E2-2014-93

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SYMMETRY AND QUANTUM EFFECTS OF EXTRA DIMENSION OF SPACE

Submitted to “Modern Physics Letters A”

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Симметрийные и квантовые эффекты дополнительной размерности пространства

Предполагается, что физическое пространство имеет дополнительную размерность. Возможные эффекты этой инновации исследованы на примере простейшего математического пространства — четырехмерного евклидова пространства. Показано, что причина и происхождение вращательного движения ясно видны только в четырехмерии. Рассмотрена пара уравнений Дирака, определяемых естественными тетрадами на четырехмерном евклидовом пространстве. Сравнивая эти уравнения с оригинальным уравнением Дирака в пространстве-времени Минковского, мы выяснили, что на четырехмерном евклидовом пространстве существует две причинные структуры. Этим дано рациональное доказательство существования лептонов и кварков, кварк-лептонной симметрии и конфайнмента. С целью проиллюстрировать некоторые вопросы, связанные с так называемой скрытой симметрией и скрытыми размерностями, изучены естественные отображения четырехмерного евклидова пространства на трехмерное евклидово пространство.

Работа выполнена в Лаборатории теоретической физики им. Н. Н. Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 2014

Pestov I. B. E2-2014-93
Symmetry and Quantum Effects of Extra Dimension of Space

It is supposed that the physical space has extra dimension. Possible effects of this innovation are investigated on the example of the simplest mathematical space — the four-dimensional Euclidian space. It is demonstrated that the origin and the nature of a rotational motion are clearly visible in four dimensions only. A pair of the Dirac equations associated with the natural tetrads on the four-dimensional Euclidian space is considered. Comparing these equations with the original Dirac equation in the Minkowski space-time, we show that there are two causal structures on the four-dimensional Euclidian space. With this, the rational proof of the existence of leptons and quarks, lepton–quark symmetry and confinement is obtained. To illustrate some questions connected with the so-called hidden symmetry and hidden dimensions, natural mappings of the four-dimensional Euclidian space onto the three-dimensional one are considered.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 2014

INTRODUCTION

One of the primary goals of physics is to understand the variety of physical reality in a unified way: a single mathematical framework, in which all fundamental forces and units of matter can be described together in a manner that is internally consistent, and consistent with current and future observations. The greatest advances have been steps towards this goal: the unification of terrestrial and celestial mechanics by Newton; of optics with the theories of electricity and magnetism by Maxwell; of geometry and the theory of gravitation by Einstein; of electromagnetism with weak interactions, but the work of unification can be completed if gravity is included. Experiments at the LHC and elsewhere should let us complete the Standard Model, but a unified theory will require first of all a solution of the conceptual problems and probably radically new ideas. Thus, motivation for our study is clear. Here, we should like to understand the physical meaning of the fourth dimension. What does it mean? The mathematical space is a smooth manifold. A smooth manifold, or C^∞-manifold, is a differentiable manifold, for which all the transition maps are smooth. That is, derivatives of all orders exist; so, it is a C^k-manifold for all k. An atlas on the topological space M is a collection of pairs $(U_\alpha, \varphi_\alpha)$ called charts, where U_α are open sets that cover M, and for each index α

$$\varphi_\alpha \colon U_\alpha \to R^n$$

is a homeomorphism of U_α onto an open subset of n-dimensional real space. An equivalence class of such atlases is said to be a smooth structure. The underlying space R^n is the space of n tuples of real numbers $(q^1, q^2, \cdots, q^n)$ with usual topology. Here, we should like to emphasize that coordinates $q^1, q^2, \cdots, q^n$ should be considered on equal footing, but space-time is a mathematical space, whose points must be specified by both space and time coordinates. However, it is clear that there is no regular method to introduce space coordinates and time coordinate in the framework of a smooth manifold alone. Our goal is to recognize a regular transition from the mathematical space to space-time and with this to derive new information about the nature of space and time. To this end, we consider the Dirac equation in the mathematical space. A comparison will be produced of the Dirac theory of the electron with spin in the simplest four-dimensional mathematical space and the original Dirac theory in the Minkowski space-time. New representations about nature of space, time, rotation, quark–lepton symmetry and confinement will be derived from this consideration.

1. DIRAC EQUATION IN A MATHEMATICAL SPACE

The Dirac equation in the four-dimensional mathematical space reads [1]:

$$i\gamma^\mu D_\mu \psi = m\psi, \tag{1}$$

where D_μ are linear differential operators

$$D_\mu = E^i_\mu \partial_i = E^1_\mu \frac{\partial}{\partial q^1} + E^2_\mu \frac{\partial}{\partial q^2} + E^3_\mu \frac{\partial}{\partial q^3} + E^4_\mu \frac{\partial}{\partial q^4}$$

and E^i_μ are quadruplets of linear independent vector fields, which will be considered as components of a tetrad field or simply tetrad. The tetrad has sixteen components. The gamma matrices γ^μ are normalized as follows:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}\,(1,\,-1,\,-1,\,-1).$$

The world indices of vectors are denoted by latin letters $i, j, k, \dots = 1, 2, 3, 4$, and the greek letters enumerate the vector fields in question $\mu, \nu \dots = 0, 1, 2, 3$. For simplicity, we consider the case when the vector part of the torsion tensor

$$H^i_{jk} = E^i_\mu(\partial_j E^\mu_k - \partial_k E^\mu_j)$$

is equal to zero, $H^i_{ik} = 0$. Since

$$\gamma^\mu D_\mu = \gamma^0 D_0 + \gamma^1 D_1 + \gamma^2 D_2 + \gamma^3 D_3,$$

then to get a regular transition from the mathematical space to space-time, we need to introduce the system of coordinates x^1, x^2, x^3, t, in which the linear differential operator D_0 takes the form

$$D_0 = E^i_0 \partial_i \to \frac{\partial}{\partial t}.$$

To this end, let us consider the system of ordinary differential equations

$$\frac{dq^i}{dt} = E^i_0(q^1, q^2, q^3, q^4).$$

It is well known that this system has a unique solution

$$q^1 = f_1(q^1_0, q^2_0, q^3_0, q^4_0, t), \cdots, q^4 = f_4(q^1_0, q^2_0, q^3_0, q^4_0, t),$$

which satisfies the condition

$$q_0^1 = f_1(q_0^1, q_0^2, q_0^3, q_0^4, t_0), \cdots, q_0^4 = f_4(q_0^1, q_0^2, q_0^3, q_0^4, t_0).$$

