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MOMENTS OF INERTIA IN **IBM**

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Моменты инерции в IBM

Для более удобной визуализации характера спектра в полосе, а также качества воспроизведения экспериментальных энергий в различных моделях удобно от энергий перейти к эффективным моментам инерции ядра J , зависящим от квадрата частоты вращения ω^2 . В этом отношении выясняется, какой характер $J(\omega^2)$ могут воспроизвести различные предельные случаи IBM, а также каковы вариации в поведении $J(\omega^2)$ при произвольных значениях параметров гамильтониана IBM. Это может предоставить дополнительную информацию об изменении природы состояний в полосах по мере роста спина.

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Moments of Inertia in IBM

For more convenient visualization of the character of the spectrum in the band, as well as the quality of reproduction of experimental energies in various models, it is convenient to move from energies to effective moments of inertia J depending on the square of the rotation frequency ω^2 . In this regard, it is found out what character of $J(\omega^2)$ can be reproduced by various limiting cases of IBM, as well as what are the variations in the behavior of $J(\omega^2)$ for arbitrary values of the IBM Hamiltonian parameters. This can provide additional information on the change in the nature of states in the bands as the spin increases.

The investigation has been performed at the Flerov Laboratory of Nuclear Reactions, JINR.

INTRODUCTION

In cases where the calculated and experimental values of energies in one band are very close, it is convenient to use the effective moments of inertia J rather than the levels energy E as a visualization of the quality of the description, as well as the presentation of the experimental data themselves. In addition, the backbending effect can be used to judge at what values of spins the bands cross. Correlating the characteristic curves $J(\omega^2)$ obtained from experimental energies with what is given by various nuclear models, including microscopic ones, can reveal the capabilities of the corresponding models and provide more reasoned judgments about the nature and character of the states.

One of the methods for reproducing and predicting energies is based on the expansion of the moment of inertia in powers of the rotation frequency, as in the Harris model [1]. This is justified when the transition to a new band has not yet occurred in the rotational band. As a rule, such a transition is realized rather quickly in one or two states of the yrast band. This is manifested in the graph of the dependence of the moment of inertia on the square of the frequency in a specific way through backbending. Among heavy nuclei, starting with thorium isotopes, three such nuclei are currently known: ^{220}Th , ^{242}Pu and ^{244}Pu . That is why in work [2], within the framework of the phenomenology of IBM1 (hereinafter simply IBM) [3, 4], a very satisfactory description of the energies of the yrast bands up to extremely high spins I in isotopes from Pu to No was obtained. The effective moment of inertia and the square of the rotation frequency are defined as

$$\frac{2J}{\hbar^2} = \frac{4I - 2}{E(I \rightarrow I - 2)},$$
$$(\hbar\omega)^2 = \frac{(E(I \rightarrow I - 2))^2}{\left((I(I + 1))^{1/2} - ((I - 2)(I - 1))^{1/2}\right)^2}. \quad (1)$$

In nuclei with stable deformation, the deviation from the energy dependence $I(I + 1)$ arises not only due to the growing influence of high-spin modes as the energy and spin of the collective state increase, but also due to the weakening of pairing with increasing rotation. If the first leads to upbending, then the second to downbending.

The paper presents a description of the moments of inertia of yrast bands in even-even heavy and superheavy nuclei using IBM in its phenomenological

aspect. In this case, a sufficiently large number of bosons is used, which was first applied to describe the properties of collective states in even isotopes of Hf [5] and heavy nuclei from Pu to No [2]. This allows us to identify characteristic features in the behavior of the moment of inertia as a function of the square of the rotation frequency, which are realized in IBM. Since both phenomena, the increase in the influence of high-spin modes and the weakening of pairing, lead to differently directed trends with respect to the energies of states and moments of inertia, they can partially compensate for themselves. In this case, a number of models describing purely collective states should correctly reproduce the energy characteristics of the bands up to sufficiently large spins.

Comparison of the effective moments of inertia obtained from experimental energies and from IBM calculations may allow one to develop a criterion for determining the spin at which the band crossing occurs. It should be noted that the absence of backbending does not guarantee the absence of band crossing, which is possible in this case as well, but then it occurs quite smoothly. This was revealed during the microscopic description of the ^{222}Th nucleus in [6].

