta_i\mathcal{D}^i - \Delta\bar\theta^j\bar{\mathcal{D}}_j + \Delta t\,\mathcal{D}_t\right]\Phi_A. \tag{2.20}$$

The covariant derivatives $\mathcal{D}^i, \bar{\mathcal{D}}_j, \mathcal{D}_H$ are read off from this definition as

$$\begin{aligned}
\mathcal{D}^i &= \left[1 + m\left(\bar\theta\cdot\theta\right) - \frac{3m^2}{4}\left(\bar\theta\cdot\theta\right)^2\right]\frac{\partial}{\partial\theta_i} - m\bar\theta^i\theta_j\frac{\partial}{\partial\theta_j} - i\bar\theta^i\frac{\partial}{\partial t}\\
&\quad + m\bar\theta^i\tilde F - m\bar\theta^j\tilde I^i_j + \frac{m^2}{2}\left(\bar\theta\cdot\theta\right)\bar\theta^j\tilde I^i_j - \frac{m^2}{2}\bar\theta^i\bar\theta^j\theta_k\tilde I^k_j,\\
\bar{\mathcal{D}}_j &= -\left[1 + m\left(\bar\theta\cdot\theta\right) - \frac{3m^2}{4}\left(\bar\theta\cdot\theta\right)^2\right]\frac{\partial}{\partial\bar\theta^j} + m\bar\theta^k\theta_j\frac{\partial}{\partial\bar\theta^k} + i\theta_j\frac{\partial}{\partial t}\\
&\quad - m\theta_j\tilde F + m\theta_k\tilde I^k_j - \frac{m^2}{2}\left(\bar\theta\cdot\theta\right)\theta_k\tilde I^k_j + \frac{m^2}{2}\theta_j\bar\theta^l\theta_k\tilde I^k_l,\\
\mathcal{D}_t &= \partial_t.
\end{aligned} \tag{2.21}$$

They form the algebra which mimics $su(2|1)$ *:

$$\{\mathcal{D}^i, \bar{\mathcal{D}}_j\} = 2m\left(\tilde{I}^i_j - \delta^i_j \tilde{F}\right) + 2i\delta^i_j \mathcal{D}_t, \qquad \left[\tilde{I}^i_j, \tilde{I}^k_l\right] = \delta^i_l \tilde{I}^k_j - \delta^k_j \tilde{I}^i_l,$$
$$\left[\tilde{I}^i_j, \bar{\mathcal{D}}_l\right] = \delta^i_l \bar{\mathcal{D}}_j - \frac{1}{2}\delta^i_j \bar{\mathcal{D}}_l, \qquad \left[\tilde{I}^i_j, \mathcal{D}^k\right] = \frac{1}{2}\delta^i_j \mathcal{D}^k - \delta^k_j \mathcal{D}^i,$$
$$\left[\tilde{F}, \bar{\mathcal{D}}_l\right] = \frac{1}{2}\bar{\mathcal{D}}_l\,, \qquad \left[\tilde{F}, \mathcal{D}^k\right] = -\frac{1}{2}\mathcal{D}^k. \tag{2.22}$$

3. THE MULTIPLET (1,4,3)

3.1. Constraints. Now we are ready to define the properly constrained $SU(2|1)$ superfields encompassing the appropriate analogs of the irreducible off-shell multiplets of the standard $\mathcal{N} = 4, d = 1$ supersymmetry. As the first example, we consider an analog of the multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ [14, 15].

This multiplet is described by the real neutral superfield G satisfying the $SU(2|1)$ covariantization of the standard $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplet constraints

$$\varepsilon^{lj}\bar{\mathcal{D}}_l\,\bar{\mathcal{D}}_j G = \varepsilon_{lj}\mathcal{D}^l\,\mathcal{D}^j G = 0. \tag{3.1}$$

They are solved by

$$\begin{aligned} G \;\; = \;\; & x - mx\left(\bar{\theta}\cdot\theta\right)\left[1 - 2m\left(\bar{\theta}\cdot\theta\right)\right] + \frac{\ddot{x}}{2}\left(\bar{\theta}\cdot\theta\right)^2 - i\left(\bar{\theta}\cdot\theta\right)\left(\theta_i\,\dot{\psi}^i + \bar{\theta}^j\,\dot{\bar{\psi}}_j\right) \\ & + \left[1 - 2m\left(\bar{\theta}\cdot\theta\right)\right]\left(\theta_i\psi^i - \bar{\theta}^j\,\bar{\psi}_j\right) + \bar{\theta}^j\theta_i\,B^i_j, \quad B^k_k = 0. \end{aligned} \tag{3.2}$$

We see that the irreducible set of the off-shell component fields is $x(t)$, $\psi^i(t)$, $\bar{\psi}_i(t)$, $B^i_j(t)(B^k_k = 0)$, i.e., G reveals just the $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ content. In the contraction limit $m = 0$, it is reduced to the ordinary $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ superfield.

The ϵ transformation law of G,

$$\delta G = -i\left[\epsilon_i Q^i - \bar{\epsilon}^j\bar{Q}_j, G\right], \tag{3.3}$$

implies the following transformation laws for the component fields:

$$\delta x = \left(\bar{\epsilon}\cdot\bar{\psi}\right) - \left(\epsilon\cdot\psi\right), \qquad \delta\psi^i = i\bar{\epsilon}^i\dot{x} - m\bar{\epsilon}^i x + \bar{\epsilon}^k B^i_k,$$
$$\begin{aligned} \delta B^i_j \;\; = \;\; & -2i\left(\epsilon_j\dot{\psi}^i + \bar{\epsilon}^i\dot{\bar{\psi}}_j - \frac{\delta^i_j}{2}\left[\epsilon_k\dot{\psi}^k + \bar{\epsilon}^k\dot{\bar{\psi}}_k\right]\right) + \\ + \;\; & 2m\left(\bar{\epsilon}^i\bar{\psi}_j - \epsilon_j\psi^i + \frac{\delta^i_j}{2}\left[\bar{\epsilon}^k\bar{\psi}_k - \epsilon_k\psi^k\right]\right). \end{aligned} \tag{3.4}$$

*When computing the anticommutator of the covariant spinor derivatives, it is assumed that the $U(2)$ matrix parts of the spinor derivative standing on the left properly act also on the doublet index of the derivative on the right.

3.2. Invariant Lagrangian. Using the definition (2.14), we can construct the general Lagrangian and action for the $SU(2|1)$ multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ as

$$\mathcal{L} = -\int d^2\theta\, d^2\bar{\theta}\left[1 + 2m\left(\bar{\theta}\cdot\theta\right)\right] f(G), \quad S = \int dt\mathcal{L}. \tag{3.5}$$

Note that the superfield Lagrangian density in (3.5) is defined up to the shift

$$f \to f + c_1\, G + c_0, \tag{3.6}$$

where c_0, c_1 are arbitrary real constants. It follows from (3.2) that after integrating over θ, $\bar{\theta}$, these additional terms yield total derivatives and so they do not contribute to the full action S.

Doing the Berezin integral in (3.5), we obtain the component off-shell Lagrangian

$$\begin{aligned}\mathcal{L} = \dot{x}^2 g(x) + i\left(\bar{\psi}_i\dot{\psi}^i - \dot{\bar{\psi}}_i\psi^i\right) g(x) + \frac{B^i_j B^j_i}{2} g(x) - B^i_j\left(\frac{\delta^j_i}{2}\bar{\psi}_k\psi^k + \bar{\psi}_i\psi^j\right) g'(x) - \\ - \frac{1}{2}\left(\bar{\psi}_i\psi^i\right)^2 g''(x) + 2m\bar{\psi}_i\psi^i g(x) + mx\bar{\psi}_i\psi^i g'(x) - m^2x^2 g(x),\end{aligned} \tag{3.7}$$

where $g := f''$ and primes mean differentiation in x , $f' = \partial_x f$, etc. The absence of the explicit f and f' in the component action reflects the freedom (3.6).

As the next standard step, we eliminate the auxiliary fields B^i_j by their algebraic equations of motion,

$$B^i_j = \frac{g'(x)}{g(x)}\left(\frac{\delta^i_j}{2}\bar{\psi}_k\psi^k - \bar{\psi}_j\psi^i\right), \tag{3.8}$$

and rewrite the Lagrangian in terms of x and ψ^i:

$$\begin{aligned}\mathcal{L} = \dot{x}^2 g(x) + i\left(\bar{\psi}_i\dot{\psi}^i - \dot{\bar{\psi}}_i\psi^i\right) g(x) - \frac{1}{2}\left(\bar{\psi}_i\psi^i\right)^2\left[g''(x) - \frac{3\left(g'(x)\right)^2}{2g(x)}\right] - \\ - m^2x^2 g(x) + 2m\bar{\psi}_i\psi^i g(x) + mx\bar{\psi}_i\psi^i g'(x).\end{aligned} \tag{3.9}$$

It is invariant, modulo a total time derivative, under the following on-shell odd transformations:

$$\begin{aligned}&\delta x = \bar{\epsilon}^k\bar{\psi}_k - \epsilon_k\psi^k,\\ &\delta\psi^i = i\bar{\epsilon}^i\dot{x} - m\bar{\epsilon}^i x - \left(\bar{\epsilon}^k\bar{\psi}_k\psi^i - \frac{1}{2}\bar{\epsilon}^i\bar{\psi}_k\psi^k\right)\frac{g'(x)}{g(x)}, \quad \text{and c.c.}\end{aligned} \tag{3.10}$$

The Lagrangian (3.9) can be simplified by passing to the new bosonic field $y(x)$ with the free kinetic term. From the equality

$$\dot{x}^2 g(x) = \frac{1}{2}\dot{y}^2 \tag{3.11}$$

we find the equation

$$[y'(x)]^2 = 2g(x) \;\Rightarrow\; y' = \sqrt{2g}\,, \tag{3.12}$$

and (3.9) is rewritten in the form

$$\begin{aligned}\mathcal{L} \;&=\; \frac{\dot{y}^2}{2} + \frac{i}{2}\left(\bar{\zeta}_i\dot{\zeta}^i - \dot{\bar{\zeta}}_i\zeta^i\right) - \frac{m^2x^2}{2}\left(y'(x)\right)^2 + m\bar{\zeta}_i\zeta^i\left(1 + \frac{xy''(x)}{y'(x)}\right) \\ &\quad -\frac{1}{2}\left(\bar{\zeta}_i\zeta^i\right)^2\left[\frac{y'''(x)y'(x) - 2\left(y''(x)\right)^2}{\left(y'(x)\right)^4}\right],\end{aligned} \tag{3.13}$$

where we defined $\zeta^i = \psi^i y'(x)$. Solving (3.12) for x, $x = x(y)$, and defining $V(y) = xy'(x) = x(y)\frac{1}{x'(y)}$, we obtain*

$$\begin{aligned}\mathcal{L} = \frac{\dot{y}^2}{2} + \frac{i}{2}\left(\bar{\zeta}_i\dot{\zeta}^i - \dot{\bar{\zeta}}_i\zeta^i\right) - \frac{m^2}{2}V^2(y) + m\bar{\zeta}_i\zeta^i V'(y) - \\ - \frac{1}{2}\left(\bar{\zeta}_i\zeta^i\right)^2\partial_y\left(\frac{V'(y)-1}{V(y)}\right).\end{aligned} \tag{3.14}$$

Thus, we have finally obtained the Lagrangian involving an arbitrary function $V(y)$ (it is only required to be regular at $y = 0$). In the new representation, the supersymmetry transformations acquire the form

$$\begin{aligned}&\delta y = \bar{\epsilon}^k\bar{\zeta}_k - \epsilon_k\zeta^k, \\ &\delta\zeta^i = i\bar{\epsilon}^i\dot{y} - m\bar{\epsilon}^i V(y) - \left(\epsilon_k\zeta^k\zeta^i + \bar{\epsilon}^k\bar{\zeta}_k\zeta^i - \bar{\epsilon}^i\bar{\zeta}_k\zeta^k\right)\frac{V'(y)-1}{V(y)}.\end{aligned} \tag{3.15}$$

After redefinitions

$$\begin{aligned}&\zeta^i = \frac{1}{\sqrt{2}}\left(\bar{\mu}^i - \mu^i\right), \qquad \bar{\zeta}_i = \frac{1}{\sqrt{2}}\left(\bar{\mu}_i + \mu_i\right), \\ &\tilde{\epsilon}_i = \frac{1}{\sqrt{2}}\left(\bar{\epsilon}_i - \epsilon_i\right), \quad \bar{\tilde{\epsilon}}^j = \frac{1}{\sqrt{2}}\left(\bar{\epsilon}^j + \epsilon^j\right), \quad \tilde{V}(y) = mV(y),\end{aligned} \tag{3.16}$$

the Lagrangian (3.14) and the transformation rules (3.15) are recognized as defining the general SQM model with «weak» $\mathcal{N} = 4$ supersymmetry [3]. Thus this model is the on-shell version of the general $SU(2|1)$ symmetric model of a single $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplet. In what follows, we will stick to our original choice of the fermionic variables.

