Anomalies and superpotential in $\mathcal{N}=1$ noncommutative gauge theories *)

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The anomaly of various currents in the noncommutative supersymmetric $N=1$, $U(1)$ gauge theory are calculated and the effective superpotential obtained.

PACS: 11.15.Bt
Key words: noncommutative $\mathcal{N}=1$ supersymmety, effective superpotential

1 Introduction

Noncommutative gauge theories emerged in the context of string theory in the presence of a nontrivial background of one of the massless states of closed string, and has been extensively studied in the last six years.

One important aspect of quantum field theories is the violation of classical symmetries of the theory due to quantum effects, anomalies. It has been found that in noncommutative gauge theories anomalies have essential new aspects. In this talk the most salient features of the axial anomaly in noncommutative QED are discussed following [1, 2].

Anomalies also play a crucial role in derivations of effective actions for supersymmetric gauge theories.

Utilization of the axial anomalies of noncommutative gauge theories and a corresponding anomaly in the supersymmetric version in the derivation of an effective action of the $\mathcal{N}=1$ supersymmetric noncommutative $U(1)$ gauge theory are briefly reviewed [3].

2 Anomalies

2.1 Noncommutative $U(1)$ gauge theory

Noncommutativity from string theory in the presence of background antisymmetric field $B_{\mu\nu}$ on a brane gives (only $B_{12}\neq 0$ is assumed here)

$$[x^1, x^2] = \mathrm{i}\Theta\,, \quad \text{where } \Theta \text{ is related to } B. \tag{1}$$

*) Talk presented by F.A. in the XI–th International Conference, Symmetry Methods in Physics, Prague, June 21, 2004

Then the low energy of the string theory gives a noncommutative gauge theory on the brane, where products are substituted by $\star$–products defined by

$$f(x)\star g(x)\equiv f(x)\exp\left(\frac{\mathrm{i}\Theta_{\mu\nu}}{2}\ \overleftarrow{\partial}_{\mu}\overrightarrow{\partial}_{\nu}\right)g(x)\,, \tag{2}$$

with the properties

$$\exp(\mathrm{i}kx)\star\exp(\mathrm{i}px) = \exp\big(\mathrm{i}(k+p)x\big)\exp\left(-\tfrac{\mathrm{i}}{2}\,\Theta_{\mu\nu}k^{\mu}p^{\nu}\right), \tag{3}$$

$$\int f\star g = \int f\,g\,, \tag{4}$$

$$\int f\star g\star h = \int h\star f\star g\,. \tag{5}$$

The noncommutative $U(1)$ gauge theory is given by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}\,F_{\mu\nu}\star F^{\mu\nu}+\bar{\psi}\star(\mathrm{i}\not{D}-m)\star\psi\,, \tag{6}$$

with

$$\begin{aligned} D_{\mu}\psi &= \partial_{\mu}\psi+\mathrm{i}gA_{\mu}\star\psi\,,\\ F_{\mu\nu} &= \partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}+\mathrm{i}g\left(A_{\mu}\star A_{\nu}-A_{\nu}\star A_{\mu}\right). \end{aligned} \tag{7}$$

A most important property of noncommutative field theories is the UV/IR mixing, where the better UV behavior due to the phase

$$\exp\left(\frac{\mathrm{i}\Theta^{\mu\nu}}{2}\,k_{\mu}p_{\nu}\right),$$

comes back to haunt as IR singularity in the "nonplanar" diagram [4]. The effective cutoff

$$\Lambda_{\text{eff.}}^{-2} = \Lambda^{-2}+p\circ p\,,\quad \text{with}\quad p\circ p\equiv p_{\mu}\Theta^{\mu\rho}\Theta_{\rho\nu}p^{\nu}\,, \tag{8}$$

becomes finite as $\Lambda\to\infty$, thus giving good UV behavior; while as $p\to 0$, singularity as IR reappears.