The objectives of this work are defined as:

1) study of the behavior of the moment of inertia in three classical IBM limits and the possibility of reproducing the linear dependence $J(\omega^2)$;

2) consideration of the possibility of IBM to describe effective moments of inertia with arbitrary parameters. Assumptions about the causes of discrepancies.

First, we will consider the nature of the moment of inertia in classical IBM variants.

1. MOMENTS OF INERTIA IN LIMITING CASES OF IBM

Let us consider possible variants of the behavior of the moment of inertia J as a function of the square of the rotation frequency $(\hbar\omega)^2$. In the simplest version of the Harris model, this dependence is linear, namely,

$$J = J_0 + J_1\omega^2, \quad (2)$$

which, for fixed values of the parameters J_0 , J_1 and together with Eqs. (1), leads to values of ω for each transition and, accordingly, the excitation energies of the yrast band are restored.

Let us consider what behavior of $J(\omega^2)$ can be given by different variants of IBM, whose Hamiltonian is of the form

$$H_{\text{IBM}} = \varepsilon_d \hat{n}_d + k_1 (d^+ \cdot d^+ s s + \text{H.c.}) + k_2 \left((d^+ d^+)^{(2)} \cdot d s + \text{H.c.} \right) + \frac{1}{2} \sum_L C_L (d^+ d^+)^{(L)} \cdot (d d)^{(L)}. \quad (3)$$

H.c. means Hermitian conjugation, the dot between the operators corresponds to the scalar product, and the quantities ε_d , k_1 , k_2 , C_0 , C_2 , C_4 are model parameters. An additional parameter is the maximum number d of bosons Ω .

In the case of the $SU(5)$ IBM limit, the parameters k_1 and k_2 are zero, and the energy of the spin I states of the yrast band is defined as

$$E_I = \frac{I}{2}\varepsilon_d + \frac{1}{8}(I^2 - 2I)C_4. \quad (4)$$

For small values of the anharmonicity parameter C_4 , the energy differences $E(I \rightarrow I - 2)$ remain close to the value ε_d . This gives a virtually vertical line in the function $J(\omega^2)$.

The next IBM limit corresponds to the $SU(3)$ case and describes the rotational bands of deformed nuclei. This limit will correspond to a special choice of the Hamiltonian parameters (3), namely,

$$H_{SU(3)} = -k \left(\frac{\widehat{C}(\lambda, \mu)}{2} - \frac{3}{8}(1 + \eta)\widehat{I}^2 \right), \quad (5)$$

where $\widehat{C}(\lambda, \mu)$ is the Casimir operator of the $SU(3)$ group; its eigenvalues are

$$C(\lambda, \mu) = \lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu), \quad (6)$$

where λ , μ are non-negative integers characterizing irreducible representations of the group $SU(3)$.

The eigenvalues of the Hamiltonian (5) are defined as

$$E_{SU(3)}(\Omega, (\lambda, \mu), K, I) = \frac{k}{2} \left(\frac{3}{4}(1 + \eta)I(I + 1) - C(\lambda, \mu) \right), \quad (7)$$

where Ω is the maximum number of d bosons and K is an additional quantum number that coincides with the minimum spin of the band. For the yrast band, $(\lambda, \mu) = (2\Omega, 0)$. The next two bands are degenerate and are determined by the quantum numbers $K = 0$, $(\lambda, \mu) = (2\Omega - 4, 2)$ (corresponds to the β band), $K = 2$, $(\lambda, \mu) = (2\Omega - 4, 2)$ (corresponds to the γ band). The energies of the yrast band states and the energy of the O_2 state are determined as

$$E_I = \frac{3k}{8}(1 + \eta)I(I + 1), \quad E_{O_2} = k(6\Omega - 3), \quad (8)$$

which gives the expression for the energy difference within the yrast band

$$E(I \rightarrow I - 2) = \frac{3k}{8}(1 + \eta)(I(I + 1) - (I - 2)(I - 1)) = \frac{3k}{4}(1 + \eta)(2I - 1). \quad (9)$$

The moments of inertia and rotation frequencies receive the corresponding expressions

$$\frac{2J}{\hbar^2} = \frac{2I - 1}{E(I \rightarrow I - 2)} = \frac{8}{3k(1 + \eta)}, \quad (10)$$

$$\hbar\omega = \frac{3k(1 + \eta)(2I - 1)}{4((I(I + 1))^{1/2} - ((I - 2)(I - 1))^{1/2})}.$$

That is, in the IBM rotational limit, the moment of inertia does not depend on the rotation frequency, which is well known for the ideal rotational spectrum.