*The x derivative of $V(y(x))$ is represented as $V_x = V'(y)y_x$.

4. QUANTUM (1,4,3) OSCILLATOR MODEL

Let us consider the simplest Lagrangian

$$\mathcal{L} \;=\; \frac{\dot{x}^2}{2} - \frac{m^2 x^2}{2} + \frac{i}{2}\left(\bar{\psi}_i \dot{\psi}^i - \dot{\bar{\psi}}_i \psi^i\right) + m\bar{\psi}_i \psi^i, \tag{4.1}$$

which corresponds to the choice

$$g = 1/2 \Rightarrow f(x) = \frac{x^2}{4} + c_0 x + c_1\,, \tag{4.2}$$

in (3.9). This Lagrangian is invariant under the transformations

$$\delta x = \left(\bar{\epsilon}\cdot\bar{\psi}\right) - \left(\epsilon\cdot\psi\right), \qquad \delta\psi^i = i\bar{\epsilon}^i \dot{x} - m\bar{\epsilon}^i x. \tag{4.3}$$

The corresponding conserved Noether charges are easily calculated to be:

$$\begin{aligned} &Q^i = \psi^i\left(p - imx\right), \qquad \bar{Q}_i = \bar{\psi}_i\left(p + imx\right), \\ &F = \frac{1}{2}\psi^k\bar{\psi}_k, \qquad I^i_j = \psi^i\bar{\psi}_j - \frac{1}{2}\delta^i_j\psi^k\bar{\psi}_k. \end{aligned} \tag{4.4}$$

The Poisson brackets are imposed as*

$$\{x, p\} = 1, \qquad \{\psi^i, \bar{\psi}_j\} = -i\delta^i_j. \tag{4.5}$$

The corresponding canonical Hamiltonian reads

$$H \;=\; \frac{p^2}{2} + \frac{m^2 x^2}{2} + m\psi^i\bar{\psi}_i. \tag{4.6}$$

Its bosonic part is just the Hamiltonian of harmonic oscillator. We quantize the brackets (4.5) in the standard way

$$[\hat{x}, \hat{p}] = i\,, \quad \{\hat{\psi}^i, \hat{\bar{\psi}}_j\} = \delta^i_j\,, \quad \hat{p} = -i\partial_x\,,\; \hat{\bar{\psi}}_j = \partial/\partial\hat{\psi}^j, \tag{4.7}$$

and use the relation

$$\left[\left(\hat{p} - im\hat{x}\right), \left(\hat{p} + im\hat{x}\right)\right] = 2m \tag{4.8}$$

to represent the quantum Hamiltonian as

$$\hat{H} \;=\; \frac{1}{2}\left(\hat{p} + im\hat{x}\right)\left(\hat{p} - im\hat{x}\right) + m\hat{\psi}^i\hat{\bar{\psi}}_i. \tag{4.9}$$

*For fermionic fields these are in fact the Dirac brackets.

By Noether charges we can also construct the remaining quantum operators

$$\hat{Q}^i = \hat{\psi}^i\left(\hat{p} - im\hat{x}\right), \quad \hat{\bar{Q}}_i = \hat{\bar{\psi}}_i\left(\hat{p} + im\hat{x}\right), \tag{4.10}$$

$$\hat{F} = \frac{1}{2}\hat{\psi}^k\hat{\bar{\psi}}_k, \quad \hat{I}^i_j = \hat{\psi}^i\hat{\bar{\psi}}_j - \frac{1}{2}\delta^i_j\hat{\psi}^k\hat{\bar{\psi}}_k. \tag{4.11}$$

One can check out that they indeed form the superalgebra $su(2|1)$:

$$\begin{aligned}
&\{\hat{Q}^i, \hat{\bar{Q}}_j\} = 2\delta^i_j\hat{H} + 2m\left(\hat{I}^i_j - \delta^i_j\hat{F}\right), \\
&\left[\hat{I}^i_j, \hat{\bar{Q}}_l\right] = \frac{1}{2}\delta^i_j\hat{\bar{Q}}_l - \delta^i_l\hat{\bar{Q}}_j\,, \quad \left[\hat{I}^i_j, \hat{Q}^k\right] = \delta^k_j\hat{Q}^i - \frac{1}{2}\delta^i_j\hat{Q}^k, \\
&\left[\hat{I}^i_j, \hat{I}^k_l\right] = \delta^k_j\hat{I}^i_l - \delta^i_l\hat{I}^k_j, \quad \left[\hat{F}, \hat{\bar{Q}}_l\right] = -\frac{1}{2}\hat{\bar{Q}}_l, \quad \left[\hat{F}, \hat{Q}^k\right] = \frac{1}{2}\hat{Q}^k.
\end{aligned} \tag{4.12}$$

Note that there is a freedom of adding some constants to $\hat{H}$ and $\hat{F}$, in such a way that the sum $\hat{H} - m\hat{F}$ remains intact. Using this freedom, one can, e.g., cast $\hat{H}$ in the form which corresponds to just making replacements (4.7) in the classical Hamiltonian (5.22). In what follows, we will deal with the quantum operators defined as in (4.9)–(4.11).

For further use, we give the expressions for the second- and third-order Casimir operators C_2, C_3 of $SU(2|1)$. The explicit form of these operators in terms of the $SU(2|1)$ generators can be found, e.g., in [9, 12]. We will use the following concise representation for the Casimirs:

$$4m^2 C_2 = C^i_i\,, \quad 12\,m^3 C_3 = 6m^3 F'(1 + 2C_2) + m\,I^i_k C^k_i, \tag{4.13}$$

where

$$F' = F - \frac{1}{m}\,H\,, \quad C^i_j = 2m^2\,\delta^i_j\,{F'}^2 - m^2\left\{I^i_l, I^l_j\right\} + m\left[Q^i, \bar{Q}_j\right]. \tag{4.14}$$

These expressions are valid irrespective of the particular realization of the $SU(2|1)$ generators. For our quantum-mechanical realization (4.9)–(4.11) they are reduced to the following nice form:

$$m^2 C_2 = \hat{H}\left(\hat{H} - m\right), \quad m^3 C_3 = \hat{H}\left(\hat{H} - m\right)\left(\hat{H} - \frac{m}{2}\right). \tag{4.15}$$

Thus they are fully specified by the energy spectrum of the quantum Hamiltonian.

4.1. Wave Functions and Spectrum. We construct the Hilbert space of wave functions in terms of wave functions of bosonic harmonic oscillator, to which the system (4.10), (4.9) is reduced when discarding the fermions.

The generic super wave function $\Omega^{(\ell)}$ at the energy level ℓ shows up the four-fold degeneracy due to the ψ-expansion*

$$\Omega^{(\ell)} = a^{(\ell)}|\ell\rangle + b_i^{(\ell)}\,\psi^i|\ell-1\rangle + \frac{1}{2}\,c^{(\ell)}\,\varepsilon_{ij}\psi^i\psi^j|\ell-2\rangle, \quad \ell \geqslant 2, \tag{4.16}$$

where $|\ell\rangle, |\ell-1\rangle, |\ell-2\rangle$ are the harmonic oscillator functions at the relevant levels and $a^{(\ell)}, b^{(\ell)}, c^{(\ell)}$ are some numerical coefficients. We treat the operators $\hat{p} \pm imx$ in (4.9) and (4.10) as the creation and annihilation operators and impose the standard physical conditions

$$\hat{\bar{\psi}}_k|\ell\rangle = 0, \qquad (\hat{p} - im\hat{x})\,|0\rangle = 0, \quad (\hat{p} + im\hat{x})\,|\ell\rangle = |\ell+1\rangle. \tag{4.17}$$

The spectrum of the Hamiltonian (4.9) is then

$$\hat{H}\,\Omega^{(\ell)} = m\,\ell\,\Omega^{(\ell)}, \qquad m > 0, \quad \ell \geqslant 0. \tag{4.18}$$

We observe that the ground state ($\ell = 0$) and the first excited states ($\ell = 1$) are special, in the sense that they encompass nonequal numbers of bosonic and fermionic states:

$$\Omega^{(0)} = a^{(0)}\,|0\rangle, \quad \Omega^{(1)} = a^{(1)}\,|1\rangle + b_i^{(1)}\,\psi^i|0\rangle. \tag{4.19}$$

The ground state is annihilated by all $SU(2|1)$ generators including Q^i and $\bar{Q}_i$, so it is an $SU(2|1)$ singlet. The states with $\ell = 1$ can be shown to form the fundamental representation of $SU(2|1)$. The action of the supercharges on these states is given by

$$Q^i\,\psi^k|0\rangle = 0\,, \qquad \bar{Q}_i\,\psi^k|0\rangle = \delta_i^k\,|1\rangle,$$
$$Q^i\,|1\rangle = 2m\,\psi^i|0\rangle, \quad \bar{Q}_i\,|1\rangle = 0. \tag{4.20}$$

It is instructive to see what values the Casimir operators (4.15) take on all these states.

The values of Casimir operators for the finite-dimensional $SU(2|1)$ representations can be written in the following generic form [6]:

$$C_2 = (\beta^2 - \lambda^2), \quad C_3 = \beta(\beta^2 - \lambda^2) = \beta C_2. \tag{4.21}$$

These representations are characterized by some positive number λ («highest weight»), which can be half-integer or integer, and an arbitrary additional real

*Our definition of super wave functions is different from that of [3] because of the different choice of the fermionic variables. The two sets are related through the similarity transformation $\psi \to \chi$, $\Omega \to \tilde{\Omega}$, with

$$\chi^k = -e^{-U}\psi^k\,e^U\,, \quad \bar{\chi}_k = e^{-U}\bar{\psi}_k\,e^U, \quad \tilde{\Omega} = e^{-U}\Omega, \quad U = \frac{\pi}{8}\,(\varepsilon_{ij}\psi^i\psi^j - \varepsilon^{ij}\bar{\psi}_i\bar{\psi}_j).$$

number β, which is related to the eigenvalues of the $U(1)$ generator F. Comparing (4.21) with the expressions (4.15) and using the formula for the energy spectrum (4.18), we find that in our case $\lambda = 1/2$ for any $\Omega^{(\ell)}$ and

$$C_2(\ell) = (\ell - 1)\,\ell, \quad C_3(\ell) = (\ell - 1/2)\,(\ell - 1)\,\ell, \quad \beta(\ell) = \ell - 1/2. \qquad (4.22)$$

The ground state with $\ell = 0$ is atypical, because Casimir operators take zero values on it. On the states with $\ell = 1$ both Casimirs vanish as well, so these states also form an atypical $SU(2|1)$ representation. The fact that the fundamental representation of $SU(2|1)$ is atypical is well known. On the $\ell > 1$ states both Casimirs are nonzero, so these states belong to the typical $SU(2|1)$ representations characterized by equal numbers of the bosonic and fermionic states.

Defining the inner product of the states as

$$\langle \Omega | \Psi \rangle = \int\limits_{-\infty}^{\infty} dx\, \Omega^{\dagger} \Psi, \qquad (4.23)$$

one can check that the states $\Omega^{(\ell)}$ for different ℓ are orthogonal with respect to this product and the norms of these states are positive-definite. For instance,

$$\langle 0|0 \rangle = \int\limits_{-\infty}^{\infty} dx\, \exp\left(-mx^2\right) = \sqrt{\frac{\pi}{m}}. \qquad (4.24)$$

The norm of the state $\Omega^{(\ell)}$ is defined as

$$||\Omega^{(\ell)}||^2 = \frac{\langle \Omega^{(\ell)} | \Omega^{(\ell)} \rangle}{\langle \ell | \ell \rangle}. \qquad (4.25)$$

Hence, for the wave functions (4.16), (4.19) we find the following manifestly positive norms:

$$\begin{aligned}
&||\Omega^{(\ell)}||^2 = \bar{a}^{(\ell)} a^{(\ell)} + \frac{\bar{b}^{(\ell)i} b^{(\ell)}_i}{2m\ell} + \frac{\bar{c}^{(\ell)} c^{(\ell)}}{4m^2\,(\ell - 1)\,\ell}, \quad \ell \geqslant 2, \\
&||\Omega^{(1)}||^2 = \bar{a}^{(1)} a^{(1)} + \frac{\bar{b}^{(1)i} b^{(1)}_i}{2m}, \quad ||\Omega^{(0)}||^2 = \bar{a}^{(0)} a^{(0)}.
\end{aligned} \qquad (4.26)$$

4.2. Exotic $SU(2)$ Symmetry. Let us define the operators $\hat{B}_{\pm}$ belonging to the universal enveloping algebra of $su(2|1)$:

$$\hat{B}_{+} = \hat{\bar{Q}}_2 \hat{\bar{Q}}_1 = (\hat{p} + im\hat{x})^2\, \hat{\bar{\psi}}_2 \hat{\bar{\psi}}_1, \quad \hat{B}_{-} = \hat{Q}^1 \hat{Q}^2 = (\hat{p} - im\hat{x})^2\, \hat{\psi}^1 \hat{\psi}^2. \quad (4.27)$$