2.2 Anomalies in commutative gauge theory

Generally symmetries of an action, e.g.,

$$I = \int\bar{\psi}\,(i\not{D})\,\psi\,, \tag{9}$$

under

$$\begin{aligned} &\psi\to \mathrm{e}^{\mathrm{i}\alpha}\psi &&\text{for }\alpha\text{ constant,}\\ \text{or}\quad &\psi\to \mathrm{e}^{\mathrm{i}\gamma_5\alpha}\psi &&\text{for }\alpha\text{ constant,} \end{aligned} \tag{10}$$

are violated upon quantization. In the path integral formulation

$$Z = \int\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\mathrm{e}^{-\mathrm{i}I}\,, \tag{11}$$

anomaly is the consequence of noninvariance of the measure; while invariance of the action

$$\delta\mathcal{L} = \alpha\ \partial_\mu j^\mu\,, \quad j^\mu = \bar\psi\gamma^\mu\psi\,, \quad \partial_\mu j^\mu = 0\,, \tag{12}$$

(same with $j_5^\mu = \bar\psi\gamma^\mu\gamma^5\psi$), leads to a conserved charge,

$$Q = \int j_0 \mathrm{d}^3x\,, \quad \dot{Q} = 0\,. \tag{13}$$

To find the anomaly it is convenient to consider a modified derivation, where α is initially to depend on x,

$$\delta\psi = \mathrm{i}\alpha(x)\psi(x)\,, \tag{14}$$

then

$$\delta I = \int \alpha(x)\partial_\mu j^\mu(x) = -\int \partial_\mu\alpha j^\mu = 0\,;$$

if α is to be a constant,

$$\partial_\mu j^\mu = 0\,. \tag{15}$$

Under this change of variable, measure changes as

$$\mathcal{D}\bar\psi\,\mathcal{D}\psi \to \mathcal{D}\bar\psi\,\mathcal{D}\psi\ \exp\left(-2\mathrm{i}\int \alpha(x)\sum_n \varphi_n^\dagger(x)\varphi_n(x)\right)\,, \tag{16}$$

where

$$\psi = \sum_n a_n\varphi_n\,, \quad \mathrm{i}\not{D}\varphi_n = \lambda_n\varphi_n\,. \tag{17}$$

Regularization of the sum,

$$\sum_n \varphi_n^\dagger\varphi_n \longrightarrow \lim_{M\to\infty}\sum_n \varphi_n^\dagger\varphi_n \exp\left(-\frac{\lambda_n^2}{M^2}\right)\,, \tag{18}$$

replaces the exponent in (17) by a gauge invariant expression

$$\lim_{M\to\infty}\sum_n \varphi_n^\dagger \exp\left(\frac{(\mathrm{i}\not{D})^2}{M^2}\right)\varphi_n\,, \tag{19}$$

which gives no anomaly for the $U(1)$ symmetry

$$\partial_\mu j^\mu = 0\,, \tag{20}$$

while giving an anomaly

$$\partial_\mu j_5^\mu = -\frac{g^2}{16\pi^2}F_{\mu\nu}\tilde{F}_{\mu\nu}\,, \quad \tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}\,, \tag{21}$$

to the axial symmetry

$$\delta\psi = \mathrm{i}\alpha\gamma_5\psi \quad \text{axial (chiral).} \tag{22}$$

2.3 Anomalies in noncommutative gauge theory

The same can now be repeated for the noncommutative gauge theory, where

$$\mathcal{L} = \bar{\psi} \star (\mathrm{i}\not{D}) \star \psi = \bar{\psi} \star (\mathrm{i}\not{\partial} + \mathrm{i}g\not{A}) \star \psi \,. \tag{23}$$

The transformation

$$\psi(x) \rightarrow \mathrm{e}^{\mathrm{i}\alpha\gamma_5}\psi(x)\,, \tag{24}$$

leads to a classical symmetry,

$$\Rightarrow \partial_\mu j_5^\mu = 0\,, \quad j_5^\mu = \bar{\psi} \star \gamma^\mu \gamma_5 \psi\,, \tag{25}$$

and

$$Q_5 = \int j_0^5 \, \mathrm{d}^3 x\,, \quad \dot{Q}_5 = 0\,. \tag{26}$$

To study its anomaly the modified procedure is used, which gives two distinct currents in contrast to ordinary theory,

a) the transformation

$$\psi(x) \rightarrow e^{i\alpha(x)\gamma_5} \star \psi(x), \tag{27}$$

leads to a covariantly conserved current J_5^μ;

$$\Rightarrow J_5^\mu = \psi_\beta \star \bar{\psi} \left(\gamma^\mu \gamma^5\right)_{\alpha\beta}\,, \qquad D_\mu J_5^\mu \equiv \partial_\mu J_5^\mu + ig\left[A_\mu, J_5^\mu\right]_\star = 0. \tag{28}$$

b) the transformation

$$\psi(x) \rightarrow \psi \star e^{i\alpha(x)\gamma_5}, \tag{29}$$

leads to an invariant conserved current j_μ^5

$$\Rightarrow j_5^\mu = \bar{\psi} \star \gamma^\mu \gamma^5 \psi, \qquad \partial_\mu j_\mu^5 = 0. \tag{30}$$

Here ψ transforms in the fundamental representation $\psi \rightarrow U \star \psi$ and $\bar{\psi} \rightarrow \bar{\psi} \star U^\dagger$.