Let the initial point $P(q_0^1, q_0^2, q_0^3, q_0^4)$ belong to the three-dimensional surface S, which is parameterized by the coordinates x^1, x^2, x^3

$$\begin{aligned} q_0^1 &= h_1(x^1, x^2, x^3), \quad q_0^2 = h_2(x^1, x^2, x^3), \\ q_0^3 &= h_3(x^1, x^2, x^3), \quad q_0^4 = h_4(x^1, x^2, x^3). \end{aligned}$$

The surface S should be chosen so that the variables x^1, x^2, x^3, t define a new system of coordinates in the mathematical space and the tetrad takes the following form:

$$\begin{aligned} E_0^i &= (0, 0, 0, 1), & E_1^i &= (E_1^1, E_1^2, E_1^3, 0), \\ E_2^i &= (E_2^1, E_2^2, E_2^3, 0), & E_3^i &= (E_3^1, E_3^2, E_3^3, 0). \end{aligned}$$

The so-defined surface will be called the characteristic surface of space-time, the variables x^1, x^2, x^3 will be called the space coordinates, and accordingly t will be the time coordinate. We conclude that the space-time is a causal structure on the mathematical space, which is defined by the vector field or congruence of lines. We recall that the congruence of lines is a set of lines characterized by that the only element of the set crosses every point of a manifold or its part. The lines belonging to the congruence do not intersect and fill either the whole manifold or its part. In the mathematical space equipped by the causal structure, Eq. (1) takes the Hamiltonian form

$$i\frac{\partial}{\partial t}\psi = H\psi,$$

where the operator H does not contain the partial derivative up to t. After this general consideration, we start to learn the simplest four-dimensional mathematical space R^4.

2. SYMMETRY AND GEOMETRICAL ASPECTS OF THE FOUR-DIMENSIONAL EUCLIDIAN SPACE

Points of R^4 have the vector

$$\mathbf{q} = (q^1, q^2, q^3, q^4)$$

and the quaternion representations

$$q = q^1 i + q^2 j + q^3 k + q^4 1,$$

with the usual linear structure. The quaternion algebra is defined as usual:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

The scalar product

$$\mathbf{p} \cdot \mathbf{q} = p^1q^1 + p^2q^2 + p^3q^3 + p^4q^4$$

can be written in the quaternion form in two ways:

$$\mathbf{p} \cdot \mathbf{q} = \frac{1}{2}(p\overline{q} + q\overline{p}) = \frac{1}{2}(\overline{p}q + \overline{q}p), \tag{2}$$

where $\overline{q} = -q^1 i - q^2 j - q^3 k + q^4 1$. The scalar product is invariant with respect to the right- and left-turn dilatations

$$q \Rightarrow \tilde{q} = s\,q, \quad \Rightarrow \tilde{q} = q\,\overline{t}, \tag{3}$$

since

$$\tilde{\mathbf{p}} \cdot \tilde{\mathbf{q}} = s\overline{s}\,(\mathbf{p} \cdot \mathbf{q}), \quad \tilde{\mathbf{p}} \cdot \tilde{\mathbf{q}} = t\overline{t}\,(\mathbf{p} \cdot \mathbf{q}).$$

We suppose that q and λq, where λ is a number, are equivalent. For a given q, equations $q = sq$, $q = q\overline{t}$ have only trivial solutions $s = \overline{t} = 1$, and the absence of fixed points under turn dilatations exhibits a fundamental property of the space in question: the existence of two simply transitive groups of transformations.

Now, we introduce two natural frames intrinsically connected with the space in question. The standard frame

$$\mathbf{c}_1 = (1,\ 0,\ 0,\ 0) \quad \mathbf{c}_2 = (0,\ 1,\ 0,\ 0) \quad \mathbf{c}_3 = (0,\ 0,\ 1,\ 0) \quad \mathbf{c}_4 = (0,\ 0,\ 0,\ 1),$$

$$c_1 = i, \quad c_2 = j, \quad c_3 = k, \quad c_4 = 1$$

gives rise to the pair of right-handled moving frames

$$m_1 = iq,\ m_2 = jq,\ m_3 = kq,\ m_4 = 1q, \quad n_1 = qi,\ n_2 = qj,\ n_3 = qk,\ n_4 = q1.$$

$$\begin{aligned}
\mathbf{m}_1 &= (\ q^4, & -q^3, && q^2, && -q^1\ ),\\
\mathbf{m}_2 &= (\ q^3, & q^4, && -q^1, && -q^2\ ),\\
\mathbf{m}_3 &= (-q^2, & q^1, && q^4, && -q^3\ ),\\
\mathbf{m}_4 &= (\ q^1, & q^2, && q^3, && q^4\ ),
\end{aligned}$$

$$\begin{aligned}
\mathbf{n}_1 &= (\ q^4, & q^3, && -q^2, && -q^1\ ),\\
\mathbf{n}_2 &= (-q^3, & q^4, && q^1, && -q^2\ ),\\
\mathbf{n}_3 &= (\ q^2, & -q^1, && q^4, && -q^3\ ),\\
\mathbf{n}_4 &= (\ q^1, & q^2, && q^3, && q^4\ ).
\end{aligned}$$

It is easy to see that

$$\mathbf{m}_a \cdot \mathbf{m}_b = q\bar{q}\,\delta_{ab}, \quad \mathbf{n}_a \cdot \mathbf{n}_b = q\bar{q}\,\delta_{ab}, \quad (a, b = 1, 2, 3, 4).$$

Let us consider the running point $T(q_1,\, q_2,\, q_3,\, q_4)$, and the twelve coherent points

$$A(q^4,\, -q^3,\, q^2,\, -q^1), \quad B(q^3,\, q^4,\, -q^1,\, -q^2), \quad C(-q^2,\, q^1,\, q^4,\, -q^3),$$

$$K(q^4,\, q^3,\, -q^2,\, -q^1), \quad L(-q^3,\, q^4,\, q^1,\, -q^2), \quad M(q^2,\, -q^1,\, q^4,\, -q^3),$$
$$\bar{A}(-q^4,\, q^3,\, -q^2,\, q^1), \quad \bar{B}(-q^3,\, -q^4,\, q^1,\, q^2), \quad \bar{C}(q^2,\, -q^1,\, -q^4,\, q^3),$$
$$\bar{K}(-q^4,\, -q^3,\, q^2,\, q^1), \quad \bar{L}(q^3,\, -q^4,\, -q^1,\, q^2), \quad \bar{M}(-q^2,\, q^1,\, -q^4,\, q^3).$$