Defining also the operator

$$\hat{B}_3 = 2m^2 C_2 \left(1 - \hat{\psi}^k \hat{\bar{\psi}}_k\right), \tag{4.28}$$

we observe that at every level with $C_2 \neq 0$, i.e., with $\ell \geqslant 2$, these three operators generate the algebra $su(2)$:

$$\left[\hat{B}_+, \hat{B}_-\right] = 2\hat{B}_3, \quad \left[\hat{B}_3, \hat{B}_\pm\right] = \pm 4m^2 C_2 \hat{B}_\pm. \tag{4.29}$$

The whole algebra is nonvanishing only on the bosonic states $(|\ell\rangle, \varepsilon_{ij}\psi^i\psi^j|\ell-2\rangle)$ which form doublets of this $su(2)$. The fermionic states are singlets. This extra $su(2)$ algebra commutes with the Hamiltonian and with the $su(2)$ subalgebra of $su(2|1)$, while the fermionic $U(1)$ charge operator $\hat{F}$ defines its outer automorphism. It is instructive to quote the Casimir operator of this $su(2)$ algebra

$$B^2 = 2m^2 C_2 \left\{B_+, B_-\right\} + (B_3)^2 = 12\, m^4 (C_2)^2 \left(1 - \hat{\psi}^k \hat{\bar{\psi}}_k\right)^2. \tag{4.30}$$

Using the definition (4.27), it is also easy to show that

$$\left\{\hat{B}_+, \hat{B}_-\right\} = 4\, m^2 C_2 \left(1 - \hat{\psi}^k \hat{\bar{\psi}}_k\right)^2. \tag{4.31}$$

On the fermionic states $\psi^i|\ell - 1\rangle$ this anticommutator vanishes, while on the bosonic states it becomes

$$\left\{\hat{B}_+, \hat{B}_-\right\} = 4\, m^2 C_2 = 4\hat{H}(\hat{H} - m). \tag{4.32}$$

Thus, the operators $\hat{B}_+, \hat{B}_-$ can be interpreted as generators of some «$\mathcal{N} = 2$ superalgebra» acting only on the bosonic states and possessing the «Hamiltonian» which is quadratic in the Hamiltonian of the original $SU(2|1)$-invariant system. Just this «superalgebra» was constructed in [3] in order to establish a link with the so-called «N-fold» supersymmetries, which are defined by the nonlinear algebras of the type (4.32) (see [16] and references therein). Our consideration in the framework of the simple oscillator model shows that the relevant product «supercharges» (4.27) are in fact generators of some extra bosonic $su(2)$ algebra which belongs to the universal enveloping of $su(2|1)$ and is such that the full space of quantum states is split into its doublets and singlets. The relevant nonlinear «Hamiltonian» proves to be the quadratic Casimir of $su(2|1)$. It is an open question whether this interpretation applies to the case of the general quantum $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ models.

5. THE MULTIPLETS (2,4,2)

5.1. Chiral $SU(2|1)$ Superspaces. The supergroup $SU(2|1)$ admits two mutually conjugated complex supercosets which can be identified with the left and right chiral subspaces:

$$(t_L, \theta_i)\,, \qquad \left(t_R, \bar{\theta}^i\right). \tag{5.1}$$

The relevant complex even coordinates are related to the real time coordinate t via

$$t_L = t + \frac{i}{2m}\ln K, \quad t_R = t - \frac{i}{2m}\ln K, \qquad K = \left[1 + 2m\left(\bar{\theta}\cdot\theta\right)\right]. \tag{5.2}$$

The Grassmann coordinates θ_i and $\bar{\theta}^i$ are the same as in (2.12). Chiral subspaces are closed under the $SU(2|1)$ transformations

$$\delta\theta_i = \epsilon_i + 2m\left(\bar{\epsilon}\cdot\theta\right)\theta_i \quad \delta t_L = 2i\left(\bar{\epsilon}\cdot\theta\right), \quad \text{and c.c.} \tag{5.3}$$

The multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ is described by a complex superfield Φ subjected to the chirality condition and possessing a fixed external $U(1)$ charge*

$$\bar{\mathcal{D}}_j \Phi_L = 0, \quad \tilde{I}^i_j \Phi = 0, \quad \tilde{F}\Phi = 2\kappa\Phi. \tag{5.4}$$

The general solution of (5.4) reads

$$\begin{aligned}
&\Phi(t,\theta,\bar{\theta}\,) = e^{2i\kappa m(t_L - t)}\Phi_L(t_L,\theta) = \left[1 + 2m\left(\bar{\theta}\cdot\theta\right)\right]^{-\kappa}\Phi_L(t_L,\theta),\\
&\Phi_L(t_L,\theta) = z + \sqrt{2}\,\theta_i\xi^i + \varepsilon^{ij}\theta_i\theta_j B.
\end{aligned} \tag{5.5}$$

In the central basis $\{t,\theta_i,\bar{\theta}^k\}$ the same superfield is written as

$$\begin{aligned}
\Phi(t,\theta,\bar{\theta}) = z + \sqrt{2}\,\theta_i\xi^i + \varepsilon^{ij}\theta_i\theta_j B + i\left(\bar{\theta}\cdot\theta\right)\nabla_t z + \sqrt{2}\,i\left(\bar{\theta}\cdot\theta\right)\theta_i\nabla_t\xi^i -\\
- \frac{1}{2}\left(\bar{\theta}\cdot\theta\right)^2\left[2im\nabla_t z + \nabla_t^2 z\right],
\end{aligned} \tag{5.6}$$

where

$$\nabla_t = \partial_t + 2i\kappa m, \qquad \bar{\nabla}_t = \partial_t - 2i\kappa m. \tag{5.7}$$

The chiral superfield having the $\tilde{F}$ charge 2κ transforms as

$$\begin{aligned}
&\delta\Phi \simeq \Phi'(t',\theta',\bar{\theta}') - \Phi(t,\theta,\bar{\theta}) = \left(i\delta\hat{h}\tilde{F} + i\delta h^j_i\tilde{I}^i_j\right)\Phi = 2\kappa m\left(\epsilon_i\bar{\theta}^i + \bar{\epsilon}^j\theta_j\right)\Phi \Longleftrightarrow\\
&\delta\Phi_L(t_L,\theta) = 4\kappa m\left(\bar{\epsilon}^j\theta_j\right)\Phi_L(t_L,\theta).
\end{aligned} \tag{5.8}$$

---

*In principle, we could ascribe to it also a nontrivial external $SU(2)$ index, but we do not consider here such complications.

The corresponding generators $Q^i, \bar{Q}_i$ can be easily found, but we do not quote them here.

The superfield transformation laws (5.8) induce the following transformations for the component fields:

$$\begin{aligned}&\delta z = -\sqrt{2}\,\epsilon_i \xi^i, \quad \delta \xi^i = \sqrt{2}\, i \bar{\epsilon}^i \nabla_t z - \sqrt{2}\,\varepsilon^{ik}\epsilon_k B,\\ &\delta B = -\sqrt{2}\,\varepsilon_{ik}\bar{\epsilon}^k \left[m\xi^i + i\nabla_t \xi^i\right].\end{aligned} \tag{5.9}$$

5.2. Lagrangian. The general Lagrangian involves the function $f\left(\Phi, \Phi^\dagger\right)$, which is an analog of the Kähler potential of the standard $\mathcal{N}=4$ mechanics based on the multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ [17]:

$$\mathcal{L}_{(k)} = \frac{1}{4}\int d^2\theta\, d^2\bar{\theta}\left[1 + 2m\left(\bar{\theta}\cdot\theta\right)\right] f\left(\Phi, \Phi^\dagger\right). \tag{5.10}$$

If $\kappa = 0$, the potential $f\left(\Phi, \Phi^\dagger\right)$ can be an arbitrary function of its arguments, without breaking of the $U(1)$ invariance generated by F. For $\kappa \neq 0$, the $U(1)$ invariance necessarily implies that $f\left(\Phi, \Phi^\dagger\right) = \tilde{f}\left(\Phi\Phi^\dagger\right)$ because the $U(1)$ generator F has a nontrivial matrix part in this case, $\delta\Phi = 2i\kappa\alpha\Phi$, $\delta\bar{\Phi} = -2i\kappa\alpha\bar{\Phi}$.

The general component Lagrangian reads

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}f_z \nabla_t^2 z - \frac{1}{4}f_{\bar{z}}\bar{\nabla}_t^2\bar{z} - \frac{1}{4}f_{zz}\left(\nabla_t z\right)^2 - \frac{1}{4}f_{\bar{z}\bar{z}}\left(\bar{\nabla}_t\bar{z}\right)^2 + \frac{1}{2}g\bar{\nabla}_t\bar{z}\nabla_t z -\\ &-\frac{im}{2}\left(f_{\bar{z}}\bar{\nabla}_t\bar{z} - f_z\nabla_t z\right) - \frac{i}{2}\left(\bar{\xi}\cdot\xi\right)\left(\bar{\nabla}_t\bar{z}g_{\bar{z}} - \nabla_t z g_z\right) + \frac{i}{2}\left(\bar{\xi}_i\nabla_t\xi^i + \xi^i\bar{\nabla}_t\bar{\xi}_i\right)g +\\ &+ m\left(\bar{\xi}\cdot\xi\right)g + \frac{1}{2}\left(\bar{\xi}\cdot\xi\right)^2 g_{z\bar{z}} + B\bar{B}g + \frac{1}{2}\varepsilon_{kl}\xi^k\xi^l\bar{B}g_z + \frac{1}{2}\varepsilon^{kl}\bar{\xi}_k\bar{\xi}_l B g_{\bar{z}},\end{aligned} \tag{5.11}$$

where $g := f_{z\bar{z}}$ is the metric on the Kähler manifold*. After eliminating the auxiliary field B by its equation of motion,

$$B = -\frac{1}{2g}\varepsilon_{kl}\xi^k\xi^l g_z, \tag{5.12}$$

the Lagrangian can be rewritten as

$$\begin{aligned}\mathcal{L} = g\dot{\bar{z}}\dot{z} + 2i\kappa m\left(\dot{\bar{z}}z - \dot{z}\bar{z}\right)g - \frac{im}{2}\left(\dot{\bar{z}}f_{\bar{z}} - \dot{z}f_z\right) - \frac{i}{2}\left(\bar{\xi}\cdot\xi\right)\left(\dot{\bar{z}}g_{\bar{z}} - \dot{z}g_z\right) +\\ + \frac{i}{2}\left(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i\right)g - m^2 V - m\left(\bar{\xi}\cdot\xi\right)U + \frac{1}{2}\left(\bar{\xi}\cdot\xi\right)^2 R,\end{aligned} \tag{5.13}$$

*Here, the lower case indices denote the differentiation in $z, \bar{z}$: $f_{z\bar{z}} = \partial_z\partial_{\bar{z}}f$.

where

$$
\begin{aligned}
V &= \kappa\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right) f - \kappa^2\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right)^2 f, \\
U &= \kappa\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right) g - (1-2\kappa)\, g, \\
R &= g_{z\bar{z}} - \frac{g_z g_{\bar{z}}}{g}.
\end{aligned} \tag{5.14}
$$

The on-shell transformations read

$$
\delta z = -\sqrt{2}\,\epsilon_i \xi^i, \quad \delta\xi^i = \sqrt{2}\, i\bar{\epsilon}^i \nabla_t z + \sqrt{2}\,\epsilon_k \xi^k \xi^i \frac{g_z}{g}. \tag{5.15}
$$

It is worth pointing out that, at $\kappa \neq 0$, one *has* to choose $f(z,\bar{z}) = \tilde{f}(z\bar{z})$ in both the off-shell and the on-shell component Lagrangians (5.11) and (5.13). Only under this restriction the $\kappa \neq 0$ Lagrangians are invariant, modulo a total derivative, with respect to the transformations (5.9) and (5.15).

To close this Subsection, let us summarize a few peculiar features of the Lagrangian (5.13) at $m \neq 0$, which distinguish it from its standard Kähler $(\mathbf{2},\mathbf{4},\mathbf{2})$ counterpart (recovered in the limit $m = 0$).

- The $m \neq 0$ Lagrangian contains the bosonic potential $V(z,\bar{z})$ which is expressed in terms of the «Kähler potential» f and vanishes at $\kappa = 0$.

- In addition, there is a new Yukawa-type coupling $\sim U$ which is also determined by f and survives at $\kappa = 0$.

- The $m \neq 0$ Lagrangian contains two $d = 1$ WZ terms $\sim \kappa m$ and $\sim m$. At $\kappa = 0$, one of them vanishes, while the other retains.

- These WZ terms necessarily accompany the Kähler kinetic term $\sim \dot{z}\dot{\bar{z}}$ and so are prescribed by the $SU(2|1)$ supersymmetry. No such terms can be defined for the standard linear $\mathcal{N} = 4, d = 1$ chiral multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ [17].