In order to relate the parameters of the Hamiltonian (3) to the parameters of (5), we note that

$$\widehat{C}(\lambda, \mu) = 2\widehat{Q}^{SU(3)} \cdot \widehat{Q}^{SU(3)} + \frac{3}{4}\widehat{I}^2, \quad (11)$$

and therefore (5) can be represented as

$$H_{SU(3)} = -k \left(\widehat{Q}^{SU(3)} \cdot \widehat{Q}^{SU(3)} - \frac{3}{4}\eta\widehat{I}^2 \right), \quad (12)$$

$$\widehat{Q}_\mu^{SU(3)} = d_\mu^+ s + s^+ d_{\bar{\mu}} \pm \frac{\sqrt{7}}{2}(d^+ d)_\mu^{(2)}, \quad \widehat{I}_\mu = \sqrt{10}(d^+ d)_\mu^{(1)}. \quad (13)$$

The Hamiltonian (3) corresponds to the Hamiltonian (12) of the $SU(3)$ limit of IBM with the following parameters (3):

$$\begin{aligned} \varepsilon_d &= \left(\frac{17}{4} - 2\Omega + \frac{9}{4}\eta \right) k, \\ k_1 &= -k, \\ k_2 &= \mp\sqrt{7}k, \\ C_0 &= \left(\frac{1}{2} - \frac{9}{2}\eta \right) k, \\ C_2 &= \left(\frac{19}{4} - \frac{9}{4}\eta \right) k, \\ C_4 &= 3(1 + \eta)k. \end{aligned} \quad (14)$$

The signs in (13) and (14) agree with each other.

The third limiting case of IBM is the $O(6)$ limit of IBM. The Hamiltonian of this limit is expressed in terms of the Casimir operators of the chain of subgroups

$$SU(6) \supset SO(6) \supset SO(5) \supset SO(3).$$

In the simplest case, it can be represented as

$$H_{O(6)} = -k_6 \left(\widehat{Q}^{O(6)} \cdot \widehat{Q}^{O(6)} - \frac{3}{8}\eta\widehat{I}^2 \right), \quad (15)$$

$$\widehat{Q}_\mu^{O(6)} = d_\mu^+ s + s^+ d_\mu^-, \quad (16)$$

where $\widehat{Q}^{O(6)}$ is one of the generators of the $SO(6)$ algebra, which is composed of seven generators of $SO(5) - (d^+ d)_\mu^{(3)}$, three $\widehat{I}_\mu = \sqrt{10} (d^+ d)_\mu^{(1)}$ and five $\widehat{Q}^{O(6)}$ operators. Eigenvalues of $H_{O(6)}$ can be obtained from the fact that the operator $\widehat{Q}^{O(6)} \cdot \widehat{Q}^{O(6)}$ is the difference between the Casimir operators of the algebra $\widehat{SO}(6) - \widehat{C}(6)$ and the algebra $\widehat{SO}(5) - \widehat{C}(5)$, where

$$\widehat{C}(6) = \widehat{C}(5) + \widehat{Q}^{O(6)} \cdot \widehat{Q}^{O(6)}, \quad (17)$$

$$\widehat{C}(5) = 2 \sum_{\lambda=1,3} \sum_{\mu} (-1)^\mu (d^+ d)_\mu^{(\lambda)} (d^+ d)_{-\mu}^{(\lambda)} = -S_+ S_- + \widehat{n}_d^2 + 3\widehat{n}_d, \quad (18)$$

$$S_+ = \sum_{\mu} d_\mu^+ d_\mu^+, \quad S_- = (S_+)^+. \quad (19)$$

This allows us to obtain the corresponding expressions

$$\widehat{C}(6)|\sigma, v, \omega, I\rangle = \sigma(\sigma + 4)|\sigma, v, \omega, I\rangle \quad (20)$$

$$E(\Omega, \sigma, v, I) = k_6 \left(v(v + 3) - \sigma(\sigma + 4) + \frac{3}{8} \eta I(I + 1) \right), \quad (21)$$