The distance function is defined as usual:

$$d^2_{PQ} = (p^1 - q^1)^2 + (p^2 - q^2)^2 + (p^3 - q^3)^2 + (p^4 - q^4)^2.$$

With this, it is easy to see that

$$d^2_{AB} = d^2_{AC} = d^2_{BC} = d^2_{TA} = d^2_{TB} = d^2_{TC} = 2q\bar{q},$$

$$d^2_{\bar{A}\bar{B}} = d^2_{\bar{A}\bar{C}} = d^2_{\bar{B}\bar{C}} = d^2_{T\bar{A}} = d^2_{T\bar{B}} = d^2_{T\bar{C}} = 2q\bar{q}$$

and

$$d^2_{KL} = d^2_{KM} = d^2_{LM} = d^2_{TK} = d^2_{TL} = d^2_{TM} = 2q\bar{q},$$
$$d^2_{\bar{K}\bar{L}} = d^2_{\bar{K}\bar{M}} = d^2_{\bar{L}\bar{M}} = d^2_{T\bar{K}} = d^2_{T\bar{L}} = d^2_{T\bar{M}} = 2q\bar{q},$$

where d_{AB} is the distance between the points A and B. We see a pair of regular tetrahedrons and a dual one with a common vertex T: $TABC$ and $TKLM$, $T\bar{A}\bar{B}\bar{C}$ and $T\bar{K}\bar{L}\bar{M}$. These tetrahedrons give a visual representation of the frames in question

$$\mathbf{m}_\mu \quad \mathbf{n}_\mu, \quad -\mathbf{m}_\mu \quad -\mathbf{n}_\mu, \quad (\mu = 1, 2, 3)$$

and discover the nature of rotational motion. Let $\mathbf{q} = \mathbf{q}(t)$ be a trajectory in R^4. When point T moves along this trajectory, the tetrahedrons $TABC$ and $TKLM$ are pulsed and rotated with respect to each other. And the same is for the dual tetrahedrons $T\bar{A}\bar{B}\bar{C}$ and $T\bar{K}\bar{L}\bar{M}$.

The matrix of scalar products

$$P_{\mu\nu} = \mathbf{m}_\mu \cdot \mathbf{n}_\nu, \quad (\mu,\, \nu = 1, 2, 3)$$

describes this relative rotational motion.

The scalar products of the tangent vector $\dot{\mathbf{q}} = d\mathbf{q}/dt$ with the vectors of dual frames $\mathbf{m}_a$ and $\mathbf{n}_a$, $(a = 1, 2, 3, 4)$

$$\mathbf{m}_1 \cdot \frac{d\mathbf{q}}{dt} = q^4 \frac{dq^1}{dt} - q^3 \frac{dq^2}{dt} + q^2 \frac{dq^3}{dt} - q^1 \frac{dq^4}{dt},$$

$$\mathbf{m}_2 \cdot \frac{d\mathbf{q}}{dt} = q^3 \frac{dq^1}{dt} + q^4 \frac{dq^2}{dt} - q^1 \frac{dq^3}{dt} - q^2 \frac{dq^4}{dt},$$

$$\mathbf{m}_3 \cdot \frac{d\mathbf{q}}{dt} = -q^2 \frac{dq^1}{dt} + q^1 \frac{dq^2}{dt} + q^4 \frac{dq^3}{dt} - q^3 \frac{dq^4}{dt},$$

$$\mathbf{n}_1 \cdot \frac{d\mathbf{q}}{dt} = q^4 \frac{dq^1}{dt} + q^3 \frac{dq^2}{dt} - q^2 \frac{dq^3}{dt} - q^1 \frac{dq^4}{dt},$$

$$\mathbf{n}_2 \cdot \frac{d\mathbf{q}}{dt} = -q^3 \frac{dq^1}{dt} + q^4 \frac{dq^2}{dt} + q^1 \frac{dq^3}{dt} - q^2 \frac{dq^4}{dt},$$

$$\mathbf{n}_3 \cdot \frac{d\mathbf{q}}{dt} = q^2 \frac{dq^1}{dt} - q^1 \frac{dq^2}{dt} + q^4 \frac{dq^3}{dt} - q^3 \frac{dq^4}{dt},$$

$$\mathbf{m}_4 \cdot \frac{d\mathbf{q}}{dt} = \mathbf{n}_4 \cdot \frac{d\mathbf{q}}{dt} = q^1 \frac{dq^1}{dt} + q^2 \frac{dq^2}{dt} + q^3 \frac{dq^3}{dt} + q^4 \frac{dq^4}{dt}$$

are invariant with respect to the left- and right-turn dilatations. The invariants

$$\Omega_\mu = \frac{1}{2}\,\mathbf{m}_\mu \cdot \frac{d\mathbf{q}}{dt}, \quad \tilde{\Omega}_\mu = \frac{1}{2}\,\mathbf{n}_\mu \cdot \frac{d\mathbf{q}}{dt}, \quad (\mu = 1, 2, 3)$$

are components of angular velocity of rotation of tetrahedron $TABC$ with respect to tetrahedron $TKLM$ and vice versa. These invariants play the important role in the rigid-body dynamics as well. Thus, a kinematics of rotational motion has an adequate representation in the four dimensions.

To quantize the rotational motion, let us introduce the four-dimensional operator ∇:

$$\nabla_4 = (\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial q^3}, \frac{\partial}{\partial q^4}),$$

and setting

$$M_\nu = \frac{1}{2}(\mathbf{m}_\nu \cdot \nabla_4), \quad N_\nu = \frac{1}{2}(\mathbf{n}_\nu \cdot \nabla_4), \quad (\nu = 1, 2, 3),$$

we have six anti-Hermitian operators of angular momentum of a rotational motion. Factor (1/2) is essential, since natural commutation relations hold valid

$$M_1 M_2 - M_2 M_1 = M_3, \quad N_1 N_2 - N_2 N_1 = -N_3,$$

and so on. The operator of dilatations

$$D = (\mathbf{m}_4 \cdot \nabla_4) = q^1 \frac{\partial}{\partial q^1} + q^2 \frac{\partial}{\partial q^2} + q^3 \frac{\partial}{\partial q^3} + q^4 \frac{\partial}{\partial q^4}$$

has important meaning as well, since it commutes with the operators of angular momentum

$$DM_\nu - M_\nu D = 0, \quad DN_\nu - N_\nu D = 0, \quad (\nu = 1,\, 2,\, 3).$$

Now, we shall introduce two natural tetrads in the space in question and consider the Dirac equations associated with these quadruplets of linear independent vector fields.