5.3. Superpotential. When $\kappa \neq 0$, we can also add to $\mathcal{L}_{(k)}$ the potential term

$$
\mathcal{L}_{(p)} = \tilde{m}\left[\int d^2\theta\, \mathcal{F}\left(\Phi_L\right) + \text{c.c.}\right]. \tag{5.16}
$$

As opposed to the case of the standard $\mathcal{N} = 4$ mechanics [17], in the $SU(2|1)$ case the superfield potential $\mathcal{F}$ is severely constrained by the requirement of compensating the nontrivial transformation of the chiral measure $dt_L d^2\theta$:

$$
\delta dt_L d^2\theta = -2m\, dt_L d^2\theta\, (\bar{\epsilon}^j \theta_j). \tag{5.17}
$$

The only possibility to ensure the invariance is to choose the potential as

$$\mathcal{L}_{(p)} = \tilde{m}\left[\int d^2\theta\,(\Phi_L)^{\frac{1}{2\kappa}} + \text{c.c.}\right] = \frac{\tilde{m}}{\kappa}\left[Bz^{(\frac{1}{2\kappa}-1)} + \bar{B}\bar{z}^{(\frac{1}{2\kappa}-1)}\right] + \\ + \frac{\tilde{m}}{2\kappa}\left(\frac{1}{2\kappa}-1\right)\left[\varepsilon_{ik}\xi^i\xi^k z^{(\frac{1}{2\kappa}-2)} + \varepsilon^{ik}\bar{\xi}_i\bar{\xi}_k\bar{z}^{(\frac{1}{2\kappa}-2)}\right], \quad (5.18)$$

where $\tilde{m}$ is an extra parameter of the mass dimension. The potential term takes the simplest form $\sim B+\bar{B}$ at $2\kappa = 1$. For $\kappa = 0$, no potential terms are possible at all. For simplicity, in what follows, we will limit our consideration to the option $\tilde{m} = 0$.

5.4. Hamiltonian Formalism. Performing the Legendre transformation, we define the classical Hamiltonian as

$$H = g^{-1}\left(p_z - \frac{i}{2}mf_z + \frac{i}{2}g_z\,\xi^k\bar{\xi}_k + 2i\kappa m\bar{z}g\right)\times \\ \times\left(p_{\bar{z}} + \frac{i}{2}mf_{\bar{z}} - \frac{i}{2}g_{\bar{z}}\,\xi^k\bar{\xi}_k - 2i\kappa mzg\right) + m^2V + m\left(\bar{\xi}\cdot\xi\right)U - \frac{1}{2}\left(\bar{\xi}\cdot\xi\right)^2 R. \quad (5.19)$$

By Noether prescription we can calculate the supercharges $\left(Q^i\right)^\dagger = \bar{Q}_i$ and the remaining bosonic charges:

$$Q^i = \sqrt{2}\,\xi^i\left(p_z - \frac{i}{2}mf_z + \frac{i}{2}g_z\,\xi^k\bar{\xi}_k\right),$$

$$\bar{Q}_j = \sqrt{2}\,\bar{\xi}_j\left(p_{\bar{z}} + \frac{i}{2}mf_{\bar{z}} - \frac{i}{2}g_{\bar{z}}\,\xi^k\bar{\xi}_k\right),$$

$$F = -2i\kappa\left(zp_z - \bar{z}p_{\bar{z}}\right) - \left(2\kappa - \frac{1}{2}\right)g\,\xi^k\bar{\xi}_k, \quad I^i_j = g\left[\xi^i\bar{\xi}_j - \frac{1}{2}\delta^i_j\xi^k\bar{\xi}_k\right]. \quad (5.20)$$

For $\kappa \neq 0$,

$$f\left(\Phi, \Phi^\dagger\right) = \tilde{f}\left(\Phi\Phi^\dagger\right) \Rightarrow f(z,\bar{z}) = \tilde{f}(z\bar{z}), \quad (5.21)$$

and we can cast the Hamiltonian (5.19) in the following form:

$$H = g^{-1}\left(p_z - \frac{i}{2}mf_z + \frac{i}{2}g_z\xi^k\bar{\xi}_k\right)\left(p_{\bar{z}} + \frac{i}{2}mf_{\bar{z}} - \frac{i}{2}g_{\bar{z}}\xi^k\bar{\xi}_k\right) - \\ - 2i\kappa m\left(zp_z - \bar{z}p_{\bar{z}}\right) - m\left(1-2\kappa\right)\left(\bar{\xi}\cdot\xi\right)g - \frac{1}{2}\left(\bar{\xi}\cdot\xi\right)^2 R. \quad (5.22)$$

The $\kappa = 0$ form of the Hamiltonian (5.19) coincides with that of (5.22) without any restrictions on the function $f(z,\bar{z})$. One of the admissible choices of $f(z,\bar{z})$ in the $\kappa = 0$ case is, as before, $f(z,\bar{z}) = \tilde{f}(z\bar{z})$.

The Poisson brackets and the Dirac brackets are imposed as

$$\{z, p_z\} = 1, \qquad \{\xi^i, \bar{\xi}_j\} = -i\delta^i_j\, g^{-1}\,. \tag{5.23}$$

To prepare the system for quantization, it is useful to make the substitution

$$\left(z, \xi^i\right) \longrightarrow \left(z, \eta^i\right), \qquad \eta^i = g^{\frac{1}{2}}\xi^i. \tag{5.24}$$

In terms of the new variables, the brackets become

$$\{z, p_z\} = 1, \qquad \{\eta^i, \bar{\eta}_j\} = -i\delta^i_j\,, \qquad \{p_z, \eta^i\} = \{p_z, \bar{\eta}_j\} = 0\,. \tag{5.25}$$

The Noether charges (5.20) and the Hamiltonian (5.22) are rewritten as

$$\begin{aligned}
Q^i &= \sqrt{2}\,\eta^i g^{-\frac{1}{2}}\left(p_z - \frac{i}{2}mf_z + \frac{i}{2}g^{-1}g_z\,\eta^k\bar{\eta}_k\right), \\
\bar{Q}_j &= \sqrt{2}\,\bar{\eta}_j\, g^{-\frac{1}{2}}\left(p_{\bar{z}} + \frac{i}{2}mf_{\bar{z}} - \frac{i}{2}g^{-1}g_{\bar{z}}\,\eta^k\bar{\eta}_k\right),
\end{aligned} \tag{5.26}$$

$$F = -2i\kappa\left(zp_z - \bar{z}p_{\bar{z}}\right) - \left(2\kappa - \frac{1}{2}\right)\eta^k\bar{\eta}_k, \quad I^i_j = \eta^i\bar{\eta}_j - \frac{1}{2}\delta^i_j\eta^k\bar{\eta}_k, \tag{5.27}$$

$$\begin{aligned}
H &= g^{-1}\left(p_z - \frac{i}{2}mf_z + \frac{i}{2}g^{-1}g_z\,\eta^k\bar{\eta}_k\right)\left(p_{\bar{z}} + \frac{i}{2}mf_{\bar{z}} - \frac{i}{2}g^{-1}g_{\bar{z}}\,\eta^k\bar{\eta}_k\right) \\
&\quad -2i\kappa m\left(zp_z - \bar{z}p_{\bar{z}}\right) + m\left(1 - 2\kappa\right)\eta^k\bar{\eta}_k - \frac{1}{2}\,g^{-2}R\left(\eta^k\bar{\eta}_k\right)^2.
\end{aligned} \tag{5.28}$$

5.5. Quantization. We quantize the brackets (5.25) in the standard way

$$\begin{aligned}
&[\hat{z}, \hat{p}_z] = i, \quad \{\hat{\eta}^i, \hat{\bar{\eta}}_j\} = \delta^i_j, \quad [\hat{p}_z, \hat{\eta}^i] = [\hat{p}_z, \hat{\bar{\eta}}_j] = 0, \\
&\hat{p}_z = -i\partial_z, \quad \hat{\bar{\eta}}_j = \frac{\partial}{\partial\hat{\eta}^j}
\end{aligned} \tag{5.29}$$

and use the relation

$$\left[\nabla_z, \bar{\nabla}_{\bar{z}}\right] = mg - \frac{1}{2}\,g^{-1}R\left(\hat{\eta}^k\hat{\bar{\eta}}_k - \hat{\bar{\eta}}_k\hat{\eta}^k\right), \tag{5.30}$$

where

$$\begin{aligned}
\nabla_z &= -i\partial_z - \frac{i}{2}mf_z + \frac{i}{2}g^{-1}g_z\left(\hat{\eta}^k\hat{\bar{\eta}}_k - 1\right), \\
\bar{\nabla}_{\bar{z}} &= -i\partial_{\bar{z}} + \frac{i}{2}mf_{\bar{z}} + \frac{i}{2}g^{-1}g_{\bar{z}}\left(\hat{\bar{\eta}}_k\hat{\eta}^k - 1\right).
\end{aligned} \tag{5.31}$$

The general scheme of passing from the classical supercharges to the quantum ones was described in [18]. It involves two steps.

1. First, one has to Weyl-order the supercharges. The Weyl-ordered supercharges act on super wave functions with the inner product

$$\langle\Omega|\Psi\rangle = \int dz\, d\bar{z} \prod_i d\eta^i\, d\bar{\eta}_i\, \exp\{\bar{\eta}_k\eta^k\}\, \Omega^\dagger\Psi. \tag{5.32}$$

2. As the next step, one passes to the covariant supercharges, which act on the Hilbert space with the more natural, geometrically motivated inner product

$$\langle\Omega|\Psi\rangle = \int g\, dz\, d\bar{z} \prod_i d\eta^i\, d\bar{\eta}_i\, \exp\{\bar{\eta}_k\eta^k\}\, \Omega^\dagger_{(cov)}\, \Psi_{(cov)}. \tag{5.33}$$

They are related to the Weyl-ordered supercharges through the similarity transformation

$$\left(\hat{Q}^i, \hat{\bar{Q}}_j\right)_{cov} = g^{-\frac{1}{2}} \left(\hat{Q}^i, \hat{\bar{Q}}_j\right) g^{\frac{1}{2}}. \tag{5.34}$$

As a result of this procedure, we obtain the following quantum operators:

$$\begin{aligned}
&\hat{Q}^i_{(cov)} = \sqrt{2}\,\hat{\eta}^i\, g^{-\frac{1}{2}}\nabla_z, \quad \hat{\bar{Q}}_{(cov)j} = \sqrt{2}\,\hat{\bar{\eta}}_j\, g^{-\frac{1}{2}}\bar{\nabla}_{\bar{z}},\\
&\hat{F} = -2\kappa\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) - \left(2\kappa - \frac{1}{2}\right)\hat{\eta}^k\hat{\bar{\eta}}_k, \quad \hat{I}^i_j = \hat{\eta}^i\hat{\bar{\eta}}_j - \frac{1}{2}\delta^i_j\hat{\eta}^k\hat{\bar{\eta}}_k.
\end{aligned} \tag{5.35}$$

They satisfy the $su(2|1)$ superalgebra (4.12) with the quantum Hamiltonian

$$\begin{aligned}
\hat{H} = g^{-1}\,\bar{\nabla}_{\bar{z}}\,\nabla_z - 2\kappa m\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) + m\,(1-2\kappa)\,\hat{\eta}^k\hat{\bar{\eta}}_k -&\\
-\frac{1}{4}\,g^{-2}R\,\varepsilon_{kl}\varepsilon^{nj}\hat{\eta}^k\hat{\eta}^l\hat{\bar{\eta}}_n\hat{\bar{\eta}}_j.&
\end{aligned} \tag{5.36}$$

Note that the second term in (5.36) can be reabsorbed into a redefinition of the external magnetic field in $\nabla_z, \bar{\nabla}_z$ at cost of appearance of some bosonic potential. We will explicitly do this in the next Section.

Let us also define one more $U(1)$ generator

$$\hat{E} = -\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) - \hat{\eta}^k\hat{\bar{\eta}}_k. \tag{5.37}$$

It commutes with all $SU(2|1)$ generators, provided that $f(z,\bar{z}) = \tilde{f}(z\bar{z})$. Thus, in this case there is an extra $U(1)$ generator playing the role of external Casimir operator of the $su(2|1)$ superalgebra. In the next Section, we will see on the simple example that the presence of this $U(1)$ generator proves crucial for finding the quantum spectrum. Note that, since both operators $\hat{E}$ and $\hat{F}$ commute with $\hat{H}$, the same is true for the fermionic number operator $\hat{\eta}^k\hat{\bar{\eta}}_k$ which is a linear combination of these two conserved $U(1)$ generators. It is seen from (5.35) that at $\kappa = 0$, when there are no restrictions on $f(z,\bar{z})$, the fermionic number operator coincides (up to the factor $1/2$) with the generator $\hat{F}$.