The measure of the path integral is again not invariant

$$\mathcal{D}\psi \; \mathcal{D}\bar{\psi} \rightarrow \mathcal{D}\psi \; \mathcal{D}\bar{\psi} \; e^{-2i\int \alpha\star\mathcal{A}}, \tag{31}$$

with $\mathcal{A}$ the corresponding anomaly.

Now the significant difference with the ordinary theory appears: The two choices for the change of variables lead to distinct anomalies for $D_\mu J_5^\mu$ and $\partial_\mu j_5^\mu$,

a) for J_μ^5,

$$\delta\psi = \mathrm{i}\alpha(x)\gamma_5 \star \psi(x)\,, \tag{32}$$

$$\mathcal{A} = \sum_n \left(\varphi_n\right)_\beta \star \left(\varphi_n^\dagger\right)_\alpha (\gamma^5)_{\alpha\beta}\,. \tag{33}$$

After regularization,

$$\mathcal{A} = \lim_{M\to\infty} \sum_n \left(\mathrm{e}_\star^{-\not{D}^2/M^2} \star \varphi_n \right)_\beta \star \left(\varphi_n^\dagger\right)_\alpha (\gamma^5)_{\alpha\beta} \tag{34}$$

and noting that J_5^μ, $D_\mu J_5^\mu$, and $\mathcal{A}$ are covariant under gauge transformation,

$$U:\ \mathcal{O} \to U \star \mathcal{O} \star U^{-1} \tag{35}$$

a tedious calculation [1, 2, 3] leads to

$$D_\mu J_5^\mu = -\frac{g^2}{16\pi^2}\, F_{\mu\nu} \star \tilde{F}^{\mu\nu}\,; \tag{36}$$

b) For j_μ^5,

$$\delta\psi = \psi(x) \star i\alpha(x)\gamma_5\,, \tag{37}$$

$$\mathcal{A} = \sum_n \varphi_n^\dagger \star \gamma^5 \varphi_n\,, \tag{38}$$

an *invariant* regularization is needed. Using Wilson line

$$\mathcal{A} = \lim_{M\to\infty} \sum_n \varphi_n^\dagger \star \gamma_5 \left[\mathrm{e}^{(\mathrm{i}\not{D})^2/M^2}\right]_{\mathrm{inv.}} \star \varphi_n(x)\,, \tag{39}$$

where

$$\left[\mathcal{O}\right]_{\mathrm{inv.}} \equiv \int \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kx} \int \mathrm{d}y \sum_n \frac{1}{n!} \prod_{i=1}^n \int \mathrm{d}\sigma_i P_\star \left[W(y,\hat{k})\mathcal{O}(y+\sigma_i\hat{k})\right] \star \mathrm{e}^{\mathrm{i}ky}\,, \tag{40}$$

$$W(y,\hat{k}) = \exp\left(\mathrm{i}\int_0^1 \mathrm{d}\sigma \hat{k}\cdot A(x+\sigma\hat{k})\right), \tag{41}$$

$\hat{k}_\mu = \Theta_{\mu\nu} k^\nu$,

$$\begin{gathered} \mathcal{O} \to U \star \mathcal{O} \star U^{-1}\,, \quad \text{but} \\ \left[\mathcal{O}\right] \to \left[\mathcal{O}\right], \end{gathered} \tag{42}$$

one gets for $\Theta p \to 0$

$$\partial_\mu j_5^\mu = -\frac{g^2}{16\pi^2}\, F_{\mu\nu} \star' \tilde{F}^{\mu\nu} \tag{43}$$

with

$$f(x) \star' g(x) \equiv f(x) \frac{\sin\left(\frac{1}{2}\overleftarrow{\partial}_\mu\, \Theta^{\mu\nu}\, \overrightarrow{\partial}_\nu\right)}{\frac{1}{2}\overleftarrow{\partial}_\mu\, \Theta^{\mu\nu}\, \overrightarrow{\partial}_\nu}\, g(x)\,. \tag{44}$$

While covariant J_5^μ anomaly involves the $\star$–product, the invariant j_5^μ anomaly involves the $\star'$–product [2, 3].