3. GLOBAL TETRAD

Let

$$\mathbf{a} = (a^1,\, a^2,\, a^3,\, a^4)$$

be a constant unit vector, then a global tetrad in R^4 is defined as follows:

$$\mathbf{E}_0 = (a^1,\, a^2,\, a^3,\, a^4), \quad \mathbf{E}_1 = (-a^4,\, -a^3,\, a^2,\, a^1),$$
$$\mathbf{E}_2 = (a^3,\, -a^4,\, -a^1,\, a^2), \quad \mathbf{E}_3 = (-a^2,\, a^1,\, -a^4,\, a^3).$$

We put

$$D_0 = \mathbf{E}_0 \cdot \nabla_4, \quad D_1 = \mathbf{E}_1 \cdot \nabla_4, \quad D_2 = \mathbf{E}_2 \cdot \nabla_4, \quad D_3 = \mathbf{E}_3 \cdot \nabla_4,$$

then the Dirac equation in the four-dimensional Euclidian space reads

$$i\gamma^\mu D_\mu \psi = \frac{mc}{\hbar}\psi. \tag{4}$$

Since

$$\gamma^\mu D_\mu = \gamma^0 D_0 + \gamma^1 D_1 + \gamma^2 D_2 + \gamma^3 D_3,$$

then to get a regular transition from the Dirac equation in question to the original Dirac equation, we need to introduce the system of coordinates x^1, x^2, x^3, t, in which the linear differential operator D_0 takes the following form:

$$D_0 = E_0^i \partial_i \to \frac{\partial}{\partial t}.$$

To this end (see Sec. 1), we need to solve the system of equations

$$\frac{dq^i}{dt} = a^i.$$

The general solution is a straight line that goes through the fixed point $\mathbf{q}_0 = (q_0^1,\, q_0^2,\, q_0^3,\, q_0^4)$:

$$\mathbf{q}(t) = \mathbf{a}(t - t_0) + \mathbf{q}_0. \tag{5}$$

We define the three-dimensional characteristic surface S in the space of initial data as follows:

$$\mathbf{a} \cdot \mathbf{q}_0 = t_0. \tag{6}$$

The general solution to Eq. (6) has the form

$$\mathbf{q}_0 = t_0\mathbf{E}_0 + x\mathbf{E}_1 + y\mathbf{E}_2 + z\mathbf{E}_3.$$

Substituting this representation into formula (5), we have

$$\mathbf{q} = t\mathbf{E}_0 + x\mathbf{E}_1 + y\mathbf{E}_2 + z\mathbf{E}_3.$$

The Dirac equation in the coordinates $t,\, x,\, y,\, z$ has the ordinary form

$$i(\gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z})\psi = \frac{mc}{\hbar}\psi.$$

One can work in either the coordinates $q^1,\, q^2,\, q^3,\, q^4$ (that are considered on equal footing) or the coordinates $t,\, x,\, y,\, z$, but the first approach looks like more fundamental, because the direction of the vector $\mathbf{a}$ is not fixed, and this distinctive degeneration is not visible in the second approach.

Now, it is important to show the definition of interval in the four-dimensional Euclidian space. The interval in R^4 is defined as follows. Let

$$\mathbf{q}_s = 2\mathbf{a}(\mathbf{a} \cdot \mathbf{q}) - \mathbf{q}$$

be the vector symmetrical to the vector $\mathbf{q}$ with respect to the vector $\mathbf{a}$. Then, in the coordinates $q^1,\, q^2,\, q^3,\, q^4$, the interval can be written as follows:

$$s^2 = \mathbf{q} \cdot \mathbf{q}_s = 2(\mathbf{a} \cdot \mathbf{q})^2 - \mathbf{q} \cdot \mathbf{q} = (\mathbf{q} \cdot \mathbf{q}) \cos 2\theta,$$

where θ is an angle between $\mathbf{a}$ and $\mathbf{q}$. It is easy to see that in the coordinates $t,\, x,\, y,\, z,$

$$s^2 = t^2 - x^2 - y^2 - z^2.$$

We see that the existence of a natural global tetrad in the four-dimensional Euclidian space presupposes the existence of the Minkowski space-time and, hence, the known causal structure discovered here as a preferred system of coordinates defined by the given direction. The causal structure may be considered in this case as spontaneous breaking of isotropy of the four-dimensional Euclidian space. The global tetrad defines a metric as usual:

$$g_{ij} = \eta_{\mu\nu} E_i^\mu E_j^\nu = 2a_i a_j - \delta_{ij}.$$

4. LOCAL TETRAD

Let

$$\mathbf{q} = (q^1,\, q^2,\, q^3,\, q^4)$$

be a radius-vector, then a natural local tetrad in the four-dimensional Euclidian space can be represented as a quadruplet of orthogonal unit vector fields

$$\mathbf{E}_0 = (\frac{q^1}{\tau},\, \frac{q^2}{\tau},\, \frac{q^3}{\tau},\, \frac{q^4}{\tau}), \quad \mathbf{E}_1 = (\frac{-q^4}{\tau},\, \frac{-q^3}{\tau},\, \frac{q^2}{\tau},\, \frac{q^1}{\tau}),$$

$$\mathbf{E}_2 = (\frac{q^3}{\tau},\, \frac{-q^4}{\tau},\, \frac{-q^1}{\tau},\, \frac{q^2}{\tau}), \quad \mathbf{E}_3 = (\frac{-q^2}{\tau},\, \frac{q^1}{\tau},\, \frac{-q^4}{\tau},\, \frac{q^3}{\tau}),$$

where

$$\tau = \sqrt{(\mathbf{q}\cdot\mathbf{q})} = \sqrt{(q^1)^2 + (q^2)^2 + (q^3)^2 + (q^4)^2}$$