where $\sigma = \Omega, \Omega - 2, \Omega - 4, \dots, 0$ or 1 is a number characterizing the $SO(6)$ group, $v = 0, 1, \dots, \sigma$; v is seniority of the $SO(5)$ group. For the yrast band, $\sigma = \Omega$. Therefore, for the excitation energies of the yrast-band states and for the energy of the 0_2 state, we have the following expressions:

$$E_I = \frac{k_6}{8} (2I(I + 6) + 3\eta)I(I + 1), \quad E_{0_2} = 18k_6, \quad (22)$$

which gives for the energy difference

$$E(I \rightarrow I - 2) = \frac{k_6}{4} (4I + 8 + 3\eta(2I - 1)), \quad (23)$$

and accordingly for the moment of inertia and rotation frequency the expressions

$$\frac{2J}{\hbar^2} = \frac{8}{k_6 \left(\frac{4I + 8}{2I - 1} + 3\eta \right)}, \quad (24)$$

$$\hbar\omega = \frac{k_6}{4} \frac{(4I + 8 + 3\eta(2I - 1))}{\left(\sqrt{I(I + 1)} - \sqrt{(I - 2)(I - 1)} \right)}.$$

It is interesting that, within the $O(6)$ limit of IBM, one can obtain an excitation spectrum similar to the vibrational case, when the energy differences $E(I \rightarrow I - 2)$ will remain unchanged. This occurs at $\eta = -2/3$ and then $E(I \rightarrow I - 2) = 5k_6/2$, and the energies of the yrast-band states are

$E_I = 5k_6I/4$. In this case, the moment of inertia as a function of the square of the frequency will be a vertical line.

The parameters of the Hamiltonian (3) correspond to the parameters of the IBM $O(6)$ limit Hamiltonian (15) as follows:

$$\begin{aligned}
 \varepsilon_d &= \left(-2\Omega + 6 + \frac{9}{4}\eta \right) k_6, \\
 k_1 &= -k_6, \\
 k_2 &= 0, \\
 C_0 &= \left(4 - \frac{9}{2}\eta \right) k_6, \\
 C_2 &= \left(4 - \frac{9}{4}\eta \right) k_6, \\
 C_4 &= (4 + 3\eta)k_6.
 \end{aligned} \tag{25}$$

For the ^{226}Th nucleus, one of the best manifestations of the linear dependence $J(\omega^2)$ is observed. For it, Fig.1 presents data obtained in accordance with the experimental energies, data corresponding to the $SU(3)$ IBM limit, and data obtained with the Hamiltonian $H_{SU(3)} + \tilde{\varepsilon}_d n_d$ with the condition of obtaining the same first excitation energy. This somewhat corresponds to the middle position between the vibrator and the rotator, but at the same time, as can be seen from Fig. 1, it still does not allow obtaining a straight line with a slope. If the energies are defined by the expression

$$E_I = AI(I + 1) - BI^2(I + 1)^2, \tag{26}$$

then the corresponding moments of inertia with parameters $A = 12.0574 \times 10^{-3}$, $B = 4 \cdot 10^{-6}$ and $A = 12.0815 \cdot 10^{-3}$, $B = 8 \cdot 10^{-6}$ (options v.1 and v.2) take the form shown in Fig.2. From this it is clear that such a