2.4 Konishi anomaly (noncommutative)

Similar considerations apply to the current of supersymmetric (SUSY) gauge theory. In $\mathcal{N}=1$ SUSY gauge theory

$$I=\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,\bar\Phi\star\mathrm{e}_\star^V\star\Phi+\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,W_\alpha\star W^\alpha+\text{h.c.}\,, \tag{45}$$

with $\Phi(x,\theta)$ the hypermuliplet and $V(x,\theta,\bar\theta)$ the gauge field,

$$\begin{aligned} W_\alpha(x,\theta) &= \bar D^2\,\mathrm{e}_\star^{-V}\star D_\alpha \mathrm{e}_\star^V\,,\\ D &= \partial_\theta+\mathrm{i}\left(\sigma^\mu\bar\theta\right)\partial_\mu\,; \end{aligned} \tag{46}$$

the symmetry,

$$\begin{aligned} \Phi &\to \mathrm{e}^{\mathrm{i}\alpha(x,\theta)}\star\phi\,,\\ \mathrm{e}^V &\to \mathrm{e}^{\mathrm{i}\bar\alpha}\star\mathrm{e}^V\star\mathrm{e}^{-\mathrm{i}\alpha}\,,\quad \bar D\alpha=0\,. \end{aligned} \tag{47}$$

$$\begin{aligned} J &= \Phi\star\bar\Phi\star\mathrm{e}^V\,,\\ j &= \bar\Phi\star\mathrm{e}^V\star\Phi \end{aligned} \tag{48}$$

is anomalous. The anomaly is given by

$$\begin{aligned} -\frac14\,\bar D^2 J &= -\frac{1}{32\pi^2}\,W_\alpha\star W^\alpha\,,\\ -\frac14\,\bar D^2 j &= -\frac{1}{32\pi^2}W_\alpha\star' W^\alpha+\cdots\,. \end{aligned} \tag{49}$$

The calculation is the same as for nonSUSY gauge theory, except for the regulator e^{L/M^2}, where

$$L=\frac{1}{16}\,\bar D^2\mathrm{e}_\star^{-V}\,D^2\,\mathrm{e}_\star^V\,. \tag{50}$$

Lowest SUSY multiplet component of L is $\not{D}^2$, regulator brings SUSY and gauge invariance [3].

The regulator for j is the same as for the nonSUSY case with $\not{D}^2\to L$ and a corresponding Wilson line.

3 Superpotential

It is known that effective superpotential of SUSY gauge theories is the sum of a perturbative and a nonperturbative part

$$\mathcal{W}_{\text{eff}}(S)=\mathcal{W}_{\text{pert}}+\mathcal{W}_{\text{non-pert}}\,, \tag{51}$$

$\mathcal{W}_{\text{pert}}$ is calculated from Konishi anomaly, and $\mathcal{W}_{\text{non-pert}}$ from the axial anomaly. Here

$$S=W_\alpha\star W^\alpha \tag{52}$$

$$S'=W_\alpha\star' W^\alpha+\cdots\,. \tag{53}$$

The ellipses are the contribution of the Wilson line attachment.

To get $\mathcal{W}_{\text{non-pert}}$, axial anomaly is used;

$$\begin{aligned} &U_A(1) : \Phi(x,\theta) \to \mathrm{e}^{\mathrm{i}\alpha}\Phi(x,\theta) && \text{and} \quad W_\alpha(x,\theta) \to W_\alpha(x,\theta)\,, \\ &U_R(1) : \Phi(x,\theta) \to \mathrm{e}^{\mathrm{i}\alpha}\Phi\left(x, \mathrm{e}^{-3\mathrm{i}\alpha/2}\theta\right) && \text{and} \quad W_\alpha(x,\theta) \to \mathrm{e}^{3\mathrm{i}\alpha/2}W_\alpha\left(x, \mathrm{e}^{-3\mathrm{i}\alpha/2}\theta\right) \end{aligned} \tag{54}$$