is the length of the radius-vector. The metric defined by the local tetrad has a simple representation $g_{ij} = \eta_{\mu\nu} E_i^\mu E_j^\nu = 2t_i t_j - \delta_{ij}$. We again put

$$D_0 = \mathbf{E}_0 \cdot \nabla, \quad D_1 = \mathbf{E}_1 \cdot \nabla, \quad D_2 = \mathbf{E}_2 \cdot \nabla, \quad D_3 = \mathbf{E}_3 \cdot \nabla,$$

but here the operator ∇ is defined as follows:

$$\nabla = \nabla_4 - \frac{3}{2\tau^2}\mathbf{q},$$

since the vector part of the torsion tensor is not equal to zero in this case. The Dirac equation describing the rotational motion on the quantum level takes the following form:

$$i\gamma^\mu D_\mu \psi = \frac{mc}{\hbar}\psi. \tag{7}$$

Let us consider how to equip the four-dimensional Euclidian space with a preferred system of coordinates in this case. The general solution of the system of equations

$$\frac{dq^i}{d\tau} = \frac{q^i}{\sqrt{(q^1)^2 + (q^2)^2 + (q^3)^2 + (q^4)^2}}$$

can be written in two ways:

$$q^i(\tau) = q_0^i \frac{\tau}{\tau_0}, \quad \tau \in [\tau_0, \infty),$$

where the initial data belong to the three-dimensional sphere

$$\mathbf{q}_0 \cdot \mathbf{q}_0 = {\tau_0}^2;$$

and

$$q^i(\tau) = q_0^i \tau, \quad \tau \in [0, \infty),$$

where the initial data belong to the unit three-dimensional sphere

$$\mathbf{q}_0 \cdot \mathbf{q}_0 = 1.$$

The characteristic surface in this case can be parameterized by the Euler angles, θ, φ, γ. In the coordinates τ, θ, φ, γ, we have

$$D_0 = \frac{\partial}{\partial \tau} - \frac{3}{2\tau}, \quad D_1 = \frac{1}{\tau}\left(-\cot\theta \cos\gamma \frac{\partial}{\partial \gamma} - \sin\gamma \frac{\partial}{\partial \theta} + \frac{\cos\gamma}{\sin\theta} \frac{\partial}{\partial \varphi} \right),$$

$$D_2 = \frac{1}{\tau}\left(-\cot\theta \sin\gamma \frac{\partial}{\partial \gamma} + \cos\gamma \frac{\partial}{\partial \theta} + \frac{\sin\gamma}{\sin\theta} \frac{\partial}{\partial \varphi} \right), \quad D_3 = \frac{1}{\tau} \frac{\partial}{\partial \gamma}.$$

Let us pay attention to the following internal properties of the four-dimensional Euclidian space: kinematical picture and nature of rotational motion; existence of two causal structures. In one case, the causal structure may be geometrically represented as the congruence of parallel three-dimensional planes and the congruence of parallel straight lines orthogonal to these planes. In the other case, the causal structure is defined as the congruence of the three-dimensional sphere with a common centre and the congruence of rays orthogonal to the three-dimensional spheres. A physical interpretation: we put forward the idea that the behavior of leptons is defined by the first causal structure, and the physics of quarks is tightly connected with the new causal structure, which represents a rotating matter. The latter provides understanding and rational proof of quark–lepton symmetry, quark confinement (confinement is not a force, because in any case there is a more powerful one), conservation of the so-called baryon number (read new causal structure). Equations (4), (7) and machinery of the electroweak theory provide the new theoretical basis for understanding the world of leptons and quarks.

The action for the point particle associated with the rotational motion can be written in the following form:

$$S = -mc \int_p^q \sqrt{1 - \tau^2 \omega^2} d\tau,$$

where $\omega = dl/d\tau$ and dl is the element of the arc on the unit three-dimensional sphere. Really, $\mathbf{du} \cdot \mathbf{du} = d\tau^2 + \tau^2 dl^2$, and $\mathbf{u} \cdot \mathbf{du} = \tau d\tau$. On this ground, one can develop the classical mechanics in the new frameworks.

In conclusion of this section, we formulate the Maxwell equations in the framework of the new causal structure. Let A_i be the vector potential of the electromagnetic field. The gauge-invariant tensor of the electromagnetic field is defined as usual $F_{ij} = \partial_i A_j - \partial_j A_i$. The strength of the electric field is a general covariant and gauge-invariant quantity that is defined by the equation $E_i = t^k F_{ik}$, where in our case $t^k = t_k = q^k/\tau$.

The rotor of the vector field $\mathbf{A} = (A_1, A_2, A_3, A_4)$ is defined as a vector product of ∇_4 and $\mathbf{A}$:

$$\operatorname{rot}\mathbf{A} = \nabla_4 \times \mathbf{A}, \quad (\operatorname{rot}\mathbf{A})^i = e^{ijkl} t_j \partial_k A_l = \frac{1}{2} e^{ijkl} t_j (\partial_k A_l - \partial_l A_k),$$

where e^{ijkl} are the contravariant components of the Levi–Civita tensor normalized as $e_{1234} = 1$. The general covariant and gauge-invariant definition of the magnetic field strength is given by the formula $\mathbf{H} = \operatorname{rot}\mathbf{A}$, $H^i = (\operatorname{rot}\mathbf{A})^i$. Thus, $H_i = t^k \overset{*}{F}_{ik}$, where $\overset{*}{F}_{ij} = g_{ik} g_{jl} \overset{*}{F}{}^{kl} = \frac{1}{2} g_{ik} g_{jl} e^{klmn} F_{mn}$. It is evident that vectors $\mathbf{E}$ and $\mathbf{H}$ are orthogonal to $\mathbf{q}$:

$$\mathbf{q} \cdot \mathbf{E} = 0, \quad \mathbf{q} \cdot \mathbf{H} = 0.$$

Below, the Maxwell equations are written in the form that is most suitable for solution:

$$(\mathbf{D}_0 \cdot \nabla_4)\mathbf{H} + \frac{2}{\tau}\mathbf{H} = -\operatorname{rot}\mathbf{E}, \tag{8}$$

$$(\mathbf{D}_0 \cdot \nabla_4)\mathbf{E} + \frac{2}{\tau}\mathbf{E} = \operatorname{rot}\mathbf{H} + e\mathbf{J}, \tag{9}$$

$$\nabla_4 \cdot \mathbf{E} = e\bar{\psi}\gamma^0\psi, \quad \nabla_4 \cdot \mathbf{H} = 0, \tag{10}$$

where the current $\mathbf{J}$ is given by the expression

$$\mathbf{J} = \mathbf{E}_1 \bar{\psi}\gamma^1\psi + \mathbf{E}_2 \bar{\psi}\gamma^2\psi + \mathbf{E}_3 \bar{\psi}\gamma^3\psi.$$