6. THE MODEL ON A PLANE

6.1. Lagrangian and Hamiltonian. The model on a plane corresponds to the simplest choice of the Kähler potential in (5.10):

$$f\left(\Phi, \Phi^{\dagger}\right) = \Phi\Phi^{\dagger}. \tag{6.1}$$

For this particular case, the general component Lagrangian (5.13) is reduced to

$$\mathcal{L} = \dot{\bar{z}}\dot{z} + im\left(2\kappa - \frac{1}{2}\right)\left(\dot{\bar{z}}z - \dot{z}\bar{z}\right) + \frac{i}{2}\left(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i\right) + 2\kappa\left(2\kappa - 1\right)m^2\bar{z}z + \\ + \left(1 - 2\kappa\right)m\left(\bar{\xi}\cdot\xi\right). \tag{6.2}$$

It is invariant under the transformations

$$\delta z = -\sqrt{2}\,\epsilon_i\xi^i, \quad \delta\xi^i = \sqrt{2}\,i\bar{\epsilon}^i\dot{z} - 2\sqrt{2}\,\kappa m\bar{\epsilon}^i z. \tag{6.3}$$

In accordance with the notations of the previous Section, we will deal with the set of variables $\left(z, \eta^i\right)$ (in the considered case $\xi^i \equiv \eta^i$, because $g = 1$). The corresponding canonical Hamiltonian (5.28) is reduced to the expression

$$H = \left[p_z - \frac{i}{2}\left(1 - 4\kappa\right)m\bar{z}\right]\left[p_{\bar{z}} + \frac{i}{2}\left(1 - 4\kappa\right)mz\right] + 2\kappa\left(1 - 2\kappa\right)m^2\bar{z}z + \\ + \left(1 - 2\kappa\right)m\,\eta^k\bar{\eta}_k, \tag{6.4}$$

or to the alternative expression

$$H = \left(p_z - \frac{i}{2}m\bar{z}\right)\left(p_{\bar{z}} + \frac{i}{2}mz\right) - 2i\kappa m\left(zp_z - \bar{z}p_{\bar{z}}\right) + \\ + m\left(1 - 2\kappa\right)\eta^k\bar{\eta}_k. \tag{6.5}$$

Quantization is performed in the standard way

$$[\hat{z}, \hat{p}_z] = i, \quad \{\hat{\eta}^i, \hat{\bar{\eta}}_j\} = \delta^i_j, \quad [\hat{p}_z, \hat{\eta}^i] = [\hat{p}_z, \hat{\bar{\eta}}_j] = 0, \\ \hat{p}_z = -i\partial_z, \quad \hat{\bar{\eta}}_j = \frac{\partial}{\partial\hat{\eta}^j}. \tag{6.6}$$

The quantum Hamiltonian

$$\hat{H} = \bar{\nabla}_{\bar{z}}\,\nabla_z - 2\kappa m\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) + m\left(1 - 2\kappa\right)\hat{\eta}^k\hat{\bar{\eta}}_k \tag{6.7}$$

and the quantum operators

$$\hat{Q}^i = \sqrt{2}\,\hat{\eta}^i\nabla_z, \quad \hat{\bar{Q}}_j = \sqrt{2}\,\hat{\bar{\eta}}_j\bar{\nabla}_{\bar{z}}, \tag{6.8}$$

$$\hat{F} = -2\kappa\left(z\partial_z - \bar{z}\partial_{\bar{z}}\right) - \left(2\kappa - \frac{1}{2}\right)\hat{\eta}^k\hat{\bar{\eta}}_k, \quad \hat{I}^i_j = \hat{\eta}^i\hat{\bar{\eta}}_j - \frac{1}{2}\delta^i_j\hat{\eta}^k\hat{\bar{\eta}}_k, \tag{6.9}$$

form the $su(2|1)$ superalgebra (4.12). Here,

$$\nabla_z = -i\partial_z - \frac{i}{2}m\bar{z}, \quad \bar{\nabla}_{\bar{z}} = -i\partial_{\bar{z}} + \frac{i}{2}mz, \quad \left[\nabla_z, \bar{\nabla}_{\bar{z}}\right] = m. \tag{6.10}$$

The Hamiltonian (6.7) can be rewritten, up to a constant shift $2m\kappa$, in the form analogous to the classical expression (6.4)

$$\hat{H} = -\left[\partial_z + \frac{1}{2}(1-4\kappa)m\bar{z}\right]\left[\partial_{\bar{z}} - \frac{1}{2}(1-4\kappa)mz\right] + \\ + 2\kappa(1-2\kappa)m^2 z\bar{z} + (1-2\kappa)\, m\, \hat{\eta}^k \hat{\bar{\eta}}_k. \tag{6.11}$$

It is seen from this representation that we are dealing with a superextension of the two-dimensional harmonic oscillator with the strength $2\kappa(1-2\kappa)m^2$, supplemented by a coupling to the external magnetic field $\mathcal{A}_z = -\frac{i}{2}(1-4\kappa)m\,\bar{z}$, $\mathcal{A}_{\bar{z}} = \frac{i}{2}(1-4\kappa)m\,z$.

For further use, it will be instructive to know the explicit expressions of the $SU(2|1)$ Casimir operators defined in (4.13). For the specific realization of the quantum $SU(2|1)$ generators (6.7) and (6.9) they are

$$m^2 C_2 = \left(\hat{H} - 2\kappa m\hat{E}\right)\left(\hat{H} - 2\kappa m\hat{E} - m\right), \\ m^3 C_3 = \left(\hat{H} - 2\kappa m\hat{E}\right)\left(\hat{H} - 2\kappa m\hat{E} - m\right)\left(\hat{H} - 2\kappa m\hat{E} - \frac{m}{2}\right). \tag{6.12}$$

Comparing these expressions with those for the $(\mathbf{1},\mathbf{4},\mathbf{3})$ oscillator model, Eqs. (4.15), we observe that they involve, besides the Hamiltonian $\hat{H}$, also the extra $U(1)$ generator $\hat{E}$ defined in (5.37) and commuting with all $SU(2|1)$ generators.

6.2. Wave Functions and Spectrum. It is convenient to seek for the bosonic wave function Ω as an eigenfunction of the mutually commuting $U(1)$ operator (5.37) and the Hamiltonian (6.7). The corresponding eigenvalue problem is set by the equations

$$\text{(a)}\;\; \hat{E}\,\Omega = n\,\Omega, \quad \text{(b)}\;\; \hat{H}\,\Omega = \mathcal{E}\,\Omega = mq\,\Omega. \tag{6.13}$$

The equation (a) yields*

$$\Omega = \bar{z}^n A\,(w)\,, \quad w \equiv z\bar{z}. \tag{6.14}$$

*We could equally choose, from the very beginning, the solution with negative n, $\Omega' = z^{|n|}A'(w)$. The corresponding sets of wave functions are related through the complex conjugation.

Then the equation (b) amounts to the following one for $A\,(w)$:

$$\left[-w\partial_w^2 - (1+n)\partial_w + \frac{m^2}{4}w - \frac{m}{2}\right] A\,(w) = m\left(q - 2\kappa n + \frac{n}{2}\right)\,A\,(w)\,. \tag{6.15}$$

It is solved by

$$A(w) = e^{-\frac{mw}{2}}\,L^{(n)}_{q-2\kappa n}(mw), \tag{6.16}$$

where $L^{(n)}_{q-2\kappa n}(mw)$ are the generalized Laguerre polynomials. Thus the eigenvalue problem for $\hat{H}$ can be rewritten as

$$\hat{H}\,\Omega^{(\ell;n)} = \mathcal{E}^{(\ell;n)}\,\Omega^{(\ell;n)}, \tag{6.17}$$

with

$$\mathcal{E}^{(\ell;n)} = m(\ell + 2\kappa n) \tag{6.18}$$

and

$$\Omega^{(\ell;n)} = \bar{z}^n e^{-\frac{mz\bar{z}}{2}}\,L^{(n)}_{\ell}(mz\bar{z}) = \frac{z^{-n}}{\ell!}e^{\frac{mz\bar{z}}{2}}\;\frac{d^\ell}{dw^\ell}\left(e^{-mw}w^{n+\ell}\right)\Big|_{w=z\bar{z}}\,. \tag{6.19}$$

According to the definition of Laguerre polynomials, ℓ is a nonnegative integer, $\ell \geqslant 0$.

The orthogonality of $\Omega^{(\ell;n)}$ with respect to the inner product,

$$\langle\Omega^{(\ell_1;n_1)}|\Omega^{(\ell_2;n_2)}\rangle := \int dz\,d\bar{z}\left(\Omega^{(\ell_1;n_1)}\right)^\dagger\,\Omega^{(\ell_2;n_2)} = \\ = \frac{\pi(n+\ell)!}{\ell!\,m^{n+1}}\,\delta^{\ell_1\ell_2}\,\delta^{n_1n_2}, \tag{6.20}$$

is necessary for the super wave functions to form the complete orthogonal set. This orthogonality condition constrains n to the integer values $n \geqslant -\ell^*$. The integral in (6.20) is convergent for $m > 0^{**}$. The energies are positive and $\hat{H}$ (6.11) is bounded from below only under the following restriction on the parameter κ:

$$0 \leqslant \kappa \leqslant 1/2. \tag{6.21}$$

---

*The wave functions (6.19) remain regular at $z = 0$ for these negative values of n.

**For $m < 0$, we can take advantage of the equivalent redefinition (2.2) to bring all the quantum relations and formulas to the same form as for $m > 0$, with $m \to |m|$.

The wave functions $\Omega^{(\ell;n)}$ satisfy the relations

$$
\begin{aligned}
&\nabla_z\,\Omega^{(\ell;n)} = im\Omega^{(\ell-1;n+1)}, \quad (\nabla_z + im\bar{z})\,\Omega^{(\ell;n)} = im\Omega^{(\ell;n+1)},\\
&\bar{\nabla}_{\bar{z}}\,\Omega^{(\ell;n)} = -i(\ell+1)\Omega^{(\ell+1;n-1)}, \quad \left(\bar{\nabla}_{\bar{z}} - imz\right)\Omega^{(\ell;n)} = -i(\ell+n)\Omega^{(\ell;n-1)},\\
&\nabla_z\,\Omega^{(0;n)} = 0, \quad \left(\bar{\nabla}_{\bar{z}} - imz\right)\Omega^{(\ell;-\ell)} = 0,
\end{aligned}
\tag{6.22}
$$

which follow from the definition (6.19). The operators $(\nabla_z + im\bar{z})$, $(\bar{\nabla}_{\bar{z}} - imz)$ commute with the covariant momenta ∇_z, $\bar{\nabla}_{\bar{z}}$. Using (6.22), we can obtain the convenient representation for the generic $\Omega^{(\ell;n)}$ as

$$
\Omega^{(\ell;n)} = \frac{(-i)^n}{\ell! m^{\ell+n}}\left(\bar{\nabla}_{\bar{z}}\right)^{\ell}(\nabla_z + im\bar{z})^{\ell+n}\,\Omega^{(0;0)}(w),
\tag{6.23}
$$

where $\Omega^{(0;0)}(w)$ is the ground state wave function:

$$
\Omega^{(0;0)}(w) = e^{-\frac{mz\bar{z}}{2}}\,.
\tag{6.24}
$$

Acting on $\Omega^{(\ell;n)}$ by the supercharges Q^i , we can produce all other common eigenstates of the Hamiltonian $\hat{H}$ and the external $U(1)$ charge operator $\hat{E}$. In the process, one should take account of the physical condition:

$$
\bar{\eta}_j\,\Omega^{(\ell;n)} = 0 \Rightarrow \bar{Q}_j\,\Omega^{(\ell;n)} = 0.
\tag{6.25}
$$

Using the relations (6.22), it is easy to find

$$
\begin{aligned}
&Q^i\,\Omega^{(\ell;n)} = im\sqrt{2}\,\eta^i\,\Omega^{(\ell-1;n+1)},\\
&\varepsilon_{ij}Q^i\,Q^j\,\Omega^{(\ell;n)} = -2m^2\varepsilon_{ij}\eta^i\eta^j\,\Omega^{(\ell-2;n+2)}.
\end{aligned}
\tag{6.26}
$$

Then the super wave functions,

$$
\begin{aligned}
&\Psi^{(\ell;n)} = a^{(\ell;n)}\Omega^{(\ell;n)} + b_i^{(\ell;n)}\eta^i\,\Omega^{(\ell-1;n+1)} + \frac{1}{2}\,c^{(\ell;n)}\,\varepsilon_{ij}\eta^i\eta^j\,\Omega^{(\ell-2;n+2)}, \quad \ell \geqslant 2,\\
&\Psi^{(1;n)} = a^{(1;n)}\,\Omega^{(1;n)} + b_i^{(1;n)}\eta^i\,\Omega^{(0;n+1)},\\
&\Psi^{(0;n)} = a^{(0;n)}\,\Omega^{(0;n)},
\end{aligned}
\tag{6.27}
$$

span the full Hilbert space of quantum states of the model. We observe that the «ground states» ($\ell = 0$) and the first excited states ($\ell = 1$) are special, in the sense that they encompass nonequal numbers of bosonic and fermionic states. The eigenvalues of the operators $\hat{E}$ and $\hat{H}$ are given by

$$
\hat{E}\,\Psi^{(\ell;n)} = n\,\Psi^{(\ell;n)}, \;\; \hat{H}\,\Psi^{(\ell;n)} = \mathcal{E}^{(\ell;n)}\,\Psi^{(\ell;n)}, \;\; \mathcal{E}^{(\ell;n)} = m\,(2\kappa n + \ell)\,.
\tag{6.28}
$$

The «ground states» are annihilated by both supercharges

$$
Q^i\Omega^{(0;n)} = \bar{Q}_i\Omega^{(0;n)} = 0.
\tag{6.29}
$$

The true ground state annihilated also by $\hat{H}$ corresponds to $n = 0$ or $\kappa = 0$, $n \neq 0$. The second option shows up a degeneracy parametrized by the number n (see below). For generic κ, there is an infinite tower of the «ground states» parametrized by n, with the energy $\mathcal{E}^{(0;n)} = 2\kappa m n$. They all are annihilated by both supercharges. The combination $\hat{H} - m\hat{F}$ yields zero on all these states, but it cannot be chosen as the «genuine» Hamiltonian, since it generically *does not* commute with the supercharges (e.g., when acting on the states with $\ell \neq 0$). These surprising features of the quantum picture are in a sharp contrast with what happens in the standard $\mathcal{N} = 4$ SQM based on the chiral $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ multiplet (see, e.g., [18, 19]).