$\Rightarrow$

$$\delta_A \mathcal{L} = \begin{cases} 0\,, & \dfrac{p\circ p}{4} \gg \dfrac{1}{M^2}\,, \\ 2N_f\alpha\mathcal{A}'\,, & \dfrac{p\circ p}{4} \ll \dfrac{1}{M^2} \end{cases} \tag{55}$$

and

$$\delta_R \mathcal{L} = \begin{cases} 2\alpha R(\lambda)\mathcal{A}\,, & \dfrac{p\circ p}{4} \gg \dfrac{1}{M^2}\,, \\ 2N_f\alpha R(\psi)\mathcal{A}'\,, & \dfrac{p\circ p}{4} \ll \dfrac{1}{M^2}\,, \end{cases} \tag{56}$$

Here $\mathcal{A}$ and $\mathcal{A}'$ are the planar and nonplanar ABJ anomalies. They are defined by $\mathcal{A} \equiv -\dfrac{1}{32\pi^2}\,F_{\mu\nu}\star\tilde{F}^{\mu\nu}$ and $\mathcal{A}' \equiv -\dfrac{1}{32\pi^2}\,F_{\mu\nu}\star'\tilde{F}^{\mu\nu} + \cdots$ with the extra terms denoting the contribution of the open Wilson line.

In the "small" Θp limit

$$W_{\text{non-pert}}(T, S'; \Lambda_{N_f}, \Lambda_\Theta) = S'\left(\log\left(\frac{S'^{N_f}\Lambda_\Theta^{+(N_f+6)}}{\Lambda_{N_f}^{+2(3+N_f)}\det T}\right) - N_f\right), \tag{57}$$

with $\Lambda_\Theta \equiv \Theta^{-1/2}$.

In the "large" Θp limit

$$W_{\text{dyn}}(S; \Lambda_{N_f}, \Lambda_\Theta) = -S\left(\log\left(\frac{S\Lambda_\Theta^{3-2N_f}}{\Lambda_{N_f}^{+2(3-N_f)}}\right) - 1\right). \tag{58}$$

Here $\Lambda_\Theta = \Theta^{-1/2}$, and $T_{ij} = \tilde{\Phi}_i \star \Phi_j$.

To get $\mathcal{W}_{\text{pert}}$; choose

$$\mathcal{W}_{\text{tree}} = mT_{ii} + \lambda T_{ii}^2\,; \tag{59}$$

and use Konishi anomaly to solve for T in terms of S' (small Θp limit) and S (large Θp limit):

In the "small" Θp limit

$$
\begin{aligned}
&W_{\text{eff}}\left(S';m,\lambda;\hat{\Lambda}_0,\hat{\Lambda}_\Theta\right)\bigg|_{\sqrt{\Theta}|p|\ll 1}= \\
&\quad = 6S'\log\frac{\hat{\Lambda}_\Theta}{\Lambda_0}-\frac{N_f}{2}\,S'-N_f\,\frac{m^2}{8\lambda}+(N_f^+-N_f^-)\,\frac{m^2}{8\lambda}\sqrt{1+\frac{8\lambda S'}{m^2}}+ \\
&\quad +S'\log\left[\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8\lambda S'}{m^2}}\right)^{N_f^+}\left(\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{8\lambda S'}{m^2}}\right)^{N_f^-}\right].
\end{aligned}
\tag{60}
$$

In the "large" Θp limit

$$
W_{\text{eff}}\left(\langle\operatorname{tr}T\rangle=-\frac{N_f m}{2\lambda};m,\lambda;\hat{\Lambda}_0,\hat{\Lambda}_\Theta\right)\bigg|_{\sqrt{\Theta}|p|\gg 1}=-\frac{N_f m^2}{4\lambda}-S\left(\log\frac{S\hat{\Lambda}_\Theta^3}{\hat{\Lambda}_0^6}-1\right).
\tag{61}
$$

Acknowledgment. This work was partially supported by Iranian TWAS chapter based at ISMO.

References

[1] F. Ardalan and N. Sadooghi: Int. J. Mod. Phys. A **16** (2001) 3151.

[2] F. Ardalan and N. Sadooghi: Int. J. Mod. Phys. A **17** (2002) 123.

[3] F. Ardalan and N. Sadooghi: submitted to *Phys. Rev. D* (2004).

[4] S. Minwalla, M. Van Raamsdonk and N. Seiberg: JHEP **0002** (2000) 020.