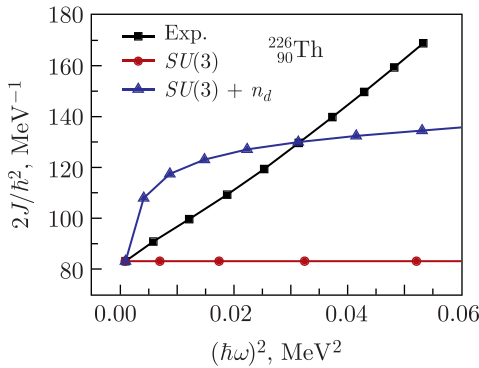


Fig. 1. Moment of inertia for $SU3$ and close modification

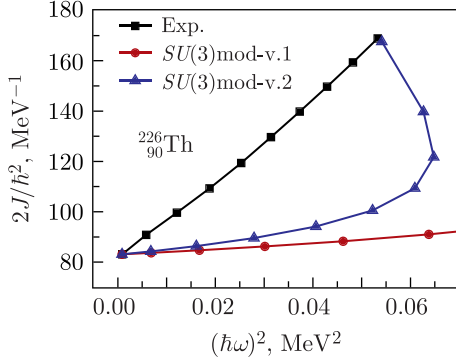


Fig. 2. Moments of inertia in another modification of $SU3$

representation of the energies does not give the required slope, and for large values of the parameter B (v.2) a reverse bend is obtained, but it is not associated with the real intersection of the bands. At the same time, looking at Figs.1 and 2, one can assume that in the $H_{SU(3)} + \tilde{\epsilon}_d n_d - BI^2(I+1)^2$ model one can obtain a curve $J(\omega^2)$ close to a straight line. We are interested in the question of the possibility of realizing the linear dependence $J(\omega^2)$ using the traditional IBM Hamiltonian (3), where there is no term $I^2(I+1)^2$.

From Fig. 3 it is evident that the $O(6)$ limit cannot reproduce the linear function $J(\omega^2)$. Depending on the parameter η , the situation for the yrast band is reproduced from the vibrational case to the case when the moment of inertia stops changing at large spin values.

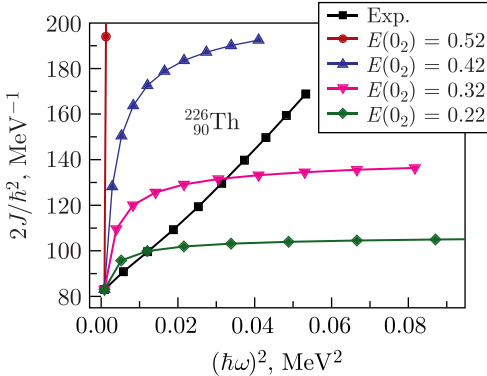


Fig. 3. Moments of inertia for the yrast band with different sets of parameters of the $O(6)$ IBM limit in ^{226}Th ; in this case, the energy $E(2^+_1) = 0.0722$, $E(0^+_2) = 0.52$, 0.42 , 0.32 , 0.22 MeV, which corresponds to the parameters k_6 , $(\eta) = 0.0289(-0.667)$; $0.0233(-0.4025)$; $0.0178(0.0272)$; $0.0122(0.8477)$

2. MOMENTS OF INERTIA IN IBM WITH ARBITRARY SET OF PARAMETERS

The question remains about the possibility of reproducing the linear function $J(\omega^2)$ within the traditional IBM Hamiltonian with an arbitrary set of parameters, that is, outside the known limits of IBM. For this purpose, we will select a series of nuclei for which the linear dependence of J on ω^2 is realized in the best possible way. As relevant examples, we consider the nuclei ^{226}Th , $^{226,230}\text{U}$, $^{238,240,246}\text{Pu}$. Therefore, along with the moments of inertia corresponding to the experimental data, the figures show the moments of inertia obtained from the IBM energies with parameters obtained phenomenologically based on the energies taken from [7]. The parameters, including the maximum number of bosons, are given in Table 1.

Table 1. Parameters of the IBM Hamiltonian for a number of heavy nuclei, describing the linear behavior of the moment of inertia as a function of the square of the rotation frequency

Parameter	^{226}Th	^{226}U	^{230}U	^{238}Pu	^{240}Pu	^{246}Pu
Ω	28	25	25	24	24	24
ε_d	0.36799	-0.059896	-0.587053	-0.793156	-0.733679	-0.792760
k_1	-0.030993	-0.037406	-0.064351	-0.050154	-0.055681	-0.060414
k_2	0.003373	0.004168	0.049507	0.038541	0.047567	0.052828
C_0	-0.08073	0.001010	0.895851	0.583019	0.694680	0.620469
C_2	-0.022909	0.011875	0.053841	0.097487	0.088359	0.043172
C_4	-0.035625	-0.020448	0.019587	0.045934	0.036018	0.029228

In ^{226}Th up to the state 10^+ , a linear dependence is reproduced; at large spins there are differences, see Fig. 4. In ^{226}U , the linear nature of the function is reproduced quite well. For ^{230}U , the linear character of $J(\omega^2)$ is realized up to the state with $I = 16^+$, then a break in the straight line is observed, and the calculated curve has a weakly parabolic character. The experimental dependence yields a function that is closer to the straight line than the calculated one within the framework of IBM phenomenology.