Since for each n we are dealing with finite-dimensional representations of $su(2|1)$ realized on the super wave functions, the Casimir operators are given by the same general expression as in (4.21). Using the formulas (6.28) and (6.12), we find that $\lambda = 1/2$ for any $\Psi^{(\ell;n)}$ and

$$C_2(\ell) = (\ell - 1)\ell, \quad C_3(\ell) = (\ell - 1/2)(\ell - 1)\ell, \quad \beta(\ell) = \ell - 1/2. \tag{6.30}$$

These values coincide with those pertinent to the oscillator model (Eq. (4.22)). Thus in the $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ model under consideration the Hilbert space is spanned by the same irreps of $SU(2|1)$ as in the oscillator $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ model. As distinct from the latter, at any fixed level ℓ one finds an infinite tower of irreps parametrized by $n \geqslant -\ell$ and exhibiting an equidistant energy spectrum, with spacing 2κ.

Supercharges do not depend on κ and, as a result, the super wave functions $\Psi^{(\ell;n)}$ involve no κ-dependent terms in their η-expansions. The parameter κ is still present in the Hamiltonian (6.7) and in the internal $U(1)$ generator $\hat{F}$. As was already mentioned, in the anticommutator $\{\hat{Q}, \hat{\bar{Q}}\}$ there appears just the combination $\hat{H} - m\hat{F}$, which involves no dependence on κ.

The norms of all super wave functions (6.27) are positive-definite. Using the inner product (6.20), we define the norms as

$$||\Psi^{(\ell;n)}||^2 = \frac{\langle \Psi^{(\ell;n)} | \Psi^{(\ell;n)} \rangle}{\langle \Omega^{(\ell;n)} | \Omega^{(\ell;n)} \rangle}. \tag{6.31}$$

This yields the following manifestly positive norms:

$$\begin{aligned} &||\Psi^{(\ell;n)}||^2 = \bar{a}^{(\ell;n)} a^{(\ell;n)} + \frac{\bar{b}^{(\ell;n)i} b_i^{(\ell;n)}}{m\ell} + \frac{\bar{c}^{(\ell;n)} c^{(\ell;n)}}{m^2(\ell - 1)\ell}, \quad \ell \geqslant 2, \\ &||\Psi^{(1;n)}||^2 = \bar{a}^{(1;n)} a^{(1;n)} + \frac{\bar{b}^{(1;n)i} b_i^{(1;n)}}{m}, \quad ||\Psi^{(0;n)}||^2 = \bar{a}^{(0;n)} a^{(0;n)}. \end{aligned} \tag{6.32}$$

6.3. Degeneracies. At some special values of κ the considered model reveals degeneracies in the energy spectrum, which amounts to the property that the symmetry algebra $su(2|1)$ is properly enhanced in these cases. Here we consider the cases $\kappa = 0$ and $\kappa = 1/2$, leaving the discussion of two other, more complicated options for Appendix.

6.3.1. $\kappa = 0$. In this case the $su(2|1)$ superalgebra is extended by the operators $(\nabla_z + im\bar{z})$, $(\bar{\nabla}_{\bar{z}} - imz)$, which commute with the Hamiltonian (6.7) and generate what is called the «magnetic translation» algebra [22]. These operators also commute with all other $su(2|1)$ generators, so we are dealing with a direct sum of $su(2|1)$ and the magnetic translation algebra. The associated degeneracy is revealed in the property that at any level ℓ the energy does not depend on the parameter n

$$\mathcal{E}^{(\ell;n)} = m\,\ell. \tag{6.33}$$

The wave function at the energy level ℓ is given by the sum

$$\Psi_\ell = \sum_{n=-\ell}^{\infty} s_{\ell\,n}\Psi^{(\ell;n)}, \quad \hat{H}_{(\kappa=0)}\Psi_\ell = m\,\ell\,\Psi_\ell, \tag{6.34}$$

where $s_{\ell\,n}$ are arbitrary coefficients. It is easy to check, that the ground state Ψ_0 have a simple expression through some antiholomorphic function $\mathcal{S}(\bar{z})$:

$$\Psi_0 = e^{-\frac{mz\bar{z}}{2}}\mathcal{S}(\bar{z}), \quad \mathcal{S}(\bar{z}) = \sum_{n=0}^{\infty} s_{0\,n}\,\bar{z}^n. \tag{6.35}$$

From this representation, it immediately follows, in particular, that

$$\hat{Q}^i\Psi_0 = \hat{\bar{Q}}_j\Psi_0 = 0. \tag{6.36}$$

6.3.2. $\kappa = 1/2$. In this case the fermionic terms entirely drop out from the Hamiltonian (6.7), so the latter becomes purely bosonic and commuting with the fermionic operators $\hat{\eta}^i$, $\hat{\bar{\eta}}_j$. It is easy to check that the $\kappa = 1/2$ Hamiltonian also commutes with the operators ∇_z, $\bar{\nabla}_{\bar{z}}$. The full set of the additional bosonic and fermionic integrals of motion, $\hat{\eta}^i$, $\hat{\bar{\eta}}_j$, ∇_z, $\bar{\nabla}_{\bar{z}}$, does not commute with the rest of the $su(2|1)$ generators; the superalgebra of these «magnetic supertranslations» forms a semi-direct sum with $su(2|1)$. The basic new nonvanishing (anti)commutators of this extended inhomogeneous superalgebra are given by

$$\begin{aligned} &[\hat{Q}^i, \bar{\nabla}_{\bar{z}}] = \sqrt{2}\,m\hat{\eta}^i, \qquad \{\hat{Q}^i, \hat{\bar{\eta}}_j\} = \sqrt{2}\delta^i_j\nabla_z, \\ &[\hat{\bar{Q}}_j, \nabla_z] = -\sqrt{2}\,m\hat{\bar{\eta}}_j \quad \{\hat{\bar{Q}}_i, \eta^j\} = \sqrt{2}\delta^j_i\bar{\nabla}_{\bar{z}}. \end{aligned} \tag{6.37}$$

The relevant infinite degeneracy of the energy levels is manifested in the structure of the generic super wave function for $2\kappa = 1$

$$\begin{gathered} \Psi_q = \sum_{\ell=0}^{\infty} d_{\ell q}\Psi^{(\ell;q-\ell)}, \quad \hat{H}_{(\kappa=1/2)}\Psi_q = mq\,\Psi_q, \\ q = 0, 1, 2\ldots, \quad (\mathcal{E} = mq), \end{gathered} \tag{6.38}$$

where $d_{\ell\,q}$ are some numerical coefficients. As distinct from the case (6.34), here the degeneracy arises between super wave functions belonging to different ℓ

levels, in accord with the property that the new symmetry generators $\hat{\eta}^i$, $\hat{\bar{\eta}}_j$, ∇_z, $\bar{\nabla}_{\bar{z}}$ mix various terms in the sum (6.38), for instance,

$$\hat{\eta}^i\Psi^{(\ell;q-\ell)} \sim \Psi^{(\ell+1,q-\ell-1)}, \quad \hat{\bar{\eta}}_j\Psi^{(\ell;q-\ell)} \sim \Psi^{(\ell-1,q-\ell+1)}, \quad \hat{\bar{\eta}}_j\Psi^{(0;q)} = 0.$$

The action of the bosonic operators ∇_z, $\bar{\nabla}_{\bar{z}}$ can be found from the relations (6.22). The $SU(2|1)$ supercharges take each super wave function in the sum (6.38) into itself.

The ground state Ψ_0 ($\mathcal{E} = 0, q = 0$) in this case is

$$\Psi_0 = \sum_{\ell=0}^{\infty} d_{\ell\,0}\Psi^{(\ell;-\ell)}. \tag{6.39}$$

Using the formula

$$\Omega^{(\ell;-\ell)} = e^{-\frac{mz\bar{z}}{2}}\,\frac{(-m)^\ell}{l!}\,z^\ell,$$

one can represent $\Psi^{(\ell;-\ell)}$ for a given ℓ as

$$\Psi^{(\ell;-\ell)} = e^{-\frac{mz\bar{z}}{2}}\left[\tilde{a}_{(\ell)}\,z^\ell + \tilde{b}_{(\ell)\,i}\,\eta^i z^{\ell-1} + \tilde{c}_{(\ell)}\,\varepsilon_{ij}\,\eta^i\eta^j\,z^{\ell-2}\right], \tag{6.40}$$

where $\tilde{a}_{(\ell)}, \tilde{b}_{(\ell)\,i}, \tilde{c}_{(\ell)}$ are arbitrary numerical coefficients, with the only restrictions $\tilde{b}_{(0)\,i} = \tilde{c}_{(0)} = \tilde{c}_{(1)} = 0$. Substituting this into the sum (6.39), we can present the ground-state wave function as

$$\Psi_0 = e^{-\frac{mz\bar{z}}{2}}\left[\mathcal{D}_0(z) + \mathcal{D}_{1i}(z)\,\eta^i + \mathcal{D}_2(z)\,\varepsilon_{ij}\,\eta^i\eta^j\right], \tag{6.41}$$

where $\mathcal{D}_0(z), \mathcal{D}_{1i}(z), \mathcal{D}_2(z)$ are arbitrary holomorphic functions, analytic at $z = 0$.

Clearly, this infinitely degenerated ground state is not annihilated by the supercharges $\hat{Q}^i, \bar{Q}_i$. Acting by the latter on the super wave functions (6.40), we observe that only $\Psi^{(0,0)} = \text{const}\,e^{-\frac{mz\bar{z}}{2}}$ is vanishing under this action, so it is the only $SU(2|1)$ singlet ground-state wave function. For any other ℓ we encounter nontrivial finite-dimensional representations of $SU(2|1)$. For $\ell = 1$, it is an atypical fundamental representation, with one bosonic and two fermionic vacuum states. At any $\ell \geqslant 2$, the vacuum states are grouped into the typical multiplets, with two bosonic and two fermionic states. The Casimir operators (6.12) take the values (6.30) on all these multiplets. Though $\hat{H}$ in (6.12) is zero for the vacuum states, the extra $U(1)$ charge generator $\hat{E}$ is nonvanishing, $\hat{E}\Psi^{(\ell;-\ell)} = -\ell\,\Psi^{(\ell;-\ell)}$.

7. CONCLUSIONS AND OUTLOOK

By this paper, we initiated the systematic study of new class of the deformed models of supersymmetric quantum mechanics, based upon the superfield approach. We constructed $d = 1$ superspace realizations of the simplest supergroup

$SU(2|1)$ which can be treated as a deformation of the $\mathcal{N}=4, d=1$ super Poincaré symmetry by mass parameter m. We showed that $SU(2|1), d=1$ supersymmetry admits off-shell realizations on the multiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{2},\mathbf{4},\mathbf{2})$, like its standard $\mathcal{N}=4, d=1$ prototype. The relevant most general superfield and component actions were constructed, the quantization was performed and, in a few simple cases, the eigenvalue problems for the relevant Hamiltonians were solved. In the $(\mathbf{1},\mathbf{4},\mathbf{3})$ case, our results basically coincide with those of [3], and we identify the weak supersymmetry models proposed there with the $SU(2|1)$ SQM models of single $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplet. The $SU(2|1)$ invariant off- and on-shell $(\mathbf{2},\mathbf{4},\mathbf{2})$ models are essentially new. Their basic novel features, as compared to the standard $\mathcal{N}=4$ supersymmetric $(\mathbf{2},\mathbf{4},\mathbf{2})$ SQM models, are the in-built presence of WZ terms in the component action and appearance of one more free parameter κ, that is the $U(1)$ charge associated with the internal $U(1)$ generator F. The presence of this parameter has a salient impact on the structure of the space of quantum states. For special values of κ there appear additional interesting degeneracies. For instance, in a simple model without WZ term and $\kappa=1/4$, the worldline $SU(2|1)$ symmetry is enhanced to $SU(2|2)$ (see Appendix).