5. CONNECTION WITH THREE-DIMENSIONAL SPACE

To complete the picture of rotational motion and to throw light on some other questions, we consider here the properties of natural mappings of the four-dimensional Euclidian space onto the three-dimensional one. Let $\varphi(x, y, z)$ be a differentiable function of the Cartesian coordinates x, y, z of the three-dimensional Euclidian space, and three differentiable functions

$$x = x(q^1, q^2, q^3, q^4), \quad y = y(q^1, q^2, q^3, q^4), \quad z = z(q^1, q^2, q^3, q^4)$$

define a mapping of R^4 onto E^3. Let us calculate the result of the action of the linear differential operator

$$L = \xi^i(q^1, q^2, q^3, q^4)\frac{\partial}{\partial q^i}$$

on the function $\varphi(x, y, z)$. Using the chain rule, we have

$$L\varphi(x, y, z) = (Lx)\frac{\partial\varphi}{\partial x} + (Ly)\frac{\partial\varphi}{\partial y} + (Lz)\frac{\partial\varphi}{\partial z}.$$

If functions

$$Lx = \xi^i\frac{\partial x}{\partial q^i}, \quad Ly = \xi^i\frac{\partial y}{\partial q^i}, \quad Lz = \xi^i\frac{\partial z}{\partial q^i}$$

of the variables q^1, q^2, q^3, q^4 can be presented as functions of the variables x, y, z, then setting

$$Lx = v_x(x, y, z), \quad Ly = v_y(x, y, z), \quad Lz = v_z(x, y, z),$$

one can calculate the result of the action of the operator L with the help of the new differential operator

$$V = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z},$$

which can be considered as transform of the operator L under the mapping in question. After these general remarks let us consider natural mapping of R^4 onto E^3. It is well known that a rotation with dilatation of the vector $v = v_1 i + v_2 j + v_3 k$ can be presented as follows:

$$v \to sv\bar{s}.$$

Let us consider the quaternions

$$R_1 = qi\bar{q}, \quad R_2 = qj\bar{q}, \quad R_3 = qk\bar{q},$$

$$T_1 = \bar{q}iq, \quad T_2 = \bar{q}jq, \quad T_3 = \bar{q}kq.$$

Under the left-turn dilatations $q \to sq$, the quaternions R_1, R_2, R_3 transform as follows: $R_\mu \to sR_\mu\bar{s}$, $(\mu = 1, 2, 3)$. Under the right-turn dilatations $q \to q\bar{t}$, the quaternions T_1, T_2, T_3 transform similarly to R_μ, $T_\mu \to tT_\mu\bar{t}$, $(\mu = 1, 2, 3)$. We see that the coordinates of the quaternions in question can be considered as the Cartesian coordinates of E^3. We denote these coordinates as $x_\mu, y_\mu, z_\mu, (\mu = 1, 2, 3)$ and, respectively, $\xi_\mu, \eta_\mu, \zeta_\mu, (\mu = 1, 2, 3)$,

$$\mathbf{R}_\mu = (x_\mu, y_\mu, z_\mu), \quad \mathbf{T}_\mu = (\xi_\mu, \eta_\mu, \zeta_\mu).$$

The vectors $\mathbf{R}_\mu$ and $\mathbf{T}_\mu$ have the same length and constitute the right-handled orthogonal bases, since

$$\mathbf{R}_1 \times \mathbf{R}_2 = q\bar{q}\mathbf{R}_3, \quad \mathbf{R}_1 \cdot (\mathbf{R}_2 \times \mathbf{R}_3) = (q\bar{q})^3,$$
$$\mathbf{T}_1 \times \mathbf{T}_2 = q\bar{q}\mathbf{T}_3, \quad \mathbf{T}_1 \cdot (\mathbf{T}_2 \times \mathbf{T}_3) = (q\bar{q})^3.$$

Here, we are slightly detained to give a simple and important geometrical interpretation of the Cartan spinors [2, 3], which is tightly connected with the complex-analytic structures on R^4. To this end, let us consider the complex null vectors

$$\mathbf{W}_1 = \mathbf{R}_2 + \sqrt{-1}\mathbf{R}_3, \quad \mathbf{W}_2 = \mathbf{R}_3 + \sqrt{-1}\mathbf{R}_1, \quad \mathbf{W}_3 = \mathbf{R}_1 + \sqrt{-1}\mathbf{R}_2.$$

Calculating components of these vectors, we have

$$\mathbf{W}_1 = (u_1,\, v_1,\, w_1) = (\,2\xi_1\xi_2, \quad \xi_1^2 - \xi_2^2, \quad -\sqrt{-1}\xi_1^2 - \sqrt{-1}\xi_2^2),$$

where $\xi_1 = q^2 + \sqrt{-1}q^3, \quad \xi_2 = q^1 + \sqrt{-1}q^4,$

$$\mathbf{W}_2 = (u_2,\, v_2,\, w_2) = (\,-\sqrt{-1}\eta_1^2 - \sqrt{-1}\eta_2^2, \quad 2\eta_1\eta_2, \quad \eta_1^2 - \eta_2^2),$$

where $\eta_1 = q^3 + \sqrt{-1}q^1, \quad \eta_2 = q^2 + \sqrt{-1}q^4,$

$$\mathbf{W}_3 = (u_3,\, v_3,\, w_3) = (\,\zeta_1^2 - \zeta_2^2, \quad -\sqrt{-1}\zeta_1^2 - \sqrt{-1}\zeta_2^2, \quad 2\zeta_1\zeta_2),$$

where $\zeta_1 = q^1 + \sqrt{-1}q^2, \quad \zeta_2 = q^3 + \sqrt{-1}q^4.$

Studying the behavior of the pairs $(\xi_1,\, \xi_2)$, (η_1, η_2), $(\zeta_1,\, \zeta_2)$ under the turn dilatations, we conclude that these pairs are spinors. It is also evident that the Cartan spinor is simply the system of complex coordinates on R^4. The general theory of complex manifolds is explained in [4]. Actually, it is shown that there are three canonical systems of complex coordinates defined by the complex structures i, j, k. The turn dilatations in the complex coordinates coincide with spinor transformations. To introduce the spinor with the so-called dotted indices, one simply needs to consider the vectors

$$\bar{\mathbf{W}}_1 = \mathbf{R}_2 - \sqrt{-1}\mathbf{R}_3, \quad \bar{\mathbf{W}}_2 = \mathbf{R}_3 - \sqrt{-1}\mathbf{R}_1, \quad \bar{\mathbf{W}}_3 = \mathbf{R}_1 - \sqrt{-1}\mathbf{R}_2.$$

Thus, it is evident that the spinors do not represent a new geometrical quantity.