For ^{238}Pu up to the spin 24^+ state, both experimental and calculated data essentially yield one linear function for $J(\omega^2)$. Starting from the spin 26^+ state, the difference increases. The discrepancy is also related to the role of high-spin excitation modes. For ^{240}Pu , as for the previous nucleus, there is a very good agreement between experimental and calculated data, with the exception of the spin 32^+ state. For ^{246}Pu , information is known only up to spin 12^+ and it is reproduced.

Figure 5 shows examples of nuclei for which a complex dependence $J(\omega^2)$ is realized, but which is reproduced within the IBM framework, i.e., with the Hamiltonian (3) without any modifications and taking into account high-spin excitation modes. For ^{236}U after the linear part of the $J(\omega^2)$ dependence at spin 20^+ , a small upbending is observed, and from spin 26^+ a reverse

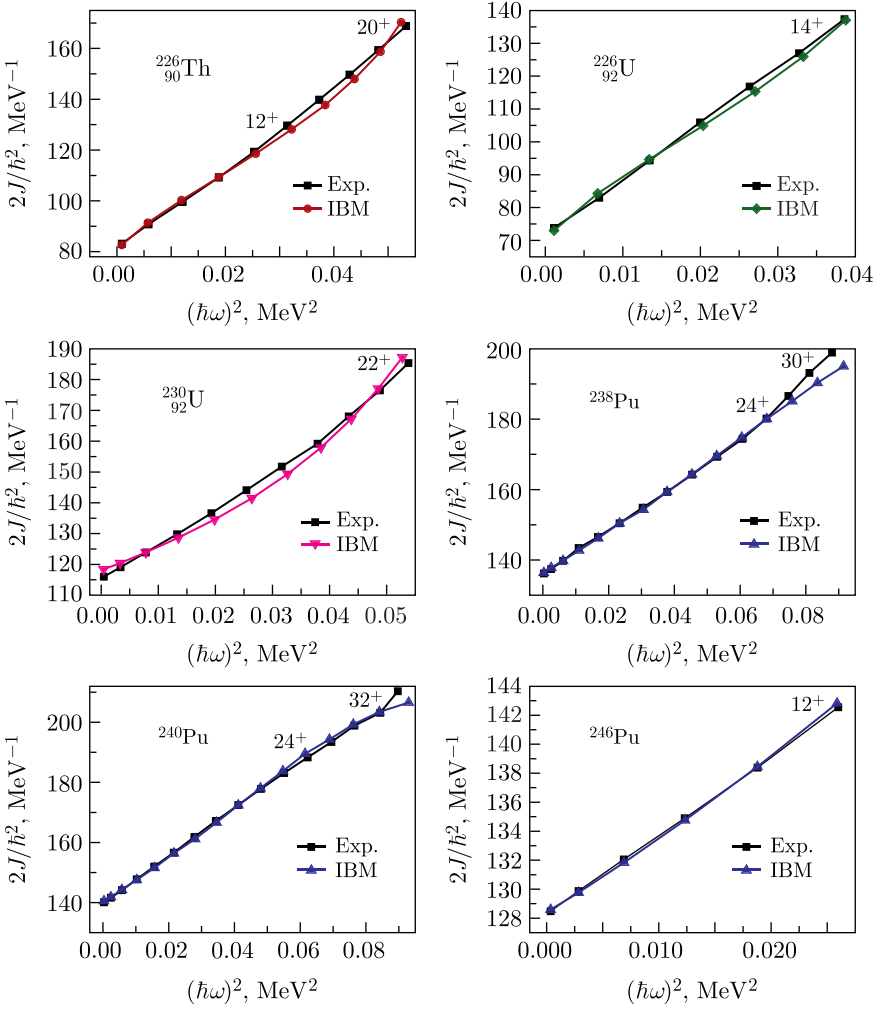


Fig. 4. The moments of inertia in the IBM phenomenology outside the analytical limits for a number of nuclei in cases where the dependence of the moments of inertia on the square of the rotation frequency is close to linear; the parameters of the boson model are given in Table 1

downbending is outlined. This situation is reproduced within the standard IBM variant.