For the oscillator $(\mathbf{1},\mathbf{4},\mathbf{3})$ and the plane $(\mathbf{2},\mathbf{4},\mathbf{2})$ SQM models we analyzed the $SU(2|1)$ representation contents of the space of quantum states and found that in both cases they necessarily involve at least one atypical $SU(2|1)$ irrep, with vanishing Casimir operators and unequal numbers of fermionic and bosonic states, apart from the singlet ground state (the equality is restored only in the special case of $\kappa=1/4$). Thus this mismatch between bosonic and fermionic excited states, observed for the first time in [3] for the $(\mathbf{1},\mathbf{4},\mathbf{3})$ models, seems to be a generic feature of the $SU(2|1)$ SQM models.

Our superfield approach enables an easy construction of the $SU(2|1)$ SQM models involving several $(\mathbf{1},\mathbf{4},\mathbf{3})$ and/or $(\mathbf{2},\mathbf{4},\mathbf{2})$ off-shell multiplets. Besides, the rest of nontrivial $\mathcal{N}=4$, $d=1$ multiplets, namely, the multiplets $(\mathbf{3},\mathbf{4},\mathbf{1})$ and $(\mathbf{4},\mathbf{4},\mathbf{0})$, seem also to have the appropriate $SU(2|1)$ counterparts, and it would be very interesting to consider the corresponding SQM models. Indeed, these multiplets are naturally described by superfields defined on the harmonic analytic $\mathcal{N}=4$ superspace [7], which has a counterpart among the admissible coset manifolds of the supergroup $SU(2|1)$. It is the following coset:

$$\frac{\{Q^i, \bar{Q}_j, H, I^i_j, F\}}{\{Q^1, \bar{Q}_2, F, I^1_2, I^1_1\}} \sim \{Q^2, \bar{Q}_1, H, I^2_1\}.$$

The corresponding coset coordinates include half the original θ coordinates, the time t, and additional harmonic coordinates of the complex internal coset $SU(2)/\{I^1_2, I^1_1\}$.

Besides setting up new $SU(2|1)$ SQM models along these lines and analyzing their hidden links with the «N-fold» supersymmetries [16], quasi-exactly solvable models [20] (a possible relation between the latter and weak supersymmetry was noticed in [3]), as well as the higher-dimensional models exhibiting curved rigid supersymmetries, there is one more intriguing problem for the future study. It would be tempting to extend our $d = 1$ superspace formalism to some higher-rank supergroups as the curved analogs of higher $\mathcal{N}$ one-dimensional Poincaré supersymmetries (1.1). The natural choice is the supergroup $SU(2|2)$ extending $SU(2|1)$. It involves eight supercharges and so is the appropriate candidate for deformed $\mathcal{N} = 8$, $d = 1$ supersymmetry. The closure of the $SU(2|2)$ supercharges contains two commuting $SU(2)$ subalgebras and so this supergroup admits as its supercosets, besides the standard harmonic analytic superspace, also analogs of the biharmonic analytic superspaces [21].

Actually, this supergroup allows three independent central charges, and two of them can be identified with two light-cone projections of the $d = 2$ translation operator. Thus such a centrally-extended $SU(2|2)$ could also be employed as a kind of $d = 2$ «weak supersymmetry», and the question is whether one can construct nontrivial $d = 2$ sigma models based on such a deformation of the flat $\mathcal{N} = (4, 4)$ $d = 2$ supersymmetry. The problem of generalizing the $d = 1$ weak supersymmetry to $d = 2$ was posed in [3]*. We would like to point out that various versions of the $SU(2|2)$ supersymmetry already appeared in the literature as the worldvolume on-shell symmetry of the Pohlmeyer-reduced $AdS_3 \times S^3$ and $AdS_5 \times S^5$ superstrings [23], as well as the worldline on-shell symmetry of $\mathcal{N} = 4$ supersymmetric Landau problem [24]. The relevant off-shell superfield formalism could help in getting further insights into the symmetry structure of these and similar $d = 1, 2$ theories of current interest.

Acknowledgements. We acknowledge support from the RFBR grants Nr. 12-02-00517 and Nr. 11-02-90445. We are indebted to Sergey Fedoruk for interest in the work. E. I. thanks Andrei Smilga for a discussion of weak supersymmetry.

Appendix

MORE ON DEGENERACIES IN THE MODEL ON A PLANE

A.1. $\kappa = 1/4$**.** This case is distinguished in that the WZ term in the Lagrangian (6.2) and, respectively, the coupling to the magnetic field in the Hamiltonian (6.11), disappear:

$$\mathcal{L}_{(\kappa=1/4)} = \dot{\bar{z}}\dot{z} + \frac{i}{2}\left(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i\right) - \frac{1}{4}m^2\bar{z}z + \frac{1}{2}m\left(\bar{\xi}\cdot\xi\right), \tag{A.1}$$

$$\hat{H}_{(\kappa=1/4)} = -\partial_{\bar{z}}\,\partial_z + \frac{m^2}{4}\,\bar{z}z + \frac{m}{2}\,\hat{\eta}^k\hat{\bar{\eta}}_k. \tag{A.2}$$

*The possible existence of such unusual $d = 2$ supersymmetric systems does not contradict the renowned Coleman–Mandula and Haag–Lopushanski–Sohnius no-go theorems, as the latter do not apply to one and two dimensions.

The Hamiltonian (A.2) is a fermionic extension of the two-dimensional oscillator Hamiltonian. It commutes with the operators

$$\hat{F}_{+} = \frac{1}{m}\nabla_z\left(\nabla_z + im\bar{z}\right), \quad \hat{F}_{-} = \frac{1}{m}\bar{\nabla}_{\bar{z}}\left(\bar{\nabla}_{\bar{z}} - imz\right). \tag{A.3}$$

Together with $\hat{F} = -(z\partial_z - \bar{z}\partial_{\bar{z}})$, they form the new algebra $su(2)$, which commutes with the original $su(2)$ generators $\hat{I}^i_j$. This extra $su(2)$ algebra is none other than the well-known hidden $su(2)$ symmetry algebra of the two-dimensional harmonic oscillator [25].

Acting by the new $su(2)$ generators on the $SU(2|1)$ supercharges, we obtain the new complex doublet of supercharges $\hat{S}^i, \hat{\bar{S}}_j$:

$$\hat{S}^i = \sqrt{2}\,\hat{\eta}^i\left(\bar{\nabla}_{\bar{z}} - imz\right), \quad \hat{\bar{S}}_j = \sqrt{2}\,\hat{\bar{\eta}}_j\left(\nabla_z + im\bar{z}\right). \tag{A.4}$$

The operators (A.3), (A.4) extend the superalgebra $su(2|1)$ to the centrally extended superalgebra $su(2|2)$, with $\hat{H}$ as the central charge generator:

$$\begin{aligned}
&\{\hat{Q}^i, \hat{\bar{Q}}_j\} = 2\delta^i_j\hat{H} + 2m\left(\hat{I}^i_j - \delta^i_j\hat{F}\right), \quad \{\hat{Q}^i, \hat{\bar{S}}_j\} = 2m\delta^i_j\hat{F}_{+},\\
&\{\hat{S}^i, \hat{\bar{S}}_j\} = 2\delta^i_j\hat{H} + 2m\left(\hat{I}^i_j + \delta^i_j\hat{F}\right), \quad \{\hat{S}^i, \hat{\bar{Q}}_j\} = 2m\delta^i_j\hat{F}_{-},\\
&\left[\hat{I}^i_j, \hat{\bar{Q}}_l\right] = \frac{1}{2}\delta^i_j\hat{\bar{Q}}_l - \delta^i_l\hat{\bar{Q}}_j, \quad \left[\hat{I}^i_j, \hat{Q}^k\right] = \delta^k_j\hat{Q}^i - \frac{1}{2}\delta^i_j\hat{Q}^k,\\
&\left[\hat{I}^i_j, \hat{\bar{S}}_l\right] = \frac{1}{2}\delta^i_j\hat{\bar{S}}_l - \delta^i_l\hat{\bar{S}}_j, \quad \left[\hat{I}^i_j, \hat{S}^k\right] = \delta^k_j\hat{S}^i - \frac{1}{2}\delta^i_j\hat{S}^k,\\
&\left[\hat{F}, \hat{\bar{Q}}_l\right] = -\frac{1}{2}\hat{\bar{Q}}_l, \quad \left[\hat{F}, \hat{Q}^k\right] = \frac{1}{2}\hat{Q}^k, \quad \left[\hat{F}_{+}, \hat{\bar{Q}}_l\right] = \hat{\bar{S}}_l, \quad \left[\hat{F}_{-}, \hat{S}^i\right] = -\hat{S}^i,\\
&\left[\hat{F}, \hat{\bar{S}}_l\right] = \frac{1}{2}\hat{\bar{S}}_l, \quad \left[\hat{F}, \hat{S}^k\right] = -\frac{1}{2}\hat{S}^k, \quad \left[\hat{F}_{-}, \hat{\bar{S}}_l\right] = \hat{\bar{Q}}_l, \quad \left[\hat{F}_{-}, \hat{Q}^i\right] = -\hat{Q}^i,\\
&\left[\hat{I}^i_j, \hat{I}^k_l\right] = \delta^k_j\hat{I}^i_l - \delta^i_l\hat{I}^k_j, \quad \left[\hat{F}_{+}, \hat{F}_{-}\right] = 2\hat{F}, \quad \left[\hat{F}, \hat{F}_{\pm}\right] = \pm\hat{F}_{\pm}.
\end{aligned} \tag{A.5}$$

The extra $U(1)$ generator $\hat{E}$ defines an outer automorphism of the extended algebra,

$$\left[\hat{E}, \hat{\bar{S}}_l\right] = 2\hat{\bar{S}}_l, \quad \left[\hat{E}, \hat{S}^k\right] = -2\hat{S}^k, \quad \left[\hat{E}, \hat{F}_{\pm}\right] = \pm 2\hat{F}_{\pm}. \tag{A.6}$$

At any fixed energy $\mathcal{E}_q = mq = m(\ell + n/2)$, the numbers n and ℓ take the values indicated in the Table

n	$2q$	$2q-2$	$\ldots$	0	$\ldots$	$-2q+2$	$-2q$
ℓ	0	1	$\ldots$	q	$\ldots$	$2q-1$	$2q$

.

The number $q = \ell + n/2$ can be nonnegative integer or half-integer:

$$q = 0, 1/2, 1, 3/2 \ldots \tag{A.7}$$

The super wave function Ψ_q with the energy $\mathcal{E}_q$ is given by the finite sum

$$\Psi_q = \sum_{\ell=0}^{2q} r_{\ell\, q} \Psi^{(\ell; 2q-2\ell)}, \tag{A.8}$$

with $r_{\ell\, q}$ being some coefficients. This means that the super wave function Ψ_q has a finite degeneracy. The ground state ($\mathcal{E} = 0, q = 0$) is a $su(2|2)$ singlet. The excited states Ψ_q, with

$$q = 1/2, 1, 3/2 \ldots, \tag{A.9}$$

are combined into the $SU(2|2)$ multiplets of dimension $8q$, whence the degeneracy $8q$.

This multiplet structure and degeneracy become manifest, when counting the numbers of states in the η-expansion of Ψ_q:

$\eta-$monomial	1	η	η^2
degeneracy	$2q+1$	$4q$	$2q-1$

.

Each excited energy level has an equal number of bosonic and fermionic states which form some «short» $SU(2|2)$ multiplet. Recall that such multiplets are characterized by the «triple» [26]

$$\langle n_1, n_2, \vec{C} \rangle, \tag{A.10}$$

where n_1, n_2 are some positive integer numbers and $\vec{C}$ is a three-dimensional vector with the three admissible $SU(2|2)$ central charges as the components. The dimensionality of such a short multiplet is given by the formula

$$d = 4(n_1 + 1)(n_2 + 1) + 4n_1 n_2. \tag{A.11}$$

Our case with one central charge corresponds to $n_1 = 2q - 1, n_2 = 0$ and $\vec{C} = \left(\hat{H}/m, 0, 0\right)$, i.e., to the triple

$$\langle 2q-1, 0, \vec{C} \rangle, \quad \vec{C} = (q, 0, 0). \tag{A.12}$$

Equation (A.11) implies just $d = 8q$ for these short $SU(2|2)$ multiplets.