Now, it is time to prolong and write out expressions for the coordinates x_μ, y_μ, z_μ, $(\mu = 1, 2, 3)$ and $\xi_\mu, \eta_\mu, \zeta_\mu$, $(\mu = 1, 2, 3)$. We have

$$\begin{aligned} x_1 &= (q^1)^2 - (q^2)^2 - (q^3)^2 + (q^4)^2, & y_1 &= 2q^1q^2 + 2q^3q^4, \\ x_2 &= 2q^1q^2 - 2q^3q^4, & y_2 &= -(q^1)^2 + (q^2)^2 - (q^3)^2 + (q^4)^2, \\ x_3 &= 2q^1q^3 + 2q^2q^4, & y_3 &= -2q^1q^4 + 2q^2q^3, \end{aligned}$$

$$\begin{aligned} z_1 &= 2q^1q^3 - 2q^2q^4, \\ z_2 &= 2q^1q^4 + 2q^2q^3, \\ z_3 &= -(q^1)^2 - (q^2)^2 + (q^3)^2 + (q^4)^2, \end{aligned}$$

and

$$(\xi_1, \eta_1, \zeta_1) = (x_1, x_2, x_3),$$
$$(\xi_2, \eta_2, \zeta_2) = (y_1, y_2, y_3),$$
$$(\xi_3, \eta_3, \zeta_3) = (z_1, z_2, z_3).$$

Thus, all natural mappings of R^4 onto E^3 are presented. Now, it is interesting to find transforms of the operators of the angular momenta of the rotational motion. Below, the results of calculations will be presented only for one case (with comments only with respect to other situations). For obviousness, let us put $x_1 = x$, $y_1 = y$, $z_1 = z$. After some calculations the following results can be presented:

$$M_1\varphi(x,\, y,\, z) = 0,$$

$$-M_2\varphi(x, y, z) = x_3\frac{\partial\varphi}{\partial x} + y_3\frac{\partial\varphi}{\partial y} + z_3\frac{\partial\varphi}{\partial z}, \quad M_3\varphi(x, y, z) = x_2\frac{\partial\varphi}{\partial x} + y_2\frac{\partial\varphi}{\partial y} + z_2\frac{\partial\varphi}{\partial z}.$$

It is visible that the operators in question have no transforms. In the other case, the picture is more interesting, since

$$N_1\varphi(x,\, y,\, z) = 0\frac{\partial\varphi}{\partial x} - z\frac{\partial\varphi}{\partial y} + y\frac{\partial\varphi}{\partial z},$$

$$N_2\varphi(x,\, y,\, z) = z\frac{\partial\varphi}{\partial x} + 0\frac{\partial\varphi}{\partial y} - x\frac{\partial\varphi}{\partial z}, \quad N_3\varphi(x,\, y,\, z) = -y\frac{\partial\varphi}{\partial x} + x\frac{\partial\varphi}{\partial y} + 0\frac{\partial\varphi}{\partial z}.$$

Let us put $\mathbf{N} = (N_1,\, N_2,\, N_3)$, and the last relations can be written as follows:

$$\mathbf{N}\varphi(x,\, y,\, z) = (\mathbf{r} \times \nabla)\varphi(x,\, y,\, z).$$

These relations are valid for all coordinates $x_\mu,\ y_\mu,\ z_\mu,\ (\mu = 1, 2, 3)$. If we consider the coordinates ξ_μ, η_μ, ζ_μ, $(\mu = 1, 2, 3)$, then the operators $\mathbf{N}$ take place of the operators $\mathbf{M}$ and vice versa. The relation

$$\mathbf{M}\varphi(\xi, \eta, \zeta) = -(\mathbf{r} \times \nabla)\varphi(\xi, \eta, \zeta)$$

exhibits this exchange. Thus, after the mappings in question we see instead of the operators of the angular momentum of the rotational motion the operators of the orbital angular momentum of the point particle. It is interesting that the relation

$$\frac{1}{2}D\varphi(x,\, y,\, z) = x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z}$$

holds valid in all instances.

Let us pay attention to the following important things. The coordinates of the four-dimensional Euclidian space are not observable, but the picture of the rotation is very detailed and beautiful in this case and can be represented in the descriptive-geometric form. The coordinates of the three-dimensional Euclidian space are quadratic functions of the coordinates of R^4 and are observable, but the harmonic picture of rotation is reduced to the operators of the orbital angular momentum of the point particle. There is an interesting problem of half-integer orbital angular momentum, which is in the sphere of interests of physicists up to now [2]. From our consideration it follows that eigenfunctions of the operators of angular momentum of the rotational motion can be the eigenfunctions of the operator of the orbital angular momentum only in the case when these functions are even, and this is the hidden reason of the integer eigenvalues. Let us give one more interesting example of four-dimensional and three-dimensional points of view on the important physical object.