For ^{238}U at spin 16^+ the linear part of the dependence $J(\omega^2)$ is replaced by quadratic or upbending, but at spin 30^+ it is definitely replaced by downbending. The fact that this situation has been successfully described within the IBM framework is quite surprising and is related to the specific

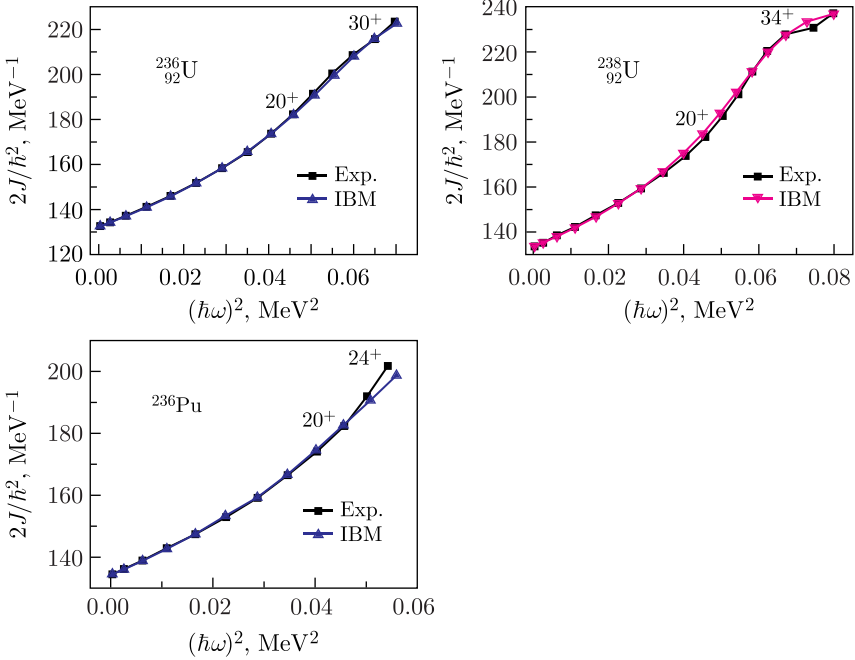


Fig. 5. The moments of inertia for a number of nuclei in cases when the moments of inertia obtained from experimental data are well reproduced in the IBM phenomenology; the parameters of the boson model are given in Table 2

values of the Hamiltonian parameters (3) given in Table 2, as well as to the fact that for spins $30^+ - 34^+$ the configuration space constructed from the set of d bosons is reduced.

In ^{236}Pu , the band is observed up to 24^+ and the moments of inertia are reproduced up to 20^+ . At high spins the calculated energies exceed the experimental ones, which may be, as was said earlier, due to the role of high-spin excitation modes, but without yet leading to the backbending effect.

Table 2. **IBM Hamiltonian parameters for a number of nuclei in which IBM describes well the nonlinear behavior of the moment of inertia as a function of the square of the rotation frequency**

Parameter	^{236}U	^{238}U	^{236}Pu
Ω	25	25	24
ε_d	-0.620910	-0.628369	-0.849539
k_1	-0.059842	-0.060053	-0.063895
k_2	0.051739	0.052799	0.047109
C_0	0.724167	0.693811	0.795943
C_2	0.043859	0.041273	0.036641
C_4	0.019167	0.016477	0.036367

CONCLUSIONS

The limiting cases of IBM are unable to reproduce the observed effective moments of inertia in heavy nuclei. Outside known limits, i.e., for arbitrary sets of parameters, it is possible. In particular, this applies to the linear nature of $J(\omega^2)$, as well as to the case when the moment of inertia grows faster than the linear dependence on ω^2 gives. It is found that the traditional IBM Hamiltonian describes both a noticeable increase in the moment of inertia — upbending, and a weakening of this increase depending on ω^2 — downbending. The latter is apparently associated with a decrease in the collective configuration space as the spin increases, that is, with the finiteness of the number of Ω bosons, although in calculations it is quite large. It should be noted that the presented moments of inertia were obtained with constant IBM parameters for all states in each of the nuclei considered, and in a number of cases, successful reproduction of the moments of inertia up to the limit of observed spins, namely, up to spin 34^+ , is achieved. This shows that even before the intersection of the bands, rather complex changes in the moments of inertia with changes in spins are possible, but their description is possible in purely collective models.

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