The simplest such multiplet, with $q = 1/2$, encompasses two super wave functions,

$$\left(\Psi^{(0;1)},\ \Psi^{(1;-1)}\right),$$

and so involves two bosonic and two fermionic states. It is a sum of the singlet and an atypical fundamental $su(2|1)$ multiplets. The additional supercharges S^i and $\bar{S}_j$ transform these wave functions into each other:

$$S^i\Psi^{(0;1)} \sim \Psi^{(1;-1)},\; S^i\Psi^{(1;-1)} = 0, \quad \bar{S}^i\Psi^{(0;1)} = 0,\; \bar{S}^i\Psi^{(1;-1)} \sim \Psi^{(0;1)}.$$

Our last observation concerning the $\kappa = 1/4$ case is as follows. Define the new supercharges

$$\hat{\Pi}^i = \hat{Q}^i + \varepsilon^{ij}\hat{\bar{S}}_j, \quad \hat{\bar{\Pi}}_i = \hat{\bar{Q}}_i - \varepsilon_{ij}\hat{S}^j, \tag{A.13}$$

one can check that they form $\mathcal{N} = 4$ Poincaré superalgebra,

$$\left\{\hat{\Pi}^i, \hat{\bar{\Pi}}_j\right\} = 4\delta^i_j\hat{H}, \quad \left\{\hat{\Pi}^i, \hat{\Pi}^j\right\} = 0, \tag{A.14}$$

which is a subalgebra of the centrally-extended $su(2|2)$, along with $su(2|1)$. This means that the Hamiltonian (A.2) possesses the standard $\mathcal{N} = 4$, $d = 1$ supersymmetry too.

The same phenomenon manifests itself as the property that the on-shell Lagrangian (A.1) can be equivalently constructed by eliminating the auxiliary fields in the simple $\mathcal{N} = 4, d = 1$ superfield action of the standard chiral $\mathcal{N} = 4, d = 1$ multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$:

$$\mathcal{L} = \frac{1}{4}\int d^2\tilde{\theta}\, d^2\tilde{\bar{\theta}}\, \tilde{\Phi}\, \tilde{\Phi}^\dagger + \frac{m}{2}\left[\int d^2\tilde{\theta}\, \tilde{\Phi}^2 + \text{c.c.}\right], \tag{A.15}$$

where $\tilde{\Phi}, \tilde{\Phi}^\dagger$ are left and right chiral superfields. Commuting $su(2|1)$ superalgebra and $\mathcal{N} = 4$ Poincaré superalgebra realized on the same on-shell set $(z, \bar{z}, \eta^i, \bar{\eta}_i)$, we recover the centrally-extended $su(2|2)$ superalgebra as the closure of these two subalgebras.

Reversing the argument, one can say that the simplest $\mathcal{N} = 4$ superextension of the two-dimensional harmonic oscillator possesses hidden symmetries $su(2|1)$ and $su(2|2)$. We have failed to find such a statement in the literature. For the time being, we do not know whether this surprising duality extends off shell.

A.2. General Rational κ. Finally, we briefly consider the case when κ takes the rational values within the range $0 < \kappa < 1/2$. Such rational κ can be represented as $\kappa = a/2b$, where a and $b > a$ are positive integers. The energies take the values

$$\hat{H}_{(\kappa=a/2b)}\Psi_q = \mathcal{E}\,\Psi_{\mathcal{E}} = mq\,\Psi_q, \quad q = \frac{1}{b}(an + b\ell) \geqslant 0. \tag{A.16}$$

Shifting n and ℓ as

$$n \to n \pm b, \quad \ell \to \ell \mp a, \tag{A.17}$$

one keeps the energy intact. Making such a shift twice or more times, we always obtain the same energy mq. According to the relations (6.22), such shifts can be accomplished by the successive action on the super wave function $\Psi^{(\ell;n)}$ by the operators which are composed as the proper products of the operators

$$\nabla_z, \quad (\nabla_z + im\bar{z}), \quad \bar{\nabla}_{\bar{z}}, \quad \left(\bar{\nabla}_{\bar{z}} - imz\right), \tag{A.18}$$

and commute with $\hat{H}_{(\kappa=a/2b)}$. These products are uniquely found to be the powers of the following «elementary» operators:

$$\begin{aligned} &J_{(ab)} = (\nabla_z)^a (\nabla_z + im\bar{z})^{b-a}, \quad \bar{J}_{(ab)} = \left(\bar{\nabla}_{\bar{z}}\right)^a \left(\bar{\nabla}_{\bar{z}} - imz\right)^{b-a}, \quad b > a > 0, \\ &J_{(ab)}\Psi^{(\ell;n)} \sim \Psi^{(\ell-a;n+b)}, \qquad \bar{J}_{(ab)}\Psi^{(\ell;n)} \sim \Psi^{(\ell+a;n-b)}. \end{aligned} \tag{A.19}$$

Note that the successive action of the operators $J, \bar{J}$ on $\Psi^{(\ell;n)}$ according to (A.19) cannot take the numbers ℓ and n out of the range of their definition, i.e., $\ell \geqslant 0$, $n \geqslant -\ell$. This can be checked, using the relations (6.22). For the boundary values $a = 0$, $\kappa = 0$ and $a = b$, $\kappa = 1/2$, these operators become the powers of the more elementary operators $(\nabla_z + im\bar{z})$, $\left(\bar{\nabla}_{\bar{z}} - imz\right)$ and $\nabla_z, \bar{\nabla}_{\bar{z}}$, which belong to the extended symmetry algebras of these special cases. For $\kappa = 1/4$ such elementary generators are just $\hat{F}_{\pm}$, Eq. (A.3), and they correspond to the simplest suitable choice $a = 1$, $b = 2$. In the generic case the operators $J_{ab}, \bar{J}_{ab}$ cannot be reduced to the more elementary operators commuting with $\hat{H}$, and so they constitute nonlinear symmetry algebras. Commuting them with the $su(2|1)$ generators, we obtain some nonlinear extensions of $su(2|1)$*.

REFERENCES

1. *Festuccia G., Seiberg N.* Rigid Supersymmetric Theories in Curved Superspace // JHEP. 2011. V. 1106. P. 114; arXiv:1105.0689 [hep-th];
   Dumitrescu T. T., Festuccia G., Seiberg N. Exploring Curved Superspace // JHEP. 2012. V. 1208. P. 141; arXiv:1205.1115 [hep-th].
2. *Witten E.* Dynamical Breaking of Supersymmetry // Nucl. Phys. B. 1981. V. 188. P. 513; Constraints on Supersymmetry Breaking // Nucl. Phys. B. 1982. V. 202. P. 253.
3. *Smilga A. V.* Weak Supersymmetry // Phys. Lett. B. 2004. V. 585. P. 173; arXiv:hep-th/0311023.
4. *Bellucci S., Nersessian A.* (Super)Oscillator on CP(N) and Constant Magnetic Field // Phys. Rev. D. 2003. V. 67. P. 065013; arXiv:hep-th/0211070.

*A nonlinear extension of $su(2|2)$ in the quantum-mechanical context was, e.g., considered in [27].

5. *Bellucci S., Nersessian A.* Supersymmetric Kähler Oscillator in a Constant Magnetic Field. arXiv:hep-th/0401232.
6. *Scheunert M., Nahm W., Rittenberg V.* Irreducible Representations of the $osp(2,1)$ and $spl(2,1)$ Graded Lie Algebras // J. Math. Phys. 1977. V. 18. P. 155.
7. *Ivanov E., Lechtenfeld O.* $N = 4$ Supersymmetric Mechanics in Harmonic Superspace // JHEP. 2003. V. 0309. P. 073; arXiv:hep-th/0307111.
8. *Delduc F., Ivanov E.* Gauging $N = 4$ Supersymmetric Mechanics // Nucl. Phys. B. 2006. V. 753. P. 211; arXiv:hep-th/0605211.
9. *Beylin A., Curtright T. L., Ivanov E., Mezincescu L., Townsend P. K.* Unitary Spherical Super Landau Models // JHEP. 2008. V. 0810. P. 069; arXiv:0806.4716 [hep-th].
10. *Ivanov E., Mezincescu L., Townsend P. K.* A Superflag Landau Model // From Fields to Strings. V. 3. P. 2123–2146; arXiv:hep-th/0404108.
11. *Ivanov E. et al.* Odd Coset Quantum Mechanics // Phys. Lett. B. 2003. V. 566. P. 175; arXiv:hep-th/0301241.
12. *Goykhman M., Ivanov E., Sidorov S.* Super Landau Models on Odd Cosets // Phys. Rev. D. 2013. V. 87. P. 025026; arXiv:1208.3418 [hep-th].
13. *Ivanov E., Krivonos S., Lechtenfeld O.* $N = 4$, $d = 1$ Supermultiplets from Nonlinear Realizations of $D(2,1;\alpha)$ // Class. Quant. Grav. 2004. V. 21. P. 1031; arXiv:hep-th/0310299.
14. *Ivanov E. A., Krivonos S. O., Leviant V. M.* Geometric Superfield Approach to Superconformal Mechanics // J. Phys. A. 1989. V. 22. P. 4201.
15. *Ivanov E. A., Krivonos S. O., Pashnev A. I.* Partial Supersymmetry Breaking in $N = 4$ Supersymmetric Quantum Mechanics // Class. Quant. Grav. 1991. V. 8. P. 19.
16. *Andrianov A. A., Ioffe M. V.* Nonlinear Supersymmetric Quantum Mechanics: Concepts and Realizations // J. Phys. A. 2012. V. 45. P. 503001; arXiv:1207.6799 [hep-th].
17. *Berezovoi V., Pashnev A.* On the Structure of the $N = 4$ Supersymmetric Quantum Mechanics in $D = 2$ and $D = 3$ // Class. Quant. Grav. 1996. V. 13. P. 1699; arXiv:hep-th/9506094;
Bellucci S., Nersessian A. A Note on $N = 4$ Supersymmetric Mechanics on Kähler Manifolds // Phys. Rev. D. 2001. V. 64. P. 021702; arXiv:hep-th/0101065.
18. *Smilga A. V.* How To Quantize Supersymmetric Theories // Nucl. Phys. B. 1987. V. 292. P. 363.
19. *Ivanov E. A., Smilga A. V.* Dirac Operator on Complex Manifolds and Supersymmetric Quantum Mechanics // Int. J. Mod. Phys. A. 2012. V. 27. P. 1230024; arXiv:1012.2069 [hep-th].
20. *Turbiner A.* Quasi-Exactly-Solvable Problems and $SL(2)$ Group // Commun. Math. Phys. V. 118. 1988. P. 467;
Ushveridze A. // Sov. J. Part. Nucl. 1989. V. 20. P. 504.
21. *Ivanov E., Niederle J.* Biharmonic Superspace for $N = 4$ Mechanics // Phys. Rev. D. 2009. V. 80. P. 065027; arXiv:0905.3770 [hep-th].
22. *Curtright T., Ivanov E., Mezincescu L., Townsend P. K.* Planar Super Landau Models Revisited // JHEP. 2007. V. 0704. P. 020; arXiv:hep-th/0612300.

23. *Hollowood T. J., Miramontes J. L.* The $AdS_5 \times S^5$ Semi-Symmetric Space sine-Gordon Theory // JHEP. 2011. V. 1105. P. 136; arXiv:1104.2429 [hep-th];
Goykhman M., Ivanov E. Worldsheet Supersymmetry of Pohlmeyer-Reduced $AdS_n \times S^n$ Superstrings // JHEP. 2011. V. 1109. P. 078; arXiv:1104.0706 [hep-th];
Schmidtt D. M. Supersymmetry Flows, Semi-Symmetric Space sine-Gordon Models and the Pohlmeyer Reduction // JHEP. 2011. V. 1103. P. 021; arXiv:1012.4713 [hep-th].
24. *Bychkov V., Ivanov E.* $\mathcal{N} = 4$ Supersymmetric Landau Models // Nucl. Phys. B. 2012. V. 863. P. 33; arXiv:1202.4984 [hep-th].
25. *Perelomov A.* Integrable Systems of Classical Mechanics and Lie Algebras: V. 1. Birkhäuser Verlag, 1990. 307 p.
26. *Beisert N.* The Analytic Bethe Ansatz for a Chain with Centrally Extended $su(2|2)$ Symmetry // J. Stat. Mech. 2007. V. 0701. P. P01017; arXiv:nlin/0610017.
27. *Correa F., Nieto L.-M., Plyushchay M. S.* Hidden Nonlinear $su(2|2)$ Superunitary Symmetry of $N = 2$ Superextended 1D Dirac Delta Potential Problem // Phys. Lett. B. 2008. V. 659. P. 746; arXiv:0707.1393 [hep-th].

Received on July 26, 2013.

Корректор *Т. Е. Попеко*

Подписано в печать 24.09.2013.
Формат $60 \times 90/16$. Бумага офсетная. Печать офсетная.
Усл. печ. л. 2,43. Уч.-изд. л. 3,27. Тираж 325 экз. Заказ № 58071.

Издательский отдел Объединенного института ядерных исследований
141980, г. Дубна, Московская обл., ул. Жолио-Кюри, 6.
E-mail: publish@jinr.ru
www.jinr.ru/publish/