6. HYDROGEN ATOM AND FOUR-DIMENSIONAL SPACE

This is the Schrödinger equation of the hydrogen atom:

$$-\frac{\hbar^2}{2m}\triangle\psi(x,\, y,\, z) - \frac{e^2}{r}\psi(x,\, y,\, z) = E\psi(x,\, y,\, z).$$

We introduce into consideration the four-dimensional Laplacian setting

$$\triangle_4 = \frac{\partial^2}{\partial {q_1}^2} + \frac{\partial^2}{\partial {q_2}^2} + \frac{\partial^2}{\partial {q_3}^2} + \frac{\partial^2}{\partial {q_4}^2}$$

and calculate a result of the action of this operator on the wave function $\psi(x,\, y,\, z)$ under the condition that $x,\, y,\, z$ are the functions of the coordinates $q^1,\, q^2,\, q^3,\, q^4$ as is explained above. The result can be written in the form of the equation

$$\triangle_4\psi(x,\, y,\, z) = 4r\triangle\psi(x,\, y,\, z), \tag{11}$$

since $r = \sqrt{x^2+y^2+z^2} = q\bar{q} = (q^1)^2 + (q^2)^2 + (q^3)^2 + (q^4)^2$. In what follows, we will put $q\bar{q} = u^2$. From this equation it follows that for any harmonic function $\psi(x,\, y,\, z)$ in the three-dimensional Euclidian space we can put in correspondence a harmonic function in the four-dimensional Euclidian space simply setting

$$x = (q^1)^2 - (q^2)^2 - (q^3)^2 + (q^4)^2, \quad y = 2q^1q^2 + 2q^3q^4, \quad z = 2q^1q^3 - 2q^2q^4.$$

For example, $1/r = 1/u^2$. Another conclusion means that a solution $\psi(x,\, y,\, z)$ of the Schrödinger equation after the substitution written above will be a solution to the equation

$$(\triangle_4 + \frac{8me^2}{\hbar^2} + \frac{8Em}{\hbar^2}u^2)\psi(x,\, y,\, z) = 0$$

in the four-dimensional Euclidian space. Seeing this equation, we have reason to consider the equation

$$(\triangle_4 + \frac{8me^2}{\hbar^2} + \frac{8Em}{\hbar^2}u^2)\psi(q^1,\, q^2,\, q^3,\, q^4) = 0 \tag{12}$$

as the Schrödinger equation for the hydrogen atom in the four-dimensional Euclidian space. The operators $\mathbf{M}$ and $\mathbf{N}$ act in the space of solution of Eq. (12), and hence it is invariant with respect to the turns but not dilatations. Below, we will give solution of Eq. (12). Let us consider the harmonic polynomials

$$D^j_{mn}(q^1,\, q^2,\, q^3,\, q^4) = \sqrt{(j+m)!(j-m)!(j+n)!(j-n)!}\,\sum_k (-1)^{m+k}\times$$
$$\times\frac{(q^1+iq^2)^{m-n+k}(q^1-iq^2)^k(q^3+iq^4)^{j-m-k}(q^3-iq^4)^{j+n-k}}{(m-n+k)!k!(j-m-k)!(j+n-k)!} \tag{13}$$

known as matrices of rotations in the theory of angular momentum [2]. Since

$$D^j_{mn}(q^1,\, q^2,\, q^3,\, q^4) = u^{2j}D^j_{mn}(q^1/u,\, q^2/u,\, q^3/u,\, q^4/u),$$

we can consider these polynomials as functions on the sphere S^3 and denote as D^j_{mn}. In R^4, Laplacian in the spherical system of coordinates can be written as follows:

$$\triangle_4 = \frac{\partial^2}{\partial u^2} + \frac{3}{u}\frac{\partial}{\partial u} + \frac{1}{u^2}\triangle_S,$$

where $\triangle_S$ is the Laplace–Beltrami operator on the unit sphere S^3. It is clear that we can look for solution of Eq. (12) in the following form:

$$\psi(q^1,\, q^2,\, q^3,\, q^4) = R(u)D^j_{mn}.$$

Since

$$\triangle_S D^j_{mn} = -2j(2j+2)D^j_{mn},$$

the function $R(u)$ satisfies the equation

$$\frac{d^2R}{du^2} + \frac{3}{u}\frac{dR}{du} - \frac{2j(2j+2)}{u^2}R + (\frac{8Zme^2}{\hbar^2} + \frac{8Em}{\hbar^2}u^2)R = 0. \tag{14}$$

The final step is to introduce the new variable $r = u^2$ and get the equation

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{j(j+1)}{r^2}R + \frac{2m}{\hbar^2}(\frac{Ze^2}{r} + E)R = 0, \tag{15}$$

well-known in quantum mechanics [5]. However, in our case, j takes not only integer but half-integer values $j = 0,\, \frac{1}{2},\, 1,\, \cdots$. The final formula for the energy levels of the hydrogen atom in the four-dimensional space can be written as follows:

$$E = -\frac{me^4}{2\hbar^2(k+j+1)^2}, \tag{16}$$

where k and j take values

$$k = 0,\, 1,\, 2,\, \cdots, \quad j = 0,\, \frac{1}{2},\, 1,\, \cdots .$$

We see that in the four-dimensional Euclidian space there are extra energy levels connected with a rotational motion, but coordinates are not observable and these extra levels have no clear physical interpretation. Equation (12) is invariant with respect to the transformation

$$\psi(q^1,\, q^2,\, q^3,\, q^4) \to \psi(-q^1,\, -q^2,\, -q^3,\, -q^4),$$

and since

$$D^j_{mn}(-q^1,\, -q^2,\, -q^3,\, -q^4) = (-1)^{2j}\, D^j_{mn}(q^1,\, q^2,\, q^3,\, q^4),$$

we again must consider only even wave functions.

CONCLUSION

In conclusion, we give the answer to the following questions. What are the new results in your article? In what way are these new results timely? Why are these new results significant? 1. A new definition of space-time is given. The origin and nature of the rotational motion are recognized. It is established that on the four-dimensional Euclidian space there are two space-time structures and one of them is tightly connected with the rotational motion and a simply transitive group of turn dilatations. On this ground, the new basic equations for description of the so-called strong interactions are suggested. 2. At the present time, quantum chromodynamics has no alternative, but in the framework of this theory, we have no answer to the set of principle questions, and hence new approaches are desirable. From this point of view, our suggestion to consider the leptons on the ground of one causal structure and to connect the quarks with the other causal structure on the same four-dimensional physical space looks like quite timely. 3. The problem of time and everything connected with this topic are always significant. The results obtained are significant, because they give a simple and evident explanation of quark–lepton symmetry, quark confinement and baryon number conservation. From the point of view in question, the baryon number conservation means that quarks cannot change the causal structure in which they live.

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Received on December 9, 2014.

Редактор *В. В. Булатова*

Подписано в печать 16.02.2015.
Формат 60 × 90/16. Бумага офсетная. Печать офсетная.
Усл. печ. л. 1,31. Уч.-изд. л. 1,87. Тираж 325 экз. Заказ № 58474.

Издательский отдел Объединенного института ядерных исследований
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